ON THE COHOMOLOGY OF THE CLASSICAL LINEAR GROUPS

BY

JACK SHAPIRO

In this paper we use the methods of [1] to partially compute the cohomology of the classical groups with coefficients in the finite field with q elements, F_q. Here q is a power of an *odd prime p*. Cohomology is the usual group cohomology of Eilenberg-MacLane [2] and coefficients are taken in Z_l, the integers mod l, where l is a prime different from p.

Inherent in this method is the equivalence between the group cohomology of G, $H^*(G)$, and the singular cohomology of BG, $H^*(BG)$, where BG is a classifying space for G (see for example [3, pp. 185–186]). In this paper we will freely interchange these two concepts.

The approach as in [1] is to tie the cohomology of BG to the cohomology of BU, where U is the infinite unitary group. This is done by the use of a virtual complex representation induced from the natural modular representation of G on F_q^n [4, Theorem 1]. Strong use is made of the classical Lie theory associated to these groups by Chevalley [5] (e.g., the action of a Weyl group on diagonal subgroups of G is critical for the analysis). In one form the main theorem says that the cohomology of G is generated by Chern classes (see [6, Appendix]).

As in [1] we must pass to a certain subfield, k_1, of the algebraic closure of F_q in order to complete the computations. Let T denote the *diagonal* subgroup of G [7, chapter 7] and W the Weyl group of G. Another form of the main theorem says that $H^*(G) \cong H^*(T)^W$, the fixed subring of $H^*(T)$ under the induced action of W. This theorem was proved in [1] for $GL_n(k_1)$ and $O_n(k_1)$, the general linear and orthogonal groups. In this paper we extend the results to the other classical groups $SL_n(k_1)$, the special linear groups, $Sp_{2m}(k_1)$, the symplectic groups and if q is an even power of p $U_n(k_1)$, the unitary groups. No attempt is made to complete the results in F_q itself as is done for $GL_n(F_q)$ in [8].

1. Definitions

Let p be any odd prime and $q = p^s$ where s is a positive integer. F_q will stand for the finite field with q elements and $GL_n(F_q)$ will be the general linear group over F_q (i.e., elements of $GL_n(F_q)$ are the $n \times n$ matrices with coefficients in F_q whose determinant is nonzero). We will consider a number of other classical linear groups and view them as subgroups of $GL_n(F_q)$.

The easiest to define is the subgroup of elements whose determinant is 1. This subgroup is denoted by $SL_n(F_q)$, *the special linear group*.

Received Feb. 28, 1974.

Now suppose V, an n-dimensional vector space over F_q is endowed with a nonsingular scalar product which is skew-symmetric (i.e., $(v, w) = -(w, v)$) then the subgroup of isometries with respect to the scalar product is called the *symplectic group*. It is well known that $n = 2m$ must be even and we denote this group by $Sp_{2m}(F_q)$. It is, up to isomorphism, independent of the choice of a skew-symmetric scalar product. We will call a basis $\{v_1, \ldots, v_{2m}\}$ for V a *symplectic basis* if $(v_i, v_{i+m}) = 1$ for $i = 1, \ldots, m$ and $(v_i, v_j) = 0$ otherwise, $i \leq j$ (i.e., the matrix of the bilinear form with respect to this basis is

$$\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

In the special case when the order of the finite field is q^2 we can define an involution of F_{q^2} by $\lambda \to \lambda^q \equiv \bar{\lambda}$, $\lambda \in F_{q^2}$. If V is now endowed with a nonsingular *hermitian scalar product* (i.e., $(v, w) = \overline{(w, v)}$), then the elements of $GL_n(F_{q^2})$ which are isometries with respect to this scalar product form a group $U_n(F_{q^2})$, the *unitary group*. Again this group is, up to isomorphism, independent of the choice of a hermetian scalar product. A basis for V, $\{v_1, \ldots, v_n\}$ will be called a *unitary basis* if $(v_i, v_j) = \delta_{ij}$.

2. Main theorems

For each group G studied in the previous section we will define a subgroup T which will play the role of the *maximal torus* in the classical Lie group theory. We will show that under the map induced by the inclusion of T in G, $H^*(G) \to H^*(T)$ is a monomorphism, where l is a prime different from p (in some cases we will also assume $l \neq 2$). In the cases discussed in this paper T will always be the *diagonal subgroup* of G.

