THE ORBITAL COUNTING FUNCTION OF A FUCHSIAN GROUP

BY

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1. Introduction and statement of results

Let Δ denote the unit disc in the complex plane, $\Delta = \{z: |z| < 1\}$, and let G be a Fuchsian group preserving Δ. For $0 < r < 1$ and $z \in \Delta$ we define the orbital counting function $n_G(r, z)$ to be the number of transforms $V \in G$ such that $|V(z)| < r$. If there is no doubt as to the group concerned we will write $n(r, z)$.

The counting function is a useful tool in the investigation of Fuchsian groups —particularly with regard to the convergence of Poincaré series [4], [12, Chapter XI] and a number of estimates are available. The earliest result is due to E. Hopf [6] and was used by him in the investigation of ergodic properties of the group.

THEOREM A. *If G is a finitely generated group of the first kind, there exists a constant A, depending only on G, such that for any $z \in \Delta$, $n(r, z) < A/(1 - r)$.*

Tsuji [12, p. 518] extended Theorem A to obtain:

THEOREM B. *For any Fuchsian group G there exists a constant A, depending only on G, such that for any $z \in \Delta$, $n(r, z) < A/(1 - r)$.*

If, further, G is finitely generated and of the first kind there exists B depending on G and z such that for $0 < r < 1$, $n(r, z) > B/(1 - r)$.

More recently Huber [7], [8] and Patterson [10] have obtained asymptotic estimates for the counting function in various cases.

THEOREM C [10]. *If G is finitely generated and of the first kind, and if A denotes the noneuclidean area of the Ford fundamental region, then*

$$n(r, z) \sim \frac{2\pi}{A}\frac{1}{(1 - r)} \quad \text{as } r \to 1.$$

In this paper we present two new estimates for the orbital counting function. To state these results we need the notions of multiplier and isometric circle of a bilinear transform [5, pp. 15–30].

THEOREM 1. *Let G be a Fuchsian group containing two hyperbolic transforms T, S such that the isometric circles of T, T^{-1}, S and S^{-1} are mutually exterior. If T fixes α, β has multiplier k and S fixes δ, γ has multiplier ρ then*

$$n(r, 0) > \frac{A}{(1 - r)^{\varepsilon_1}} \quad \text{where } \varepsilon_1 = \log 3 \left\{\log\left[\frac{256k\rho}{|\alpha - \beta|^2|\gamma - \delta|^2}\right]\right\}^{-1}.$$

Received February 4, 1976.

THEOREM 2. *Let G be a fuchsian group containing two parabolic transforms Q, V such that the isometric circles of Q, Q^{-1} are exterior to those of V, V^{-1}. If the circles have radii q, s respectively and $q, s \leq 1$ then*

$$n(r, 0) > \frac{A}{(1-r)^{1/2+\varepsilon_2}} \quad \textit{where } \varepsilon_2 = sq\left\{8 + 2\,sq \log\left(\frac{8}{sq}\right)\right\}^{-1}$$

From these results we may easily obtain bounds for $n(r, z)$—explicitly the relationship is given by:

THEOREM 3. *Let G be a Fuchsian group, suppose $z \in \Delta$ and $|z| < r < 1$; then*

$$n\left(\frac{r+|z|}{1+r|z|}, 0\right) \geq n(r, z) \geq n\left(\frac{r-|z|}{1-r|z|}, 0\right).$$

Our next result shows how small the growth of the counting function may be.

THEOREM 4. *Given any $\varepsilon > 0$ there is a non-elementary Fuchsian group G which is generated by two hyperbolic transforms, is of the second kind and whose counting function satisfies $n(r, 0)(1-r)^{\varepsilon} = 0(1)$ as $r \to 1$.*

In the opposite direction, using a refinement of the estimates used in the proof of Theorem 1, we obtain:

THEOREM 5. *There exists a Fuchsian group G which is freely generated by two hyperbolic transforms, is of the second kind and whose counting function satisfies*

$$n(r, 0) > \left(\frac{1}{1-r}\right)^{0.61}.$$

The exponent 0.61 can certainly be improved upon for groups of this type—in fact we can obtain 0.623 for such a group—but the technique appears to be inherently limited.

