ON CHEN'S ITERATED INTEGRALS

BY

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Introduction

In a series of papers, Kuo Tsai Chen has introduced his "iterated integrals"; and in particular in [1] he has related them to the homology of the loop-space of a "differential space." Here, the notion of a "differential space" is very weak-C^∞-manifolds being a special case. For a differential space X there still is a deRham complex Λ^*X and a Stokes map $\rho: \Lambda^*X \to C^*X$ but one cannot, in general, assert that ρ is a homology isomorphism. The path space P_SX and the loop space Ω_SX—slightly restricted to "smooth paths"—are again differential spaces; and the "iterated integrals" can be regarded as a morphism

$$I: B^*(\Lambda^*X) \to \Lambda^*P_SX$$

where B^* is the "bar construction." Suppose now that $A^* \subset \Lambda^*X$ is a sub DGA-algebra. Then denote the image of

$$B^*(A^*) \longrightarrow B^*(\Lambda^*X) \xrightarrow{I} \Lambda^*P_SX \xrightarrow{h} \Lambda^*\Omega_SX$$

where h is the restriction, by $\int A^*$. $\int A^*$ turns out to be a sub DGA-algebra of $\Lambda^*\Omega_SX$ and "Chen's theorem" is roughly (for a precise statement see [1, 4.7.1] or 2.3 below) that if $\rho \mid A^*: A^* \to C^*X$ is a homology isomorphism, then $H^*(\int A^*) \approx H^*(\Omega X)$. Chen proves this by a pairing of $\int A^*$ with the cobar construction, using the methods of [3]. This is fairly complicated and, at least without considerable modification, restricted to simply connected spaces.

The present paper is intended to clarify the significance of the integration map I. Also, in Chapter 2, we give a simpler proof of Chen's theorem, avoiding the use of the Adams construction, and arriving at our form of the theorem, namely (roughly again): Chen's theorem is true whenever the Adams-Eilenberg-Moore theorem $H^*(\Omega X) \approx H^*(B^*(C^*X))$ is true; it is known that this is so in certain nonsimply connected cases. In some recent papers, e.g., [2], Chen has tackled these cases by a different method. The main idea of our paper is to relate iterated integrals to the category DASH of "strongly homotopy multiplicative maps," cf. [4].

We observe that, using the proof in [5], the Stokes map ρ can be extended to a map of DASH:

$$P_B: B^*(\Lambda^*X) \to B^*(C^*X).$$

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Our form of Chen's theorem then follows immediately from the fact that the integration map induces a homology isomorphism $B^*(A^*) \to \int A^*$, where A^* means the same as above; this last result is (essentially) contained in [1].

In Chapter 3 we explain that the Adams construction—not used in Chapter 2—leads to a second map of DASH,

$$P'_B\colon B^*(\Lambda^*X) \to B^*(C^*X),$$

which is homotopic (in DASH) to P_B Chen's proof can be approximately described in our terms by saying that he uses P'_B instead of P_B.

In constructing the homotopy between P_B and P'_B we use the method of acyclic models. This forces us to prove the main results of Chapter 3 without choosing base-points or "collapsing" C^*X in any way; in particular, we are in no way restricted to simply connected X. The interest of the map P'_B lies in the fact that it is given much more explicitly in terms of the underlying geometric structure than P_B (see 4.2 below). From the form of the map P'_B it appears that one should be able to factorize it through the cubical singular cohcain-complex $CU^*(\Omega_S X)$ and the map introduced by Adams; this requires—as did, of course, the work of Adams—the use of an associative multiplication on ΩX and the complex C_1^*X based on the use of the singular complex with collapsed 1-simplexes.

Once one has such a factorization it follows easily that the isomorphism of 2.3 is an isomorphism of *algebras* if X is simply connected. There are, however, some technical difficulties in this program, and we have not carried out the details.

1. Review of Chen's theory

A *differentiable space* is a Hausdorff space X together with a certain family of continuous maps $\alpha\colon U \to X$ called *plots*, where U is a convex subset of some Euclidean space, the family being maximal subject to the conditions that with α, $\alpha\phi$ is a plot if $\phi\colon U' \to U$ is a C^∞-map between such convex regions; and every map {point} $\to X$ is a plot. A C^∞-manifold is a differentiable space in an obvious way; so is a subspace of a differentiable space. If X is a differentiable space we define the path-space $P_S X$ as the subspace of the usual path space consisting of those paths $I \to X$ which are piecewise plots; $P_S X$ is a differentiable space: We define $\alpha\colon U \to P_S X$ to be a plot if the adjoint map $\#\alpha\colon U \times I \to X$ has the property that, for some partition $0 = t_0 < t_1 < \cdots < t_r = 1$ of the unit interval I, $\#\alpha \mid U \times [t_i, t_{i+1}]$ is a plot of X for $i = 0, \ldots, p-1$.