Let N, the normalizer of T in G, act on T by conjugation. We then have a finite group $W \equiv N/T$ acting on T. W is called the *Weyl group*. Let $H^*(T)^W$ denote the fixed subring of $H^*(T)$ under the induced action of W. An inner automorphism of G induces the identity on cohomology [3, Proposition 16.2] so we will consider $H^*(G)$ as a subring of $H^*(T)^W$.

At this point we pass to a subfield, k_1, of the algebraic closure, k, of F_p which contains all the l^rth roots of unity for all r. In this case there is no odd dimensional cohomology classes to consider (see [1]). We define the analogous subgroups of $GL_n(k_1)$ and their diagonal subgroups. The "Brauer lift" of the natural modular representation of a subgroup, G, of $GL_n(k_1)$ on k^n induces a map in the homotopy category from $BG \to BU$ (see [1]; Section 1]). If c_i denotes the ith universal Chern class we get the following addendum to Theorem 4.7 of [1].

THEOREM 1. *Let $\eta: BSL_n(k_1) \to BU$ represent the homotopy class induced by the natural modular representation. Let l be a prime, $l \neq p$ and $l \nmid n$; then*

$$H^*(BSL_n(k_1); Z_l) \cong Z_l[\eta^*(c_2), \ldots, \eta^*(c_n)],$$

a polynomial algebra in $n - 1$ indeterminates.

COROLLARY 1.1. *$H^*(BSL_n(k_1))$ is generated by Chern classes.*

Note. This follows trivially from the theorem.

COROLLARY 1.2. *$H^*(SL_n(k_1)) \cong H^*(T)^W$ where T is the diagonal subgroup.* This corollary will follow from the proof of Theorem 1 (Section 3).

THEOREM 2. *Let λ: $BSp_{2m}(k_1) \to BU$ represent the homotopy class induced by the modular representation and let l be an odd prime different from p. Then*

$$H^*(BSp_{2m}(k_1); Z_l) \cong Z_l[\lambda^*(c_2), \ldots, \lambda^*(c_{2m})].$$

THEOREM 3. *Let l be an odd prime, $l \neq p$, and let δ: $BU_n(k_1) \to BU$ be the homotopy class induced by the modular representation. Then*

$$H^*(BU_n(k_1); Z_l) \cong Z_l[\delta^*(c_1), \ldots, \delta^*(c_n)].$$

The obvious corollaries analogous to those stated after Theorem 1 can be stated and proved. In addition if we use the fact proved in the appendix to [1] that λ: $BSp_{2m}(k_1) \to BU$ factors through BSp, the infinite symplectic group, we get the following additional corollaries.

COROLLARY 2.1. *λ induces an isomorphism from $H^*(BSp(k_1))$ to $H^*(BSp)$ where $Sp(k_1)$ is the infinite symplectic group over the field k_1.*

COROLLARY 3.1. *δ induces an isomorphism $H^*(BU(k_1)) \cong H^*(BU)$ where $U(k_1)$ is the infinite unitary group over k_1.*

Proof. Both corollaries follow by letting $n \to \infty$ and using the known results about the cohomology of BU and Bsp.

3. $SL_n(F_q)$

For $G = SL_n(F_q)$ we let $T = ST_{n-1}(F_q)$ be the subgroup of diagonal matrices of determinant 1. T is isomorphic to $(F_q^*)^{n-1}$, where F_q^* is the multiplicative group of non zero elements in F_q. Let $\bar{N}$ be the subgroup of the normalizer described as follows. $\bar{N}$ is generated by the elements of T together with all permutation matrices which have a ± 1 in the nth column nth row. If $\sum_n$ denotes the symmetric group on n-elements then $\bar{N}$ is isomorphic to the semi-direct product of $\sum_{n-1}$ and $(F_q^*)^{n-1}$, where $\sum_{n-1}$ acts by permuting the $n-1$ copies of F_q^*. Another way of writing this is $\bar{N} \cong \sum_{n-1} \wr F_q^*$, the wreath product of $\sum_{n-1}$ and F_q^*. The normalizer N is of order $n!(q-1)^{n-1}$ and can be described as $g \in SL_n(F_q)$ such that conjugating any diagonal matrix by g induces a permutation of the diagonal entries.