As an application of these results we consider the exponent of convergence which, for a Fuchsian group G, may be defined by

$$\delta(G) = \inf\{t > 0 : \sum_{V \in G} (1 - |V(0)|)^t < \infty\}.$$

From the following equation

$$\sum_{|V(0)|<r} (1 - |V(0)|)^t = \int_0^r (1-s)^t \, dn(s, 0)$$

we obtain:

THEOREM D. *Let G be a Fuchsian group.*

(i) *If $n(r, 0) > A(1-r)^{-\varepsilon}$ for $r \geq r_0$ say, then $\delta(G) \geq \varepsilon$.*
(ii) *If $n(r, 0) < B(1-r)^{-\varepsilon}$ for $r \geq r_0$ say, then $\delta(G) \leq \varepsilon$.*

Combining Theorems 1, 2, 4, 5, and D we obtain:

THEOREM 6. *Let G be a Fuchsian group.*

(i) *If G satisfies the hypotheses of Theorem 1 then* $\delta(G) \geq \varepsilon_1$.
(ii) *If G satisfies the hypotheses of Theorem 2 then* $\delta(G) \geq \varepsilon_2$.
(iii) *If G is a group given by Theorem 4 then* $\delta(G) \leq \varepsilon$.
(iv) *If G is the group given in Theorem 5 then* $\delta(G) \geq 0.61$.

Part (iii) is a result of Beardon [3], parts (i) and (ii) improve some results of Beardon [3], part (iv) compares with an estimate of Patterson [11] who obtained, by different methods, a group of the type described in Theorem 5.

In conclusion we give, for the sake of completeness, a result describing the behavior of the counting function for elementary groups.

THEOREM 7. *Let G be a Fuchsian group.*

(i) *If G has no limit points then* $n(r, 0) = 0(1)$ *as* $r \to 1$.
(ii) *If G has exactly one limit point then* $n(r, 0) \sim A(1 - r)^{-1/2}$ *as* $r \to 1$.
(iii) *If G has exactly two limit points then* $n(r, 0) \sim B \log(1/(1 - r))$ *as* $r \to 1$.

We conclude this section with an outline of the proof of Theorem 1. With the notation of the theorem we consider transforms of the form

$$T^{n_p} S T^{n_{p-1}} S \cdots T^{n_1} S. \tag{1.1}$$

Using results of Tsuji [12, pp. 510–511] we estimate the absolute value of the images of the origin under transforms of this type in terms of $N = \sum |n_i|$. With some combinatorial lemmas we estimate the number of sequences $\{n_1, \ldots, n_p\}$ for which N is bounded by a certain quantity (see inequality (3.6)). The number of such sequences yields the number of transforms (1.1) which map the origin into $\{|z| < r\}$.

The proofs of Theorems 1, 2, 3, 4, 5, and 7 are given in Sections 3, 4, and 5. In the next section we give some preliminary lemmas.

2. Preliminary results

We introduce a noneuclidean metric in Δ defined by

$$[z, w] = \left| \frac{z - w}{1 - \bar{z}w} \right| \quad \text{for } z, w \in \Delta.$$

We note that if S is a bilinear transform preserving Δ then $[S(z), S(w)] = [z, w]$ [12, p. 510].

If $0 < \rho < 1$ and $z \in \Delta$ then $C(z, \rho) = \{w: [w, z] = \rho\}$ is a circle and $C(z, \rho)$ is contained in a disc

$$\left\{ w: |w| \leq \frac{|z| + \rho}{1 + |z|\rho} \right\} \qquad [12, \text{p. } 511]. \tag{2.1}$$

For real numbers x, y, $0 \leq x < 1$, $0 \leq y < 1$, we define $\phi(x, y) = (x + y)/(1 + xy)$ and note that ϕ is an increasing function of x.