A *differentiable p-form* w on a differentiable space X is the assignment to each plot $\alpha\colon U \to X$ of a differentiable p-form w_α on U, this assignment to satisfy $\phi^* w_\alpha = w_{\alpha\phi}$ if $\phi\colon U' \to U$ is C^∞. We define

$$(w + w')_\alpha = w_\alpha + w'_\alpha, \qquad (w \wedge w')_\alpha = w_\alpha \wedge w'_\alpha, \qquad (dw)_\alpha = dw_\alpha.$$

The differentiable forms thus can be regarded as a graded differentiable algebra Λ^*X with unit. A map $f\colon X \to Y$ is a map of differentiable spaces if $f \circ \alpha$:

$U \to Y$ is a plot of Y whenever $\alpha: U \to X$ is a plot of X. Such a map induces a map of differentiable algebras $f^*: \Lambda^* Y \to \Lambda^* X$. Note that if X is a C^∞-manifold (with the evident structure of a differentiable space), then $\Lambda^* X$ is the classical deRham theory.

Δ^r will denote the standard r-simplex which we shall regard as the subset

$$\{(t_1, \ldots, t_r) \mid t_i \geq 0, t_1 + \cdots + t_r \leq 1\}$$

of Euclidean r-space R^r. We shall regard the coordinates as maps $t_i: \Delta^r \to I$ $(1 \leq i \leq r)$.

Now, let $w_i \in \Lambda^{p_i} X$ be a p_i-form on X and $\alpha: U \to P_S X$ a plot with adjoint $\#\alpha: U \times I \to X$. Then $w_{i\#\alpha}$ is a p_i-form which is piecewise defined on $U \times I$ and

$$(U \times t_i)^* w_{i\#\alpha} = \tilde{w}_{i\alpha}$$

say is a p_i-form on $U \times \Delta^r$. We define

$$\left(\int w_1 \cdots w_r\right)_\alpha = \int_{\Delta^r} \tilde{w}_{1\alpha} \wedge \cdots \wedge \tilde{w}_{r\alpha} \tag{1.0}$$

which is a $(p_1 + \cdots + p_r - r)$-form on U, the integration being over the "volume element" $dt_1 \wedge \cdots \wedge dt_r$. The coherency condition is easily verified and thus $\int w_1 \cdots w_r$ is a $p_1 + \cdots + p_r - r$ form on $P_S X$. Note that we have *not* assumed that $p_i > 0$; it is clear, however, that $\int w_1 \cdots w_r = 0$ if $p_i = 0$ for any i, so that, in particular, although $p_1 + \cdots + p_r - r$ may be negative, in that case $\int w_1 \cdots w_r = 0$. It is also convenient to introduce the convention that $\int w_1 \cdots w_r = 1 \in \Lambda^0 X$ if $r = 0$. Our definition agrees with that of Chen, as can be seen easily by evaluating (1.0) as an iterated integral.

If $\alpha: U \to P_S X$ is a plot and U a bounded convex set, we define

$$\int_\alpha \int w_1 \cdots w_r = \int_U \left(\int w_1 \cdots w_r \int\right)_\alpha \quad \text{if } p_1 + \cdots + p - r = \dim U$$
$$= 0 \qquad \text{otherwise.}$$

Also, we take $\int_\alpha \int w_1 \cdots w_r = \delta_0^n$ if $r = 0$ and $n = \dim U$. Notice that $\int_\alpha \int w = \int_{\#\alpha} w$ for $r = 1$. We now give a summary of some properties of these "iterated integrals"; for proofs see [1].