$$|SL_n(F_q)| = q^{\frac{n(n-1)}{2}}(q^2-1)(q^3-1)\cdots(q^n-1) \quad [7, \text{chapter } 1]$$

and as in [1, Lemma 4.2] if l is a prime dividing $q-1$ then $[SL_n(F_q): N]$ is an l-adic unit. $[N: \bar{N}] = n$ so that if we assume further that $l \nmid n$ then $H^*(SL_n(F_q)) \to H^*(\bar{N})$ will be a monomorphism [3, Theorem 16.4].

We say that a family H_i, $i \in I$, of subgroups of a group G detects the cohomology of G (mod l) if the map $H^*(G) \to \prod_i H^*(H_i)$ given by the restriction homorphisms is injective.

LEMMA 1. *Let G be a group whose* mod l *cohomology is detected by a family of abelian subgroups of exponent dividing* l^a *with* $a \geq 1$. *Then* $\sum_n \wr G$ *has the same property.*

Proof. [1, Proposition 3.4].

If $l \mid (q - 1)$ then F_q^* satisfies the hypothesis of Lemma 1 and therefore there exists abelian subgroups, A_i, of $\bar{N}$ of exponent l^a, where $l^a \mid (q - 1)$, $a \geq 1$, satisfying the conclusion. As remarked, an inner automorphism on the group level induces the identity on the cohomology level. Therefore, if we can show that each A_i is conjugate to a subgroup of $ST_{n-1}(F_q)$ in $SL_n(F_q)$ we get the following proposition:

PROPOSITION 1. *If* l *is a prime which divides* $q - 1$ *and furthermore if* $l \nmid n$ *then* $H^*(SL_n(F_q); Z_l) \to H^*(ST_{n-1}(F_q); Z_l)$ *is a monomorphism, where the map is induced by inclusion.*

Proof. By the previous remarks we must show that each A_i is conjugate to a subgroup of $ST_{n-1}(F_q)$ in $SL_n(F_q)$. A_i is abelian and has exponent dividing $q - 1$ therefore the irreducible subspaces of F_q^n under the action of A are all 1-dimensional [9, p. 272]. Since the order of A_i is prime to p the representation is completely reducible [9, p. 253]. This implies that there is a basis for F_q^n for which all of A_i is simultaneously diagonalized (i.e., A_i is conjugate to a subgroup of the diagonal matrices). Since this conjugation can be done using elements of $SL_n(F_q)$ the image lies in $ST_{n-1}(F_q)$. Q.E.D.

Proof of Theorem 1. We pass to k_1, a subfield of the algebraic closure of F_q which contains all the l^rth roots of unity for all $r \in Z$. $H^*(k_1^*) \cong Z_l[x]$ where x is the first Chern class of the 1-dimensional complex representation induced by embedding k_1^* in $S^1 \subseteq \mathbf{C}^*$. $T = ST_{n-1}(k_1)$ is isomorphic to $(k_1^*)^{n-1}$ by projecting onto the first $n - 1$ diagonal entries. In the notation of [1, Section 4], $H^*(T) \cong Z_l[x_1, \ldots, x_{n-1}]$. $W \cong \sum_n$ [12, p. 115] and acts by permuting the diagonal entries of T. If we let $x_n \equiv -(x_1 + \cdots + x_{n-1})$ then the induced action on $H^*(T)$ is the action of $\sum_n$ on $\{x_1, \ldots, x_n\}$.