LEMMA 1. *If T is a bilinear transform preserving Δ and $z \in \Delta$ then*

$$|T(z)| \leq \phi(|z|, |T(0)|).$$

Proof. It is immediate that $[z, w] \leq (|z| + |w|)/(1 + |z|\,|w|)$ for any $z, w \in \Delta$. Now $|T(z)| = [T(z), 0] = [z, T^{-1}(0)]$ and the result follows immediately when we observe that $|T^{-1}(0)| = |T(0)|$.

LEMMA 2. *Let H be a hyperbolic transform preserving Δ with fix points α, β and multiplier $k > 1$ then*

$$|H(0)| = \frac{k-1}{\{(k-1)^2 + k|\alpha - \beta|^2\}^{1/2}}.$$

Proof. Writing H in multiplier form [5, p. 16] we have

$$\frac{H(z) - \alpha}{H(z) - \beta} = k\frac{z - \alpha}{z - \beta}.$$

Thus $H(0) = \alpha\beta(k-1)(k\alpha - \beta)^{-1}$ and so $|H(0)| = (k-1)/(|k\alpha - \beta|)$ since $|\alpha| = |\beta| = 1$. Clearly, $|k\alpha - \beta|^2 = (k-1)^2 + k|\alpha - \beta|^2$ and we have the required result.

We conclude this section with three combinatorial lemmas. Lemmas 3 and 4 are routine with easy proofs by induction (see [13, p. 89] for a closely related result).

LEMMA 3. *Let R be a positive integer and define $M(R)$ to be the number of sequences of nonzero integers the sum of whose absolute values is at most R. Then $M(R) = 3^R - 1$.*

LEMMA 4. *Let p be a positive integer and R a real number greater than 1. Denote by $N(R, p)$ the number of sequences of p positive integers, $n_1, \ldots, n_p$, such that $\prod_{i=1}^{p} n_i < R$. Then $N(R, p) > R(\log R)^{p-1}/2(p-1)!$*

LEMMA 5. *Let a, R be real numbers, $0 < a < 1 < R$, and define $N(R)$ to be the number of sequences of nonzero integers, $n_1, \ldots, n_p$, such that $|\prod_{i=1}^{p} n_i| < a^p R$. Then, for $R > R_0$ say,*

$$N(R) \geq \frac{a}{6} R^{1+\delta} \quad \textit{where } \delta = \frac{2a}{1 + 2a\log(1/a)}.$$

Proof. We set $\alpha = \{1 + 2a\log(1/a)\}^{-1}$ and $x = 2a\alpha\log R$. It is easily seen that $a^x R = R^\alpha$. Now

$$N(R) > \sum_{p=1}^{[x]} N(a^p R, p) \cdot 2^p > \sum_{p=1}^{[x]} \frac{2^{p-1}[\log(a^p R)]^{p-1} a^p R}{(p-1)!}$$

by Lemma 4. But for $1 \leq p \leq [x]$ we have $a^p R > R^\alpha$ and so

$$N(R) > aR\sum_{p=1}^{[x]} \frac{(2a\alpha\log R)^{p-1}}{(p-1)!} = aR\sum_{p=1}^{[x]} \frac{x^{p-1}}{(p-1)!}. \tag{2.2}$$

A routine calculation, using Taylor's remainder formula, shows

$$N(R) > aR(\tfrac{1}{2}e^{[x]} - 1) \geq aR(\tfrac{1}{2}e^{x-1} - 1) > \frac{aR}{6}\, e^x \quad \text{if } x \text{ is large enough.}$$

But $e^x = R^{2a\alpha} = R^{2a/(1+2a\log(1/a))}$ from which we obtain the required result.