Let $\alpha: U \to P_S X$, $\alpha': U' \to P_S X$ be plots such that there is a point $x \in X$ with $\alpha(u)(1) = \alpha'(u')(0) = x$ for all $u \in U$, $u' \in U'$. Then we define the composition plot

$$\alpha \times \alpha': U \times U' \to P_S X$$

by

$$(\alpha \times \alpha')(u, u')(t) = \alpha(u)(2t) \qquad \text{for } 0 \leq t \leq \tfrac{1}{2}$$
$$= \alpha'(u')(2t - 1) \quad \text{for } \tfrac{1}{2} \leq t \leq 1.$$

1.1. LEMMA *If $\alpha \times \alpha'$ is defined on a bounded closed convex set, then*

$$\int_{\alpha\times\alpha'} \int w_1 \cdots w_r = \sum_{0\le i\le r} \left(\int_\alpha \int w_1 \cdots w_i\right)\left(\int_{\alpha'} \int w_{i+1} \cdots w_r\right) \quad [1, 1.6.2]$$

To state further properties it is convenient to introduce the bar-construction $\mathbf{B}^*(A^*)$ of a graded differential algebra A^* with a differential of grading $+1$. This is *not* the "bar construction" of [4] for instance because A^* is, at the moment, not augmented; indeed, it need not even have a unit. We can introduce the augmentation if there is one, as will be seen.

If M is a graded module, $s\colon M \to sM$ will be the "suspension," i.e., sM is the module M with grading increased by 1; if M is a differential module, so is sM with differential given by $d(sm) = -s(dm)$; similarly for s^{-1}.

As a graded module $\mathbf{B}^*(A^*)$ is $\sum_{p=0}^{\infty} \mathbf{B}_p^*(A^*)$ where $\mathbf{B}_p^*(A^*)$ is the p-fold tensor product $\otimes^p (s^{-1}A^*)$ if $p \ge 1$ and $\mathbf{B}_0^*(A^*) = R$, the underlying ring (i.e., the reals in our case). The differential is $d = d_\otimes + d_\phi$ where $d_\otimes$ is the tensor product differential and $d_\phi\colon \mathbf{B}_p^* \to \mathbf{B}_{p-1}^*$ is defined as 0 for $p \le 1$ and as

$$\sum_{i=1}^{p-1} (1 \otimes \cdots \otimes s^{-1}\phi(s \otimes s) \otimes \cdots \otimes 1)$$

for $p > 1$; in the formula the term with ϕ is in the ith position and denotes the product $A^* \otimes A^* \to A^*$. The "Koszul convention" for tensor products automatically introduces the usual complicated signs. Note that in [1], the differential is taken as $d_\otimes - d_\phi$; we use $d_\otimes + d_\phi$ in order to be consistent with the formalism of [4]. As usual, we denote $s^{-1}a_1 \otimes \cdots \otimes s^{-1}a_r$ by $[a_1, \ldots, a_r]$, and observe that $\mathbf{B}^*(\mathbf{A}^*)$ has the coproduct ψ given by

$$[a_1, \ldots, a_r] \mapsto \sum_{i=0}^{r} [a_1, \ldots, a_i] \otimes [a_{i+1} \cdots a_r].$$

If $f, g\colon \mathbf{B}^*(A^*) \to C^*$ are maps into an algebra C^* with product ϕ, we define the "cup-product" $f \cup g = \phi(f \otimes g)\psi$.

Let X be a differentiable space. We define the morphism of grading 0, $I\colon \mathbf{B}^*(\Lambda^*X) \to \Lambda^*PX$, by $I[\ \ ] = 1$ and

$$I[w_1, \ldots, w_r] = (-1)^r \int w_1 \cdots w_r \quad \text{for } r > 0.$$

By $\Pi_0, \Pi_1\colon PX \to X$ we denote the two "end-point maps"; they induce $\Pi_0^*, \Pi_1^*\colon \Lambda^*X \to \Lambda^*P_SX$. It is convenient to introduce the morphisms $\tau_0, \tau_1\colon \mathbf{B}^*(\Lambda^*X) \to \Lambda^*PX$, of grading $+1$, namely 0 on $B_r^*(\Lambda^*X)$ if $r \neq 1$ and $\tau_0[w_1] = \Pi_0^*w_1$, $\tau_1[w_1] = \Pi_1^*w_1$. Then $D\tau_0 = \tau_0 \cup \tau_0$, $D\tau_1 = \tau_1 \cup \tau_1$, c.f., [4] where, as usual, $D\tau = d \circ \tau + \tau \circ d$. Similarly we have the differential $DI = d \circ I - I \circ d$.