Since the Brauer lift of the natural modular representation restricted to $ST_{n-1}(k_1)$ is a homomorphism into the diagonal matrices of determinant 1, $\eta \mid BST_{n-1}(k_1)$ factors through $BSU \to BU$ where SU is the infinite special unitary group.

$$H^*(BSU) \cong Z_l[sc_2, \ldots, sc_n]$$

where the sc_i are the images of c_i under the map $H^*(BU) \to H^*(BSU)$. Therefore

$$\eta^*(c_i) \mid BST_{n-1}(k_1) = (\eta \mid BST_{n-1}(k_1))^*(sc_i) \quad \text{for } i \geq 2.$$

Let $\overline{T}$ be the diagonal subgroup of $SU(n)$, $\overline{T} \cong (S^1)^{n-1}$. The Weyl group acts on $\overline{T}$ by permuting the diagonal entries [12, p. 115]. If we write

$$H^*(B\overline{T}) \cong Z_l[y_1, \ldots, y_{n-1}]$$

and if we define $y_n \equiv -(y_1 + \cdots + y_{n-1})$ then the Weyl group acts on $H^*(BT)$ as the full symmetric group on the set $\{y_i\}_{i=1}^n$. We also have that $\eta^* \mid BST_{n-1}(k_1)$ pulls y_i back to x_i for all i. In this notation the sc_i are the ith elementary symmetric polynomials in the y_i. In particular $\eta^*(c_i) \mid BST_{n-1}(k_1)$ is the ith elementary symmetric polynomial in the x_i where

$$H^*(ST_{n-1}(k_1)) \cong Z_l[x_1, \ldots, x_{n-1}] \quad \text{and} \quad x_n = -(x_1 + \cdots + x_{n-1}).$$

The result now follows. Q.E.D.

4. SP_{2m}

Let $\tilde{T}_m(F_q)$ be the intersection of the diagonal subgroup of $GL_{2m}(F_q)$ with $Sp_{2m}(F_q)$. If the matrices are written with respect to a symplectic basis then a diagonal matrix $((\lambda_i))_{i=1}^{2m}$ will be in $Sp_{2m}(F_q)$ if $\lambda_{i+m} = \lambda_i^{-1}$. This implies that $\tilde{T}_m(F_q) \cong (F_q^*)^m$. The normalizer of $\tilde{T}_m(F_q)$ in $Sp_{2m}(F_q)$ is generated by: (a) matrices of the form

$$\begin{pmatrix} p_m & 0 \\ 0 & p_m \end{pmatrix}$$

where p_m is an $m \times m$ permutation matrix; (b) matrices which by conjugation on a diagonal matrix transpose the ith and $(i + m)$th diagonal entries; and (c) $\tilde{T}_m(F_q)$. Therefore $|N| = 2^m m!(q - 1)^m$. The order of the group is

$$|Sp_{2m}(F_q)| = q^{m^2} \prod_{j=1}^{m} (q^{2j} - 1) \quad [7, \text{chapter } 1].$$

If l is an odd prime which divides $q - 1$ then as before $[Sp_{2m}(F_q) : N]$ is an l-adic unit and $H^*(Sp_{2m}(F_q)) \to H^*(N)$ is a monomorphism.

If $\overline{N}$ is the subgroup of N generated by matrices of type (a) and (c) then $\overline{N} \cong \sum_m \wr F_q^*$ and $[N : \overline{N}] = 2^m$. Since l is odd this implies that

$$H^*(Sp_{2m}(F_q)) \to H^*(\overline{N})$$

is a monomorphism. As in the previous case, [1, Lemma 3.4] assures the existence of abelian subgroups, A_i, of $\overline{N}$ of exponent l^a where $l^a \mid q - 1$, $a \geq 1$, such that $H^*(\overline{N}) \to \prod_i H^*(A_i)$ is 1-1. It then follows, as described previously, that A_i is conjugate to a subgroup of diagonal matrices. In order to complete this case, we must show that this conjugation can be carried out inside $Sp_{2m}(F_q)$ (i.e., there is a *symplectic basis* under which all elements of A_i are simultaneously diagonalized).

Let $v_1, v_2, \ldots, v_{2m}$ be a basis of V under which all of A_i is simultaneously diagonalized. Such a basis exists since A_i is conjugate to a subgroup of diagonal matrices. If $a \in A_i$ then $av_i = \lambda_i(a)v_i$ where $\lambda_i(a) \in F_a^*$. Since $p \neq 2$ our scalar product is alternate (i.e., $(v, v) = 0$ for all $v \in V$). Therefore there is a v_i, $2 \leq i \leq 2m$, for which $(v_1, v_i) \neq 0$. We might as well assume that $i = 1 + m$ and that $(v_1, v_{1+m}) = 1$. Since a is symplectic, $\lambda_{1+m}(a) = \lambda_1(a)^{-1}$. If we now complete $\{v_1, v_{1+m}\}$ to a basis