3. Proofs of Theorems 1 and 2

Let $n_1, \ldots, n_p$ be a sequence of nonzero integers. We form the transformation

$$T^{n_p}ST^{n_{p-1}}S \cdots T^{n_1}S \tag{3.1}$$

which belongs to G. It is immediate that, with the hypotheses of Theorem 1, a different sequence would yield a different transformation of type (3.1) [9, p. 118]. For such a sequence and $1 \leq j \leq p$ we set

$$\phi_j = |T^{n_j}ST^{n_{j-1}}S \cdots T^{n_1}S(0)|, \qquad \mu_j = |T^{n_j}S(0)|.$$

From Lemma 2,

$$1 - |T(0)|^2 = \frac{k|\alpha - \beta|^2}{(k-1)^2 + k|\alpha - \beta|^2} \geq \frac{k|\alpha - \beta|^2}{(k+1)^2} > \frac{|\alpha - \beta|^2}{4k}.$$

Thus,

$$1 - |T(0)| > |\alpha - \beta|^2/8k. \tag{3.2}$$

Now

$$\mu_j = |T^{n_j}S(0)| \leq \frac{|T^{n_j}(0)| + |S(0)|}{1 + |T^{n_j}(0)|\,|S(0)|} \tag{3.3}$$

from Lemma 1.

The multiplier of T^{n_j} is $k^{|n_j|}$ and $\phi(x, y)$ is an increasing function of x so we obtain from (3.2) and (3.3),

$$\begin{aligned}\mu_j &\leq \left\{1 - \frac{|\alpha - \beta|^2}{8k^{|n_j|}} + |S(0)|\right\}\left\{1 + \left(1 - \frac{|\alpha - \beta|^2}{8k^{|n_j|}}\right)|S(0)|\right\}^{-1} \\ &= 1 - \frac{(1 - |S(0)|)|\alpha - \beta|^2}{8k^{|n_j|}(1 + |S(0)|) - |S(0)|\,|\alpha - \beta|^2} \\ &< 1 - \frac{(1 - |S(0)|)|\alpha - \beta|^2}{16k^{|n_j|}}.\end{aligned}$$

Thus

$$1 - \mu_j > (1 - |S(0)|)|\alpha - \beta|^2/16k^{|n_j|}. \tag{3.4}$$

Applying Lemma 1 with T replaced by $T^{n_p}S$ and z replaced by

$$T^{n_{p-1}}ST^{n_{p-2}}S \cdots T^{n_1}S(0)$$

we obtain,

$$\begin{aligned}\phi_p &\le \frac{\mu_p + \phi_{p-1}}{1 + \mu_p\phi_{p-1}} \\ &= 1 - \frac{(1 - \mu_p)(1 - \phi_{p-1})}{1 + \mu_p\phi_{p-1}} \\ &< 1 - \frac{(1 - \mu_p)(1 - \phi_{p-1})}{2}.\end{aligned}$$

So

$$1 - \phi_p > \frac{1 - \mu_p}{2}(1 - \phi_{p-1}) > \left(\frac{1}{2}\right)^{p-1} \prod_{i=1}^{p} (1 - \mu_i).$$

Using inequality (3.4) we obtain

$$1 - \phi_p > \left\{\frac{|\alpha - \beta|^2(1 - |S(0)|)}{32}\right\}^p k^{-N} \quad \text{where } N = \sum_{i=1}^{p} |n_i|. \tag{3.5}$$

Clearly $p \le N$ (since N is the sum of the absolute values of p nonzero integers) and so

$$1 - \phi_p > \left\{\frac{|\alpha - \beta|^2(1 - |S(0)|)}{32k}\right\}^N.$$

If

$$N \le \log\left(\frac{1}{1 - r}\right)\left[\log\left\{\frac{32k}{|\alpha - \beta|^2(1 - |S(0)|)}\right\}\right]^{-1} \tag{3.6}$$

then we note that $\phi_p < r$ and consequently $|T^{n_p}ST^{n_{p-1}}S \cdots T^{n_1}S(0)| < r$.