1.2 LEMMA. $DI = \tau_0 \cup I - I \cup \tau_1$.

For a proof, see 4.1.2 in [1]. The term with τ_0 is missing in Chen's formula; this is because he calculates in $P(X; x_0, *)$, the paths with a fixed initial point

x_0; a slight difference in the *signs* arises from our different choice of the differential on $\mathbf{B}^*(A^*)$.

It is interesting to remark, cf., 3.2.1$_*$ in [4] that according to 1.2, I is a homotopy in DASH between Π_0^* and Π_1^*.

Now choose a base-point $* \in X$; then we have the augmentation $\varepsilon = i^*$: $\Lambda^*X \to \Lambda^*(*) = R$ and we write $\bar{\Lambda}^*X = \ker \varepsilon$. Now we have the usual bar construction, as in [4] for instance:

$$B^*(\Lambda^*X) = \mathbf{B}^*(\bar{\Lambda}^*X) \overset{i}{\subset} \mathbf{B}^*(\Lambda^*X).$$

By $\Omega_S X \subset P_S X$ we denote the subspace of loops at $*$ and observe that the compositions

$$\Lambda^*X \xrightarrow{\Pi_t^*} \Lambda^*P_SX \xrightarrow{h} \Lambda^*\Omega_SX \quad (t = 0, 1)$$

factor through the augmentation. Hence, if I_0: $B^*(\Lambda^*X) \to \Lambda^*\Omega_SX$ denotes the composition

$$B^*(\Lambda^*X) \xrightarrow{i} \mathbf{B}^*(\Lambda^*X) \xrightarrow{I} \Lambda^*P_SX \xrightarrow{h} \Lambda^*\Omega_SX$$

then 1.2 gives:

1.21 COROLLARY. $DI_0 = 0$. *In other words, I_0 is a chain map.*

If A^* is commutative (i.e., "skew commutative") then the "shuffle homomorphism" induces a product structure in $\mathbf{B}^*(A^*)$ with [ ] as unit, as is well known.

1.3 LEMMA. I: $\mathbf{B}^*(\Lambda^*X) \to \Lambda^*P_SX$ *is a morphism of algebras.*

This is 4.1.1 of [1], and is proved in [6].

1.31 COROLLARY. I_0: $B^*(\Lambda^*X) \to \Lambda^*\Omega_SX$ *is a morphism of DGA-algebras.*

Let $A^* \subset \Lambda^*X$ be a sub DGA-algebra such that $dA^0 = A^1 \cap d\Lambda^0X$. The image $I_0(B^*(A^*))$, i.e., the submodule of $\Lambda^*\Omega X$ generated by integrals $\int w_1 \cdots w_r$ where $w_i \in A^*$, is a sub DGA-algebra by 1.21 and 1.31. We shall denote it by $\int A^*$.

1.4 PROPOSITION. *If the differentiable space X is plotwise connected (i.e. by paths which are piecewise plots), then I_0: $B^*(A^*) \to \int A^*$ is a homology-isomorphism.*

Proof. We filter $B^*(A^*)$ by $\oplus_{j \le p} B_j^*(A^*)$ and $\int_0 A^*$ by the I_0-image of this filtration. By $\bar{A}^*$ we denote $A^* \cap \bar{\Lambda}^*X$, and we define $\bar{\bar{A}}^* = s^{-1}(\bar{A}^*/\bar{A}^0 + d\bar{A}^0)$. It is easily seen that $\bar{A}^0 + d\bar{A}^0$ is acyclic and hence $\bar{A}^* \to \bar{\bar{A}}^{*+1}$ is a homology isomorphism. In the spectral sequence of the filtration,

$$E_p^1 B^*(A^*) = \otimes^p H(\bar{\bar{A}}^{*-1}) \simeq \otimes^p H(\bar{A}^*).$$

Now, in [1, 4.3.2] it is shown, by a geometric argument, that I_0 induces an isomorphism $\otimes^p H(\bar{A}^*) \to E_p^1(\int A^*)$. (Note that our $\bar{A}$, $\int A^*$ are denoted by $\bar{A}$, A^1 in [1].) Hence $E_p^1(I_0)$ is an isomorphism, and our result follows from the completeness of the filtrations.