$$\{v_1, v_{1+m}, w_2, \ldots, \hat{w}_{i+m}, \ldots, w_{2m}\}$$

for V so that $(v_1, w_i) = (v_{1+m}, w_i) = 0$ for all i [10, pp. 79–80] then the space spanned by the $\{w_i\}$ forms a subrepresentation space for A_i. For if

$$aw_i = \mu_1 v_1 + \mu_{1+m} v_{1+m} + \cdots,$$

then $\mu_{m+1} = (aw_i, v_1) = (w_i, a^{-1}v_1) = 0$ and similarly for μ_1. By finite induction we can find our desired symplectic basis and we get the following proposition.

PROPOSITION 4. *If l is an odd prime which divides $q - 1$ then*

$$H^*(Sp_{2m}(F_q); Z_l) \to H^*(\tilde{T}_m(F_q); Z_l)$$

is a monomorphism.

Proof of Theorem 2. We again pass to k_1 and get $\tilde{T}_m(k_1) \cong (k_1^*)^m$. Therefore

$$H^*(\tilde{T}_m(k_1)) \cong Z_l[x_1, \ldots, x_m].$$

Let us choose as the isomorphism from $(k_1^*)^m$ to $\tilde{T}_m(k_1)$ the projection onto the first m diagonal entries. Then W acts by permuting the first m diagonal entries (simultaneously permuting the last m diagonal entries in the identical manor) and by transposing the ith and $(i + m)$th entries. Since the first Chern class of a dual representation is equal to minus the first Chern class of the representation [6, Appendix] W acts by permuting the x_i and by sending $x_i \to -x_i$. It follows then that $H^*(\tilde{T}_m(k_1))^W$ is generated by symmetric polynomials in the x_i^2.

The induced complex representation restricted to $\tilde{T}_m(k_1)$ is a homomorphism into a diagonal subgroup of $U_{2m}(\mathbf{C})$. This is the subgroup of all diagonal matrices whose $(i + m)$th diagonal entry is the inverse of the ith diagonal entry, $1 \leq i \leq m$.

Let

$$Sp_{2m}(\mathbf{C}) \overset{j}{\hookrightarrow} U_{2m}(\mathbf{C})$$

be the natural inclusion. Then the diagonal subgroup of $Sp_{2m}(\mathbf{C})$, T', is the subgroup of diagonal matrices in $U_{2m}(\mathbf{C})$ just described. Suppose j also represents the induced map from $BS_p \to BU$; then

$$H^*(BSp) \cong Z_l[e_1, e_2, \ldots],$$

where e_i is the ith universal symplectic Pontryagin class and within sign $e_i = j^*(c_{2i})$ [11, 9.6]. Let T be the diagonal subgroup of $U_{2m}(\mathbf{C})$ then

$$H^*(T) \cong Z_l[y_1, \ldots, y_{2m}], \quad H^*(T')) \cong Z_l[v_1, \ldots, v_m]$$

and $j^*(y_i) = v_i$, $j^*(y_{i+m}) = -v_i$, $1 \leq i \leq m$. With this notation e_i is the ith elementary symmetric polynomial on $\{v_1, \ldots, v_m\}$.

The above analysis implies that $\lambda \mid B\tilde{T}_m(k_1)$ factors through BSp and in fact

$$\lambda^*(c_{2i}) \mid B\tilde{T}_m(k_1) = (\lambda \mid B\tilde{T}_m(k_1))^*(e_i).$$

It now follows from the product formula for Chern classes and from the previous remarks that $\lambda^*(c_{2i}) \mid B\tilde{T}_m(k_1)$ is the ith elementary symmetric polynomial in the x_i^2 where $H^*(\tilde{T}_m(k_1)) \cong Z_l[x_1, \ldots, x_m]$. Q.E.D.

Note. As remarked previously, in the appendix to [1] it is shown that

$$\lambda: BSp_{2m}(k_1) \to BU$$

factors through BSp. Letting λ also designate the map $BSp_{2m}(k_1) \to BSp$ then

$$H^*(BSp_{2m}(k_1) \cong Z_l[\lambda^*(e_1), \ldots, \lambda^*(e_n)]$$

and Corollary 2.1 follows from the fact that $H^*(BSp) \cong Z_l[e_1, e_2, \ldots]$.