Thus $n(r, 0)$ is at least as big as the number of sequences of nonzero integers the sum of whose absolute values satisfy (3.6). From Lemma 3 we compute the number of such sequences to be

$$\exp\left\{\log 3 \log\left(\frac{1}{1 - r}\right)\left[\log\left\{\frac{32k}{|\alpha - \beta|^2(1 - |S(0)|)}\right\}\right]^{-1}\right\} - 1$$

which is $(1/(1 - r))^{\varepsilon_1} - 1$ where

$$\varepsilon_1 = \log 3\left[\log\left\{\frac{32k}{|\alpha - \beta|^2(1 - |S(0)|)}\right\}\right]^{-1}:$$

We conclude the proof of Theorem 1 by noting, from Lemma 2, that

$$1 - |S(0)| > |\delta - \gamma|^2/8\rho$$

and by adding an equivalent of 0, namely 0 itself, which is not equivalent under a transform of type (3.1).

To prove Theorem 2, let $n_1, \ldots, n_p$ be a sequence of nonzero integers. With the hypotheses of Theorem 2 we know that each such sequence defines a unique transformation in G which is of the form $V^{n_p}QV^{n_{p-1}}Q \cdots V^{n_1}Q$. Since V is parabolic an easy estimate yields

$$1 - |V^{n_j}(0)| > s^2/4n_j^2. \tag{3.7}$$

We now follow the method of Theorem 1 and obtain from Lemma 1 and (3.7),

$$|V^{n_p}QV^{n_{p-1}}Q \cdots V^{n_1}Q(0)| < 1 - (sq/8)^{2p} \cdot \prod_{i=1}^{p} n_i^{-2}. \tag{3.8}$$

If

$$\left|\prod_{i=1}^{p} n_i\right| \le (sq/8)^p(1 - r)^{-1/2} \tag{3.9}$$

then we note that $|V^{n_p}QV^{n_{p-1}}Q \cdots V^{n_1}Q(0)| < r$. Thus $n(r, 0)$ is at least as big as the number of sequences of nonzero integers satisfying (3.9). From Lemma 5 we compute the number of such sequences to be at least

$$\frac{sq}{48}(1 - r)^{-1/2-\delta/2} \quad \text{where } \delta = \frac{sq}{4 + sq \log(8/sq)};$$

now with $\varepsilon_2 = \delta/2 = sq/(8 + 2sq \log(8/sq))$ we have the required result.

4. Proof of Theorem 5

Let T, S be two transforms defined as follows: T is hyperbolic, fixes ± 1, and has multiplier 6, S is hyperbolic, fixes $\pm i$, and has multiplier 6. It is easily verified that the isometric circles of T and T^{-1} are centered at $\pm 7/5$ and have radii $2\sqrt{6}/5$ while those of S and S^{-1} are centered at $\pm 7i/5$ and have radii $2\sqrt{6}/5$. Thus the four circles are mutually exterior and T, S freely generate a Fuchsian group G (we could achieve this result with a slightly smaller multiplier —any number bigger than $3 + 2\sqrt{2}$ would do—and a slight improvement in the estimate of Theorem 5 would result).

Assuming that 1 is the attractive fix point of T and i is the attractive fix point of S we write T^n and S^n in multiplier form:

$$\frac{T^n(z) + 1}{T^n(z) - 1} = 6^n \frac{z + 1}{z - 1}, \qquad \frac{S^n(z) + i}{S^n(z) - i} = 6^n \frac{z + i}{z - i}. \tag{4.1}$$

Thus, for a positive integer n,

$$T^n(0) = \frac{6^n - 1}{6^n + 1}, \qquad S^n(0) = \frac{(6^n - 1)i}{6^n + 1}. \tag{4.2}$$

