LOCAL-BELONGING SETS AND MULTIPLIER-INDUCED IDEALS IN GROUP ALGEBRAS

BY

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1. Introduction

Let G be a locally compact abelian group (lcag) with dual group Γ. The group and measure algebras of G will be denoted by $L^1(G)$ and $M(G)$, respectively, and $A(\Gamma)$ and $B(\Gamma)$ will denote the corresponding transform algebras, in which the norms are those induced by $L^1(G)$ and $M(G)$. The notation will generally follow that of Rudin [5].

Let f be a complex-valued function on Γ, and let I be an ideal of $A(\Gamma)$. If f agrees with a function in I on some neighborhood of $\gamma \in \Gamma$, then f is said to belong locally to I at γ. The concept of local-belonging is of great value in determining whether or not a given function belongs to an ideal of $A(\Gamma)$. (See, for example, [2] and [6].) In [1] Edwards proves that if Γ is nondiscrete, then there exists a continuous function on Γ which does not belong locally to $A(\Gamma)$ at some point of Γ. If, for $f \in C(\Gamma)$, we define the local-belonging set of f, denoted by $L(f)$, to be the set of those elements of Γ at which f belongs locally to $A(\Gamma)$, then Edwards' result may be restated: If Γ is nondiscrete, then there exists a function $f \in C(\Gamma)$ for which $L(f)$ is a proper subset of Γ. The question naturally arises as to the characterization of those (necessarily open) subsets of Γ which are of the form $L(f)$ for some $f \in C(\Gamma)$. In part 2 we will characterize such sets. We would like to thank Walter Rudin for many valuable suggestions concerning this result.

For a function f in $C(\Gamma)$ define $I(f) = \{g \in A(\Gamma): fg \in A(\Gamma)\}$. Clearly $I(f)$ is an ideal of $A(\Gamma)$, and f may be regarded as a multiplier of $I(f)$ in the sense of Meyer [4]. We shall refer to an ideal of the form $I(f)$ as a multiplier-induced ideal, induced by f. In Section 3 we will characterize (Theorem 5′) those closed, multiplier-induced ideals of $A(\Gamma)$ induced by elements of $C(\Gamma)$. The result which links Sections 2 and 3 is that the spectrum of $I(f)$ coincides with $L(f)$.

2. A characterization of local-belonging sets

Recall that for $f \in C(\Gamma)$, $L(f)$ denotes the set of all $\gamma \in \Gamma$ at which f belongs locally to $A(\Gamma)$. Our principal result is:

THEOREM 1. *Let Γ be a nondiscrete lcag, and let U be an open subset of Γ. Then $U = L(f)$ for some $f \in C(\Gamma)$ if and only if ∂U, the boundary of U, is a G_δ subset of Γ. Furthermore, the function f may be chosen to be uniformly continuous*

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and bounded on Γ *and equal to zero on* U. *If, in addition,* Γ *is* σ*-compact, then* f *may be chosen to be an element of* $C_0(\Gamma)$.

If Γ is assumed also to be metric, then we have an immediate corollary:

COROLLARY. *If* Γ *is a nondiscrete metric lcag, then every open subset of* Γ *is of the form* $L(f)$ *for some bounded, uniformly continuous function* f *on* Γ.

The proof of Theorem 1 depends upon the following four lemmas.

LEMMA 1. *Let* Γ *be a lcag, and let* $f \in C_u(\Gamma)$, *where* $C_u(\Gamma)$ *denotes the collection of uniformly continuous functions on* Γ. *There exists a compact subgroup* N *of* Γ *such that* Γ/N *is metric and* f *is constant on the cosets of* N.

Proof. Since f is uniformly continuous, a sequence $\{U_n: n = 1, 2, \ldots\}$ of neighborhoods of the identity in Γ can be chosen to satisfy the following conditions:

(i) $\overline{U_1}$ is compact.
(ii) $U_{n+1} + U_{n+1} \subseteq U_n$ for each n.
(iii) $\overline{U_{n+1}} \subseteq U_n$ for each n.
(iv) If $\alpha, \beta \in \Gamma$ and $(\alpha - \beta) \in U_n$, then $|f(\alpha) - f(\beta)| < 1/n$.

Define $N = \bigcap_1^\infty U_n$; then N is a compact subgroup of Γ. Moreover,

$$\{\pi(U_n): n = 1, 2, \ldots\}$$

is a countable neighborhood base of the identity in Γ/N, where $\pi: \Gamma \to \Gamma/N$ is the quotient mapping, and so Γ/N is metric. Finally, if α and β lie in the same coset of N, then $(\alpha - \beta) \in N$. Hence $|f(\alpha) - f(\beta)| < 1/n$ for each n, and thus $f(\alpha) = f(\beta)$.