5. BU_n

For the final case, $G = U_n(F_{q^2}) \leq GL_n(F_{q^2})$. Let $T = UT_n(F_{q^2})$ be the subgroup of diagonal matrices. If matrices are written with respect to a unitary basis then the diagonal matrix $((\lambda_i))$ is in $U_n(F_{q^2})$ iff $\lambda_i\bar{\lambda}_i = \lambda_i^{q+1} = 1$. The elements $\lambda_i \in F_{q^2}$ which have the above property form a cyclic subgroup of order $q + 1$, Z_{q+1}, in $F_{q^2}^*$. This implies that $UT_n(F_{q^2}) \cong (Z_{q+1})^n$. Since the permutation matrices are all unitary, it follows that N, the normalizer of $UT_n(F_{q^2})$ in $U_n(F_{q^2})$ is isomorphic to $\sum_n \wr Z_{q+1}$ and $|N| = n!(q + 1)^n$.

$$|U_n(F_{q^2})| = q^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} (q^j - (-1)^j).$$

Therefore

$$[U_n(F_{q^2}): N] = q^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} \frac{q^j - (-1)^j}{j(q+1)}.$$

LEMMA. *If l is odd and $l \mid q + 1$, $l \neq p$, then $(q^j - (-1)^j)/j(q + 1)$ is an l-adic unit.*

Proof. Suppose $q + 1 = kl^n$ where $l \nmid k$. Then

$$\begin{aligned} q^j - (-1)^j &= (kl^n - 1)^j - (-1)^j \\ &= \sum_{s=0}^{j} \binom{j}{s} (kl^n)^s(-1)^{j-s} - (-1)^j \\ &= \sum_{s=1}^{j} \binom{j}{s} (kl^n)^s(-1)^{j-s}. \end{aligned}$$

Therefore (1)

$$\frac{q^j - (-1)^j}{j(q+1)} = \frac{j(-1)^{j-1} + \sum_{s=2}^{j} k^{s-1} l^{n(s-1)}(-1)^{j-s}}{j}.$$

If j is an l-adic unit then the result is obvious. So suppose $j = bl^\mu$, $\mu \geq 1$, and b is prime to l. Dividing in formula (1) gives us

$$\frac{q^j - (-1)^j}{j(q+1)} = 1 + \frac{1}{bl^\mu} \sum_{s=2}^{l^\mu} \binom{l^\mu}{s} k^{s-1} l^{n(s-1)} (-1)^{j-s}.$$

This will be an l-adic unit if

$$l^{\mu+1} \,\Big|\, \binom{l^\mu}{s} l^{n(s-1)} \quad \text{where } 2 \leq s \leq l^\mu.$$

We will prove this for $n = 1$, which implies all other cases (i.e., we will show that

$$l^{\mu+2} \,\Big|\, \binom{l^\mu}{s} \cdot l^s,$$

for $2 \leq s \leq l^\mu$).

$$\binom{l^\mu}{s} = \prod_{r=1}^{s} \frac{l^\mu - (r-1)}{r}.$$

If s is prime to l then for every term in the denominator of the form $t \cdot l^m$ (t prime to l) there corresponds in a 1-1 fashion the term $l^\mu - t \cdot l^m$ in the numerator. Taking into consideration the first term in the numerator, l^μ, and the fact that $s \geq 2$ we conclude that

$$l^{\mu+2} \,\Big|\, \binom{l^\mu}{s} l^s$$

if s is prime to l.

Suppose $s = t \cdot l^m$, $m \geq 1$.

$$\binom{l^\mu}{tl^m} = \binom{l^\mu}{tl^m - 1} \cdot \frac{l^\mu - (tl^m - 1)}{tl^m}.$$

Since $tl^m - 1$ is prime to l it follows that

$$l^{\mu - m} \,\Big|\, \binom{l^\mu}{tl^m}.$$

To finish the proof we note that $l^{m+1} \mid l^{tl^m}$, since $x + 1 \leq l^x$, $l \geq 3$, for all real x. Q.E.D.