2. The Stokes map

Let X be a differentiable space; by C_*X we define the subcomplex of the usual singular complex generated by those singular simplexes $v: \Delta^n \to X$ which are plots; in [1] these are called the "smooth" simplexes. The corresponding cochain-complex $\mathrm{Hom}_R\,(C_*X, R)$ is denoted by C^*X; the pairing is denoted by $\langle\ \ \rangle$. We shall adhere strictly to the "Koszul convention" for signs; in particular a cochain $x \in C^p(X)$ will be regarded as a map of grading $-p$ so that the differential is given by

$$\langle dx, v\rangle = (-1)^{p+1}\langle x, \partial v\rangle.$$

We define the "Stokes map" $\rho = \rho(X)\colon \Lambda^*X \to C^*X$ by

$$\langle \rho w, v\rangle = (-1)^{p(p+1)/2} \int_{\Delta^p} w_v$$

if $w \in \Lambda^p X$. We shall also write $\langle w, v\rangle$ for $\langle \rho w, v\rangle$. We easily verify that $d\rho = \rho d$, i.e., $D\rho = 0$, using Stoke's theorem. We cannot, of course, assume that ρ is a homology isomorphism; it is, classically, if X is a differentiable manifold.

2.1 PROPOSITION. *There is a morphism* $P\colon \mathbf{B}(\Lambda^*X) \to C^{*+1}X$ *of grading* $+1$ *such that* $P[\ \ ] = 0$, $P[w] = \rho w$ *if* $w \in \Lambda^*X$ *and* $DP = P \cup P$.

In the language of [4]—at least after we change to the augmented case—this means that ρ can be extended to a map P of DASH; in the notation of [5],

$$P[w_1, \ldots, w_r] = \rho_r(w_1 \otimes \cdots \otimes w_r)$$

so that ρ_r has grading $-r + 1$.

The proof of 2.1 in [5] by the method of acyclic models applies, even though Λ^*X is neither of the deRham complexes considered in that paper. This is so because the proof depends only on three facts:

(i) ρ is multiplicative when restricted to $\Lambda^0 X$.
(ii) Λ^* is acyclic on simplexes.
(iii) C^* is "corepresentable."

(i) is evident; (ii) follows because on simplexes, Λ^* is the classical theory; and (iii) follows because the identity map $\Delta^n \to \Delta^n$ is a plot.

P can be regarded as a morphism $P_B\colon \mathbf{B}(\Lambda^*X) \to \mathbf{B}(C^*X)$ which, in the augmented case restricts to $B(\Lambda^*X) \to B(C^*X)$, as is easily seen. This is explained in [4]. From the usual spectral sequence argument we obtain:

2.2 PROPOSITION. *Let $A^* \subset \Lambda^*X$ be a sub DGA-algebra such that $\rho \mid A^*$: $A^* \to C^*$ is a homology isomorphism. Then $P_B \mid B(A^*)$: $B^*(A^*) \to B^*(C^*X)$ is a homology isomorphism.*

Recalling 1.4 we thus obtain the following version of the theorem of Chen [1, 4.7.1]:

2.3 THEOREM. *Let X be a plotwise connected (cf. 1.4) differentiable space and let $A^* \subset \Lambda^*X$ be a sub DGA-algebra such that $dA^0 = A^1 \cap d\Lambda^0 X$. Suppose also that:*

(i) *$\rho \mid A^*$: $A^* \to C^*X$ is a homology isomorphism.*
(ii) *C^*X is homology isomorphic to the usual (continuous) cochain complex so that $HC^*X = H(X, R)$.*
(iii) *The Adams-Eilenberg-Moore theorem, namely $H^*(B^*(C^*X)) \approx H^*(\Omega X)$ applies, where ΩX is the (continuous) loop-space.*
Then $H^*(\Omega X, R) \approx H^*(\int A^*)$ as R-modules.

3. The Adams construction

Let us denote by I^n the n-dimensional unit cube, by λ_i^ε the face operators in the cubical singular complex, by $P(X, x_0, x_1)$ the paths (which are piecewise plots) from x_0 to x_1, by v_i the ith vertex of the standard simplex, by ∂_i the face operators of the simplicial singular complex, by

$$f_i^n: \Delta^i \to \Delta^n, \qquad l_i^n: \Delta^i \to \Delta^n$$

the standard injections for the first and last $i + 1$ vertices, and by $\varepsilon^i: \Delta^{n-1} \to \Delta^n$ the adjoint of ∂_i. Adams and Chen have constructed maps $\theta_n: I^{n-1} \to P(\Delta^n, v_0, v_n)$ such that $\theta_1 I^0$ is the identity path on Δ^1 and

$$\lambda_i^1\theta_n \equiv P(\varepsilon^i)\theta_{n-1}, \qquad \lambda_i^0\theta_n \equiv P(f_i^n)\theta_i \times P(l_{n-i}^n)\theta_{n-i} \qquad (n > 1)$$

where $\times$ denotes the composition product of plots introduced earlier, and $\equiv$ means equality up to a reparametrization.