A function $\tilde{g}$ on Γ/N induces a function $g = \tilde{g} \circ \pi$ on Γ, where $\pi: \Gamma \to \Gamma/N$ is the quotient map. This notation will be used throughout the remainder of Section 2.

LEMMA 2. *Let* Γ *be a lcag, let* N *be a closed subgroup of* Γ, *and let* $\tilde{g}$ *and* $\tilde{h}$ *be functions on* Γ/N.

(a) *If* $\tilde{g} \in B(\Gamma/N)$, *then* $g \in B(\Gamma)$.

(b) *If* $\tilde{h} \in C(\Gamma/N)$ *and* $L(\tilde{h}) = \emptyset$, *then* $h \in C(\Gamma)$ *and* $L(h) = \emptyset$. *Furthermore, if* $\tilde{h}$ *is uniformly continuous (bounded), then* h *is uniformly continuous (bounded).*

Proof. (a) Let $\gamma_1, \ldots, \gamma_n \in \Gamma$, and let

$$\tilde{f}(x) = \sum_1^n c_i(x, \gamma_i + N);$$

then $f(x) = \sum_1^n c_i(x, \gamma_i)$ is a trigonometric polynomial on G. Since $\tilde{g} \in B(\Gamma/N)$, we may apply Eberlein's Theorem (see [5, p. 32]) to obtain

$$\left|\sum_1^n c_i g(\gamma_i)\right| = \left|\sum_1^n c_i \tilde{g}(\gamma_i + N)\right| \leq \|\tilde{g}\| \, \|\tilde{f}\|_\infty \leq \|g\| \, \|f\|_\infty.$$

Hence, using Eberlein's Theorem again, we have that $g \in B(\Gamma)$.

(b) Let $\tilde{h} \in C(\Gamma/N)$. We shall prove that if $L(h) \neq \emptyset$, then $L(\tilde{h}) \neq \emptyset$. Because $B(\Gamma)$ is closed under translation, it follows that $L(h)$ is a union of cosets of N. Let W_1 and W_2 be nonempty open subsets of Γ with compact closures such that $\overline{W_1} \subset W_2 \subset \overline{W_2} \subset L(h)$, and let $\tilde{g} \in B(\Gamma/N)$ be such that $\tilde{g} = 1$ on $\pi(\overline{W_1})$ and $\tilde{g} = 0$ off $\pi(W_2)$. Since $\overline{W_2}$ is a compact subset of $L(h)$, there exists a function $f \in B(\Gamma)$ such that $h = f$ on $\overline{W_2}$. Now $g \in B(\Gamma)$ by (a) above, and $g = 1$ on W_1 and $g = 0$ on the complement of W_2. Defining $f_0 = gf$, we have that $f_0 \in B(\Gamma)$. Moreover, $f_0 = gh$, and hence f_0 is constant on the cosets of N (since both g and h are). By [5, p. 53] there exists $\tilde{f}_0 \in B(\Gamma/N)$ such that $f_0 = \tilde{f}_0 \circ \pi$. Furthermore, $\tilde{f}_0 = (\widetilde{gh}) = \tilde{h}$ on $\pi(W_1)$, and therefore $L(\tilde{h}) \neq \emptyset$.

The remainder of the lemma is obvious.

LEMMA 3. *Let Γ be a nondiscrete lcag. There exists $h \in C_u(\Gamma) \cap L^\infty(\Gamma)$ such that $L(h) = \emptyset$.*

Proof. (a) Assume first that Γ is also separable and metric, and let $\{\beta_p : p = 1, 2, \ldots\}$ be a countable dense subset of Γ. For $\beta \in \Gamma$ and $r > 0$ let $N(\beta, r)$ denote the ball of radius r centered at β. For positive integers k, m, and p define

$$F(k, m, p) = \left\{ f \in C_0(\Gamma) : \left|\sum_1^n a_i f(\gamma_i)\right| \leq m \left\|\sum_1^n a_i \gamma_i\right\|_\infty \right.$$
$$\left. \text{for all } \gamma_1, \ldots, \gamma_n \in N(\beta_p, 1/k) \text{ and all complex numbers } a_1, \ldots, a_n \right\}.$$

Clearly $F(k, m, p)$ is closed in $C_0(\Gamma)$. It is also nowhere dense, for if U is any open subset of $C_0(\Gamma)$, we may choose $f \in A(\Gamma) \cap U$ and then use Edwards' result [1] to find $f_2 \in C_0(\Gamma)$ such that $\beta_p \notin L(f_2)$ and $(f_1 + f_2) \in U$. It then follows from [3, p. 215] that $(f_1 + f_2) \notin F(k, m, p)$. Hence the Baire Category Theorem implies the existence of a function $h \in C_0(\Gamma)$ which does not belong to any $F(k, m, p)$.