The previous lemma implies that $H^*(U_n(F_{q^2}); Z_l) \to H^*(N; Z_l)$ is a monomorphism if l is an odd prime dividing $q + 1$. Since $N \cong \sum_n \wr Z_{q+1}$ there are abelian subgroups A_i, of N, of exponent $l^a \mid q + 1$, $a \geq 1$, with the property that $H^*(N) \to \prod_i H^*(A_i)$ is 1-1. By the usual argument A_i is conjugate to a subgroup of diagonal matrices. We now have to show that this conjugation can be carried out inside $U_n(F_{q^2})$ (i.e., there is a unitary basis which diagonalizes all of A_i).

If we can find an eigenvector v such that $(v, v) \neq 0$ then we can construct our unitary basis of eigenvectors by finite induction. Suppose $v_1, \ldots, v_n$ is a diagonalizing basis for A_i and suppose $(v_1, v_1) = 0$. Let us look at the set $\delta = \{v_i \mid (v_1, v_i) \neq 0\}$, nonempty by the nonsingularity of the scalar product. We might as well assume that $\delta = \{v_2, \ldots, v_s\}$ $s \geq 2$ and that $(v_1, v_i) = 1$, $2 \leq i \leq s$. If $av_i = \lambda_i(a)v_i$ for $1 \leq i \leq s, a \in A_i, \lambda_i(a) \in F_{q^2}$ then $1 = (v_1, v_i) = (av_1, av_i) = \lambda_1(a)\overline{\lambda_i(a)}$. Since the exponent of A_i divides $q + 1$, $\lambda_i(a)^{q+1} = 1$ which implies that $\lambda_i(a) = \lambda_1(a)$, $2 \leq i \leq s$. Let V^1 be the subspace generated by δ. The scalar product restricted to V^1 must also be nonsingular and since every vector in V^1 is an eigenvector we are done.

PROPOSITION 5. *If l is an odd prime which divides $q + 1$ then*

$$H^*(U_n(F_{q^2}); Z_l) \to H^*(UT_n(F_{q^2}))$$

is a monomorphism.

Proof of Theorem 3. As in [1, Theorem 4.7], $\delta^*(c_i) \mid BUT_n(k_1)$ is the ith elementary symmetric polynomial in the x_i where

$$H^*(BUT_n(k_1)) \cong Z_l[x_1, \ldots, x_n].$$

Therefore the argument is completely analogous to the case $GL_n(k_1)$ of [1].

BIBLIOGRAPHY

1. D. QUILLEN, *The Adams conjecture*, Topology, vol. 10 (1971), pp. 67–80.
2. S. EILENBERG AND S. MACLANE, *Cohomology theory in abstract groups, I*; Ann. of Math, vol. 48 (1947), pp. 51–78.
3. P. J. HILTON AND U. STAMMBACH, *A course in homological algebra*, Springer, New York, 1971.
4. J. A. GREEN, *The characters of the finite general linear groups*, Trans. Amer. Math. Soc., vol. 80 (1955), pp. 402–447.
5. C. CHEVALLEY, *Sur certains groupes simples*, Tôhoku Math. J. (2), vol. 7 (1955), pp. 14–16.
6. M. F. ATIYAH, *Characters and cohomology of finite groups*, Publ. Math. l'I.H.E.S., no. 9, 1961.
7. R. W. CARTER, *Simple groups of Lie type*, Wiley, New York, 1972.
8. D. QUILLEN, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (2), vol. 96 (1972), pp. 552–586.
9. M. HALL, *The theory of groups*, Macmillan, New York, 1959.
10. N. BOURBAKI, *Algebra*, Éléments de Mathématique, Livre II, Chapter IX, Herman, 1959.
11. A. BOREL AND F. HIRZEBRUCH, *Characteristic classes and homogenous spaces, I.*, Amer. J. Math., vol. 80 (1958), pp. 458–538.
12. J. F. ADAMS, *Lectures on Lie groups*, W. A. Benjamin, Reading, Mass., 1969.

WASHINGTON UNIVERSITY
ST. LOUIS, MISSOURI
ISRAEL INSTITUTE OF TECHNOLOGY
HAIFA, ISRAEL