Chen's modification was needed to make sure that all the maps are piecewise C^∞. In [1] the roles of λ_i^0, λ_i^1 are exchanged: We return to the formulas as originally given by [3].

Suppose X is a differentiable space and $v: \Delta^{n+1} \to X$ a plot. We define the plot $c(v): I^n \to P_S X$ as the composition

$$I^n \xrightarrow{\theta_{n+1}} P(\Delta^{n+1}, v_0, v_{n+1}) \xrightarrow{P(v)} P(X, v(v_0), v(v_{n+1}))$$

and verify that

$$c(v)I^0 = v \text{ regarded as a path in } X \text{ if } n = 0,$$

$$\lambda_i^1 c(v) \equiv c(\partial_i v), \qquad \lambda_i^0 c(v) \equiv c(vf_i^{n+1}) \times c(vl_{n+1-i}^{n+1}) \tag{3.0}$$

$$(1 \leq i \leq n).$$

We shall regard c as a morphism $c\colon C_*X \to CU_{*-1}X$ where CU denotes the (smooth) cubical complex, and where we put $c \mid C_0X = 0$. Now we introduce the morphism of grading 1, $\sigma\colon \Lambda^*P_SX \to C^{*+1}(X)$, by

$$\langle \sigma W, v\rangle = (-1)^n\langle W, cv\rangle = (-1)^{n(n+3)/2}\int_{I^n} W_{cv}$$

where $W \in \Lambda^nP_SX$ and $v\colon \Delta^{n+1} \to X$ is a plot.

Next we define morphisms

$$1\colon \mathbf{B}^*(\Lambda^*X) \to \Lambda^*PX \quad \text{and} \quad e\colon \mathbf{B}^*(\Lambda^*X) \to C^{*+1}(X)$$

as follows:

$$1[\ ] = 1 \in \Lambda^0PX, \quad 1 \mid \mathbf{B}^*_p(\Lambda^*X) = 0 \quad \text{if } p > 0;$$
$$e[w] = \rho w \quad \text{if } w \in \Lambda^0X \text{ (cf. Chapter 2)}$$
$$= 0 \qquad \text{otherwise;}$$
$$e \mid \mathbf{B}^*_p(X) = 0 \quad \text{if } p \neq 1.$$

Next, we define

$$\bar{I}\colon \mathbf{B}^*(\Lambda^*X) \to \Lambda^*P_SX \quad \text{and} \quad P'\colon \mathbf{B}^*(\Lambda^*X) \to C^{*+1}(X)$$

by

$$\bar{I} = I - 1 \text{ (cf. Chapter 1)}, \qquad P' = \sigma\bar{I} + e.$$

3.1 PROPOSITION. $DP' = P' \cup P'$.

Proof. A straightforward calculation using 3.0 shows that

$$\langle (D\sigma)(W, v)\rangle = \langle W, c(\partial_0 v)\rangle + (-1)^{n+2}\langle W, c(\partial_{n+2}v)\rangle + \sum_{i=1}^{n+1}(-1)^i\langle W, c(vf_i^{n+2}) \times c(vl^{n+2}_{n+2-i})\rangle$$

where $W \in \Lambda^nP_SX$, $v\colon \Delta^{n+2} \to X$ is a plot and $n \geq 0$.

In this formula we substitute $W = \int w_1 \cdots w_r$ where $w_i \in \Lambda^{p_i}X$ and $p_i + \cdots + p_r - r = n$, $r \geq 1$. From 1.1 and making due allowance for the signs we have introduced, we get

$$\left\langle \int w_1 \cdots w_r, c(vf_i^{n+2}) \times c(vl^{n+2}_{n+2-i} \right\rangle$$
$$= (-1)^{in+n+i+1}\sum_{j=0}^{r}\left\langle \int w_1 \cdots w_j, c(vf_i^{n+2})\right\rangle \left\langle \int w_{j+1} \cdots w_r, c(vl^{n+2}_{n+2-i})\right\rangle.$$