We will show that $L(h) = \emptyset$. Let $\gamma \in \Gamma$, and let V be an open neighborhood of γ. If $h = g$ on V for some $g \in A(\Gamma)$, then $h = g$ on $N(\beta_p, 1/k)$ for some positive integers p and k. But by taking $m = \|g\|$, we are led to the contradiction that $h \in F(k, m, p)$.

(b) Now assume only that Γ is a nondiscrete, σ-compact lcag. By Lemma 1 there exists a compact subgroup N of Γ such that Γ/N is metric. Since Γ/N is also σ-compact, it is separable. Thus by (a) above there exists $\tilde{h} \in C_0(\Gamma/N)$ such that $L(\tilde{h}) = \emptyset$. By Lemma 2,

$$h = \tilde{h} \circ \pi \in C_u(\Gamma) \cap L^\infty(\Gamma) \quad \text{and} \quad L(h) = \emptyset.$$

(c) Finally, let Γ be any nondiscrete lcag. Let U_1 by a symmetric neighborhood of the identity having compact closure. For $n = 2, 3, \ldots$ define $U_n = U_{n-1} + U_{n-1}$, and let $H = \bigcup_1^\infty U_n$. Then H is an open σ-compact subgroup of Γ, and so by (b) above there exists

$$h_0 \in C_u(H) \cap L^\infty(H)$$

such that $L(h_0) = \emptyset$. Now Γ is a disjoint union of cosets $\gamma_\alpha + H$, where α belongs to some indexing set. For $\gamma \in \Gamma$ find an index α and an element β in H such that $\gamma = \gamma_\alpha + \beta$. By defining $h(\gamma) = h_0(\beta)$, we have $h \in C_u(\Gamma) \cap L^\infty(\Gamma)$ and $L(h) = \emptyset$.

The final lemma, contained in [7], is proved here for completeness.

LEMMA 4. *Let Γ be a σ-compact lcag, and let F be a closed, nonempty G_δ subset of Γ. There exists $g \in A(\Gamma)$ such that g equals zero precisely on F.*

Proof. Let $F = \bigcap_1^\infty U_n$, where U_n is open; then $F' = \bigcup_1^\infty U_n'$, where $'$ denotes complementation. Since F is σ-compact, each U_n' is a countable union of compact sets, and hence F' is also a countable union of compact sets $\{K_m: m = 1, 2, \ldots\}$. Choose $f_m \in A(\Gamma)$ such that $f_m > 0$ on K_m, $\|f_m\| < 2^{-m}$, and $f_m = 0$ on F. Then $g = \sum_m f_m$ is the desired function.

Proof of Theorem 1. Suppose first that $U = L(f)$ for some function f which is uniformly continuous on Γ. If $U = \emptyset$, then ∂U is a G_δ subset of Γ; so assume that $U \neq \emptyset$. Choose a compact subgroup N for f as in Lemma 1. As we have observed previously, U (and hence ∂U) is a union of cosets of N. Since Γ/N is metric, $\pi(\partial U)$ is a G_δ subset of Γ/N, and hence there exist open sets V_n ($n = 1, 2, \ldots$) in Γ/N such that $\pi(\partial U) = \bigcap_1^\infty V_n$. Thus

$$\partial U = \pi^{-1}(\pi(\partial U)) = \bigcap_1^\infty \pi^{-1}(V_n),$$

so that ∂U is a G_δ subset of Γ.

Now suppose that $U = L(f)$ for some continuous real-valued function f on Γ. As in the proof of Lemma 3, let H be a σ-compact open subgroup of Γ, and write Γ as a disjoint union of cosets $\bigcup_\alpha (\gamma_\alpha + H)$. Use the construction of Lemma 4 to find $g_0 \in C_0(H)$ with $g_0 > 0$ and $L(g_0) = H$, and define $g \in C(\Gamma)$ by the rule $g(\gamma_\alpha + \gamma) = g_0(\gamma)$ for each $\gamma \in H$. Letting $h = ge^{if}$, we have $h \in C_u(\Gamma)$ since $g_0 \in C_0(H)$ and e^{if} is bounded on Γ. Moreover, $L(h) = L(e^{if}) = L(f)$ because $g > 0$ on Γ and $L(g) = \Gamma$. Hence the preceding paragraph shows that $\partial L(f) = \partial L(h)$ is a G_δ subset of Γ.