A routine calculation using (4.1) and (4.2) shows that for positive integers n, m,

$$|T^nS^m(0)|^2 = 1 - \frac{8 \cdot 6^{m+n}}{6^{2n+2m} + 6^{2m} + 6^{2n} + 1 + 4 \cdot 6^{n+m}}. \tag{4.3}$$

Now for positive integers n and m, $6^{2m} + 6^{2n} + 1 + 4 \cdot 6^{n+m} < \frac{1}{5} \cdot 6^{2m+2n}$ and so from (4.3), $|T^n S^m(0)|^2 < 1 - 20/3 \cdot 6^{m+n}$ and

$$(4.4) \qquad 1 - |T^n S^m(0)| > 10/3 \cdot 6^{|n|+|m|}.$$

It is geometrically evident that (4.4) is valid for any nonzero integers n, m.

Now consider a sequence of nonzero integers $n_1, \ldots, n_{2p}$ and for $1 \leq j \leq p$ define

$$\mu_j = |T^{n_{2j}} S^{n_{2j-1}}(0)|, \qquad \phi_j = |T^{n_{2j}} S^{n_{2j-1}} \cdots T^{n_2} S^{n_1}(0)|.$$

We obtain, as in the proof of Theorem 1, $1 - \phi_p > (1/2)^{p-1} \prod_{i=1}^{p} (1 - \mu_i)$. Setting $N = |n_1| + |n_2| + \cdots + |n_{2p}|$ and noting that $N \geq 2p$ we obtain from (4.4),

$$(4.5) \qquad 1 - \phi_p > (1/6)^N.$$

Thus $\phi_p < r$ provided

$$(4.6) \qquad N < \log\left(\frac{1}{1 - r}\right)[\log 6]^{-1}.$$

Consequently $n(r, 0)$ is at least as big as the number of sequences of nonzero integers which have an even number of terms and the sum of whose absolute values satisfies (4.6). It follows immediately from Lemma 3 that the number of such sequences is at least $(1/4)3^R$ where

$$R = \log\left(\frac{1}{1 - r}\right)[\log 6]^{-1}.$$

Thus

$$n(r, 0) > A\left(\frac{1}{1 - r}\right)^{\delta} \quad \text{where } \delta = \frac{\log 3}{\log 6} > 0.61$$

which completes the proof of Theorem 5.

5. Proofs of Theorems 3, 4, and 7

Suppose G is a Fuchsian group and $z \in \Delta$; then for any $V \in G$ we have by Lemma 1,

$$(5.1) \qquad |V(z)| \leq \phi(|z|, |V(0)|).$$

If, for $|z| < r < 1$, we have that $|V(0)| < (r - |z|)/(1 - r|z|)$ then it follows from (5.1) that

$$|V(z)| < \phi\left(|z|, \frac{r - |z|}{1 - r|z|}\right) = r.$$

Thus

$$n(r, z) \geq n\left(\frac{r - |z|}{1 - r|z|}, 0\right).$$

If $|V(0)| < r$ then by (5.1), $|V(z)| < (r + |z|)/(1 + r|z|)$ and we obtain

$$n\left(\frac{r + |z|}{1 + r|z|}, 0\right) \geq n(r, z)$$

which completes the proof of Theorem 3.

To prove Theorem 4 we will use some estimates of Akaza [1] and Beardon [2]. Let R be a real number, $0 < R < \frac{1}{2}$, and we choose two transforms S, T preserving Δ such that the isometric circles of S, S^{-1}, T, and T^{-1} are mutually exterior, all have radius R and are symmetrically placed around the unit circle. Let G be the Fuchsian group generated by S and T.