Now, for $j = 0$ we get $\delta_{0,i-1}\langle \int w_1 \cdots w_r, c(vl^{n+2}_{n+2-i})\rangle$ which is nonzero only if $i = 1$. Then $v(l^{n+2}_{n+2-i}) = \partial_0 v$ and we get

$$\left\langle \int w_1 \cdots w_r, c(\partial_0 v)\right\rangle$$

and this cancels with the term $\langle w, c(\partial_0 v)\rangle$ in the formula. Similarly, the term for $j = r$ cancels with $\langle w, c(\partial_{n+2}v)\rangle$ and we have

$$\left\langle (D\sigma)\int w_1 \cdots w_r, v\right\rangle$$
$$= \sum_{i=1}^{n+1}\sum_{j=1}^{r-1}(-1)^{in+n+1}\left\langle \int w_1 \cdots w_j, c(vf_i^{n+2})\right\rangle\left\langle \int w_{j+1} \cdots w_r, c(vl_{n+2-i}^{n+2})\right\rangle.$$

The same formula is obviously true if $\int w_1 \cdots w_r$ is replaced by $I[w_1, \ldots, w_r]$, etc. For the moment, let us denote $\sigma\bar{I}$ by $\bar{P}$, so that $P' = \bar{P} + e$. We compute $\bar{P} \cup \bar{P}$:

$$(\bar{P} \cup \bar{P})[w_1 \cdots w_r] = \sum_{j=0}^{r} \cup (\bar{P} \otimes \bar{P})([w_1 \cdots w_j] \otimes [w_{j+1} \cdots w_r])$$

where the terms $j = 0$ and $j = r$ are zero because $\bar{P}[\ \ ] = 0$. Thus

$$\langle(\bar{P} \cup \bar{P})[w_1 \cdots w_r], v\rangle$$
$$= \sum_{j=1}^{r-1}\sum_{i=0}^{n+2}(-1)^{in+n+1}\langle\bar{I}[w_1 \cdots w_j], c(vf_i^{n+2})\rangle\langle\bar{I}[w_{j+1} \cdots w_r], c(vl_{n+2-i}^{n+2})\rangle$$

which we obtain by evaluating the $\cup$-product by the standard Whitney formula. The terms with $i = 0$ and $i = n + 2$ are zero. Hence, comparing our formulas

$$(D\sigma)\bar{I} = \bar{P} \cup \bar{P} \tag{3.11}$$

where we need merely add that both sides are zero on $[\ \ ]$. Next, we prove the formulas

$$e \cup \bar{P} = -\sigma(\tau_0 \cup \bar{I}), \tag{3.12}$$

$$\bar{P} \cup e = \sigma(\bar{I} \cup \tau_1) \tag{3.13}$$

where τ_0, τ_1 are as in 1.2. To prove 3.12, note that both sides are 0 on $[\ \ ]$. Now, let $w_i \in \Lambda^{p_i}X$ $(i = 1, \ldots, r, r \geq 1)$. Both sides of 3.12 are zero on $[w_1, \ldots, w_r]$ if $r = 1$. Thus, let $r > 1$.

$$\begin{aligned}(e \cup \bar{P})[w_1, \ldots, w_r] &= U(e \otimes \bar{P})([w_1] \otimes [w_2, \ldots, w_r])\\ &= (-1)^{p_1-1}e[w_1] \cup \bar{P}[w_2 \cdots w_r],\\ \sigma(\tau_0 \cup \bar{I})[w_1, \ldots, w_r] &= \sigma \cup (\tau_0 \otimes \bar{I})([w_1] \otimes [w_2, \ldots, w_r])\\ &= \sigma\{\tau_0[w_1] \wedge \bar{I}[w_2, \ldots, w_r]\}.\end{aligned}$$

Now, if $v\colon \Delta^{n+2} \to X$ where $n = p_1 + \cdots + p_r - r$ is a plot, then

$$\langle\sigma\{\tau_0[w_1] \wedge \bar{I}[w_2 \cdots w_r]\}, v\rangle = \langle\Pi_0^* w_1 \wedge \bar{I}[w_2, \ldots, w_r], cv\rangle(-1)^{n+1}.$$

Now, $(\Pi_0^* w_1)_{cv} = (w_1)_{\pi_0 cv}$ and $\pi_0 cv$ is the constant plot at $v(v_0)$. Hence we get 0 unless $p_1 = 0$, as required by our identity. Thus, let $p_1 = 0$. Then

$$\langle\sigma\{\tau_0[w_1] \wedge \bar{I}[w_2, \ldots, w_r]\}, v\rangle = w_1(v(v_0))\langle I[w_2, \ldots, w_r], c(v)\rangle(-1)^{n+1}.$$