Finally, suppose that $U = L(f)$ for an arbitrary continuous function f on Γ. Write $f = f_1 + if_2$, where f_1 and f_2 are continuous real-valued functions on Γ.

Since the boundaries of $L(f_1)$ and $L(f_2)$ are G_δ subsets of Γ by the preceding paragraph, there exist bounded, uniformly continuous, real-valued functions g_1 and g_2 on Γ such that $L(g_1) = L(f_1)$ and $L(g_2) = L(f_2)$. Let $g = g_1 + ig_2$; then

$$L(g) = L(g_1) \cap L(g_2) = L(f_1) \cap L(f_2) = L(f) = U.$$

But since g is a bounded, uniformly continuous function of Γ, the first paragraph of the proof shows that ∂U is a G_δ subset of Γ.

To prove the converse, assume first that Γ is σ-compact. Lemma 3 proves the desired result if $U = \emptyset$; so suppose that $U \neq \emptyset$. If ∂U is a G_δ subset of Γ, then so is $\bar{U}$. Hence by Lemma 4 there exists $g \in A(\Gamma)$ such that $g = 0$ precisely on $\bar{U}$. Use Lemma 3 to select $h \in C_u(\Gamma) \cap L^\infty(\Gamma)$ such that $L(h) = \emptyset$. Define $f = gh$; then $f \in C_0(\Gamma)$, $f = 0$ on U, and $L(f) = U$.

Now assume only that Γ is a nondiscrete lcag and that U is a G_δ subset of Γ, where $\partial U = \bigcap_1^\infty U_n$ for open sets U_n. Construct an open σ-compact subgroup H of Γ as in the proof of Lemma 3. Since U and H are open, we have

$$\partial_H(U \cap H) = (\partial U) \cap H = \left(\bigcap_1^\infty U_n\right) \cap H = \bigcap_1^\infty (U_n \cap H),$$

where ∂_H denotes the boundary relative to H. Thus $U \cap H$ has a G_δ boundary in H, and so the preceding paragraph guarantees the existence of $f_0 \in C_0(H)$ such that $f_0 = 0$ on $U \cap H$ and $L(f_0) = U \cap H$. Write Γ as a disjoint union of cosets of H, say $\Gamma = \bigcup_\alpha (\gamma_\alpha + H)$. Given $\gamma \in \Gamma$, find an index α and an element $\beta \in H$ such that $\gamma = \gamma_\alpha + \beta$. Defining $f(\gamma) = f_0(\beta)$ (as in Lemma 3) gives a function $f \in C_u(\Gamma) \cap L^\infty(\Gamma)$ with the desired properties.

3. A characterization of closed multiplier-induced ideals in $A(\Gamma)$

We seek to classify those closed ideals of $A(\Gamma)$ which are of the form $I(f) = \{g \in A(\Gamma): fg \in A(\Gamma)\}$ for some $f \in C(\Gamma)$.

First, however, observe that $I(f)$ need not be closed even for $f \in C_u(\Gamma) \cap L^\infty(\Gamma)$. For example, we need only select such an f with $L(f) = \Gamma$ and $f \notin B(\Gamma)$. Then $I(f)$ is dense in $A(\Gamma)$ since Theorem 2 below implies that the spectrum of $I(f)$ equals Γ, but $I(f) \neq A(\Gamma)$ lest Theorem 3.8.1 of [5] imply that $f \in B(\Gamma)$. For noncompact Γ a function f satisfying these conditions can be defined by letting $f = 1/\phi$, where ϕ is a function having the properties in Theorem 5.3.4 of [5].

Our characterization (Theorem 5′) of closed ideals of $A(\Gamma)$ having the form $I(f)$ utilizes Theorem 1 and a result of Meyer [4] which asserts that the multipliers of an ideal depend only on the spectrum of the ideal. Our next result determines the spectrum of $I(f)$.

THEOREM 2. *Let Γ be a lcag, and let $f \in C(\Gamma)$. Then $L(f) = sp(I(f))$, the spectrum of $I(f)$.*

Proof. Let $\gamma \in sp(I(f))$. Then there exists $g \in I(f)$ such that $g(\gamma) \neq 0$. Choose a neighborhood U of γ and a function h in $A(\Gamma)$ such that $gh = 1$ on U. Since $gh \in I(f)$, we have $fgh \in A(\Gamma)$ and $f = fgh$ on U. Thus $\gamma \in L(f)$.