Any transform in G is of the form

$$S^{n_1}T^{n_2}S^{n_3} \cdots T^{n_p} \tag{5.2}$$

for some sequence of integers $n_1, \ldots, n_p$. The *grade* of such a transform is defined to be $|n_1| + |n_2| + \cdots + |n_p|$. Let $V \in G$ and denote by $r(V)$ the radius of the isometric circle of V. If λ denotes the least distance between any two isometric circles of the four transforms of grade one and if V is of grade m then [1, p. 53]

$$r(V) < (R/\lambda)^{m-1}R. \tag{5.3}$$

Now from elementary geometry, $|V(0)| = (1 + r(V)^2)^{-1/2}$ and so, from (5.3), if V is of grade m,

$$|V(0)| > \{1 + (R/\lambda)^{2(m-1)}R^2\}^{-1/2}. \tag{5.4}$$

Define, for any r in $(0, 1)$, $M(r)$ to be the largest integer m such that

$$\{1 + (R/\lambda)^{2(m-1)}R^2\}^{-1/2} < r.$$

Clearly if V is of grade m, where $m > M(r)$, then $|V(0)| \geq r$. Thus $n(r, 0)$ is not greater than the number of transforms of grade not larger than $M(r)$. However, the number of transforms of grade m is $4 \cdot 3^{m-1}$ [1, p. 53] and so

$$n(r, 0) \leq 4(1 + 3 + \cdots + 3^{M(r)-1}) < 2 \cdot 3^{M(r)}. \tag{5.5}$$

From the definition of $M(r)$, $(1 + (R/\lambda)^{2M(r)}R^2)^{-1} \geq r^2$, from which

$$M(r) \leq \log\left(\frac{1 - r^2}{R^2r^2}\right)\left[2 \log\left(\frac{R}{\lambda}\right)\right]^{-1}.$$

Thus, by (5.5)

$$n(r, 0) < 2\left(\frac{r^2R^2}{1 - r^2}\right)^{(\log 3)/2 \log (\lambda/R)}.$$

Given any $\varepsilon > 0$ we choose R so small that $(\log 3)/2 \log (\lambda/R) < \varepsilon$ which completes the proof of Theorem 4.

Theorem 7(i) is trivial—as a group with no limit points contains only finitely many transforms.

If G has one limit point then G is cyclic and is generated by a parabolic transform S. Let ρ be the radius of the isometric circle of S then $|S^n(0)| < r$ if and only if $|n| < r\rho/(1 - r^2)^{1/2}$. Thus $n(r, 0) \sim \sqrt{2}\,\rho(1 - r)^{-1/2}$ as $r \to 1$, which proves Theorem 7(ii).

If G has two limit points then we may write $G = G_1 \cup G_2$ where G_1 is a cyclic subgroup generated by a hyperbolic element H and G_2 is a set of elliptic transforms.

Let $E \in G_2$ it is easy to show that $G_2 = \{H^n E\colon n \text{ is an integer}\}$ and it follows immediately that

$$n_G(r, 0) = n_{G_1}(r, 0) + n_{G_1}(r, E(0)). \tag{5.6}$$

From Lemma 2 we obtain after a routine computation that

$$n_{G_1}(r, 0) \sim \frac{2 \log (r^2|\alpha - \beta|^2/(1 - r^2))}{\log k} \quad \text{as } r \to 1. \tag{5.7}$$

From Theorem 3 we note that, for $r > |E(0)|$,

$$n_{G_1}\left(\frac{r - |E(0)|}{1 - r|E(0)|}, 0\right) \leq n_{G_1}(r, E(0)) \leq n_{G_1}\left(\frac{r + |E(0)|}{1 + r|E(0)|}, 0\right). \tag{5.8}$$

Now

$$\frac{1}{1 - (r + |E(0)|)/(1 + r|E(0)|)} = \frac{1 + r|E(0)|}{1 - |E(0)|} \cdot \frac{1}{1 - r}$$

and

$$\frac{1}{1 - (r - |E(0)|)/(1 - r|E(0)|)} = \frac{1 - r|E(0)|}{1 + |E(0)|} \cdot \frac{1}{1 - r}.$$

Thus from (5.6), (5.7), and (5.8) we deduce $n_G(r, 0) \sim A \log (1/(1 - r))$. This completes the proof of Theorem 7.

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