Now suppose that $\gamma \in L(f)$. Then there exists a neighborhood U of γ and a function h in $A(\Gamma)$ such that $f = h$ on U. Choose $g \in A(\Gamma)$ such that $g(\gamma) = 1$ and $g = 0$ on U'. Then $fg = hg \in A(\Gamma)$, and hence $g \in I(f)$. Thus $\gamma \in sp(I(f))$.

For a closed subset E of Γ we define $I_E = \{f \in A(\Gamma): f = 0 \text{ on } E\}$.

THEOREM 3. *Let Γ be a lcag, and let $f \in C(\Gamma)$. If $I(f)$ is closed, then $I(f) = I_E$, where $E = (L(f))'$.*

Proof. Since f is a multiplier of $I(f)$, it follows from Meyer [4] and Theorem 2 that f is also a multiplier of I_E. That is, $I(f) = I_E$.

The question of when $I(f)$ is closed may be reformulated as in the following result.

THEOREM 4. *Let Γ be a nondiscrete lcag, and let $f \in C(\Gamma)$. Then $I(f)$ is closed if and only if $I(f) = I(g)$ for some $g \in C(\Gamma)$ such that g is the restriction of a Fourier-Stieltjes transform on $L(f)$.*

Proof. Assume first that $I(f) = I(g)$, where g is the restriction of a Fourier-Stieltjes transform g_0 on $L(f)$. We shall prove that $I(g) = I_E$, where $E = (L(f))'$. Let $h \in I_E$; then $hg = hg_0$ since $h = 0$ on E. But $h \in A(\Gamma)$ and $g_0 \in B(\Gamma)$, and hence $hg \in A(\Gamma)$. Thus $h \in I(g)$. So $I(f) = I(g) = I_E$.

Conversely, assume that $I(f)$ is closed, and let $U = L(f)$. Since ∂U is a G_δ subset of Γ, Theorem 1 implies that there exists $g \in C(\Gamma)$ such that $g = 0$ on U and $L(g) = U$. Thus g is the restriction of 0 on $L(f)$, and $I(g) = I_E = I(f)$, where $E = U'$, by the preceding paragraph and Theorem 3.

Theorem 4 cannot be strengthened to conclude that if $I(f)$ is closed, then f is actually the restriction of a Fourier-Stieltjes transform. In fact, there exists $f \in C_u(\Gamma) \cap L^\infty(\Gamma)$ which is not the restriction of a Fourier-Stieltjes transform on $L(f)$, and yet $I(f)$ is closed. Applying the technique of Meyer [4] to $\Gamma = R$ and $E = (0, 1)'$, we can construct a continuous function f on $[0, 1]$ such that $f(0) = f(1) = 0$ and f is a multiplier of I_E. The desired function is obtained by extending f to $C_0(R)$ via Lemma 3 in such a way that $L(f) = (0, 1)$.

THEOREM 5. *Let Γ be a nondiscrete lcag, and let E be a closed subset of Γ having a G_δ boundary. Then there exists $f \in C_u(\Gamma) \cap L^\infty(\Gamma)$ such that $I_E = I(f)$.*

Proof. By Theorem 1 we may select $f \in C_u(\Gamma) \cap L^\infty(\Gamma)$ such that $f = 0$ on E' and $L(f) = E'$. By Theorem 4, $I(f)$ is closed and $I(f) = I_E$.

We may combine Theorem 3 and Theorem 5 into a more compact form:

THEOREM 5′. *Let Γ be a nondiscrete lcag, and let I be a closed ideal of $A(\Gamma)$ with cospectrum E. Then $I = I(f)$ for some $f \in C_u(\Gamma) \cap L^\infty(\Gamma)$ if and only if $I = I_E$ and ∂E is a G_δ subset of Γ.*

COROLLARY. *Let Γ be a nondiscrete metric lcag, and let I be a closed ideal of $A(\Gamma)$ having cospectrum E. Then $I = I(f)$ for some $f \in C_u(\Gamma) \cap L^\infty(\Gamma)$ if and only if $I = I_E$.*

We will conclude with an example of an ideal which is not of the form $I(f)$ for any $f \in C(\Gamma)$. From the last theorem it follows that if Γ were an uncountable product of circles and I were the ideal of all functions vanishing at the origin, then I would not be of the form $I(f)$ for any $f \in C(\Gamma)$ since $\{0\}$ is not a G_δ subset of Γ.

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