Also, in this case

$$\begin{aligned}\langle (e \cup \bar{P})[w_1, \dots, w_r], v\rangle &= -\langle \rho w_1 \cup \sigma \bar{I}[w_2, \dots, w_r], v\rangle \\ &= -w_1(v(v_0))\langle \bar{I}[w_2, \dots, w_r], c(v)\rangle(-1)^{n+1}\end{aligned}$$

and our proof is complete. The proof of 3.13 is similar. From 1.2 and $D1 = 0$ we easily deduce

(3.14) $$D\bar{I} = \tau_0 \cup \bar{I} - \bar{I} \cup \tau_1 + \tau_0 - \tau_1.$$

We now calculate

$$\begin{aligned}D\bar{P} &= D(\sigma\bar{I}) \\ &= (D\sigma)\bar{I} - \sigma D\bar{I} \\ &= \bar{P} \cup \bar{P} - \sigma(\tau_0 \cup \bar{I} - \bar{I} \cup \tau_1 + \tau_0 - \tau_1)\end{aligned}$$

by 3.11 and 3.14. Hence

$$DP' = D\bar{P} + De = \bar{P} \cup \bar{P} - \sigma(\tau_0 \cup \bar{I} - \bar{I} \cup \tau_1 + \tau_0 - \tau_1) + De$$

and

$$P' \cup P' = (\bar{P} + e) \cup (\bar{P} + e) = \bar{P} \cup \bar{P} + e \cup \bar{P} + \bar{P} \cup e + e \cup e$$

and by 3.12, 3.13 it remains to prove that

(3.15) $$-\sigma(\tau_0 - \tau_1) + De = e \cup e.$$

Now, both sides of 3.15 are clearly zero on $[w_1, \dots, w_r]$ unless $r = 1$ or 2. For $r = 1$, note $\langle \sigma\tau_0[w_1], v\rangle = (-1)^{p_1}\langle \Pi_0^* w_1, cv\rangle$ which is zero unless $p_1 = 0$, as before. Similarly for $\sigma\tau_1$, and thus 3.15 is true for $r = 1$ unless $p_1 = 0$; and in that case

$$\langle \Pi_0^* w_1, cv\rangle - \langle \Pi_1^* w_1, cv\rangle = w_1(c(v_0)) - w_1(v(v_1)) = -\langle de[w_1], v\rangle$$

as required. Finally, we prove that $de = e \cup e$ on $[w_1, w_2]$, which is easy. This completes the proof of 3.1.

Comparison of 2.1 and 3.1 suggests some relationship between P and P'. Suppose $w_i \in \Lambda^{p_i}X$ so that $P[w_1, \dots, w_r] \in C^n X$ where $n = p_1 + \cdots + p_r - r + 1$.

3.2. LEMMA *Suppose* $r > 1$.

(i) $P[w_1, \dots, w_r] = 0$ *if* $p_i > n$ *for any* i.

(ii) $P[w_1, \dots, w_r] = 0$ *if* $p_1 + \cdots + p_r < r$.

(iii) *If* $p_1 = \cdots = p_r = 1$, *then*

$$\langle P[w_1, \dots, w_r], v\rangle = (-1)^r \left(\int w_1 \cdots w_r\right)_v$$

where $v: \Delta^1 \to X$ *is a plot.*

The proof of this follows easily from the inductive construction of P in [5]. The iterated integration in (iii) arises from the use of the chain homotopy S derived from the standard contraction of Δ^1 to v_0. We omit these details. It was the discovery of the relationship (iii) which led to the present paper; it is interesting to observe that the case $p_1 = \cdots = p_r = 1$ is the only one arising in Chen's theory of the fundamental group.

3.21 COROLLARY. $P \mid \mathbf{B}^m(\Lambda^*X) = P' \mid \mathbf{B}^m(\Lambda^*X)$ *if* $m \leq 0$.

Proof. With the notation of the lemma, $m = p_1 + \cdots + p_r - r = n - 1$. Consider $P, P' \mid \mathbf{B}_r^m(\Lambda^*X)$. For $r = 0$, the result is immediate from the definitions. Now, let $r = 1$ so that $m = p_1 - 1$. For $p_1 = 0$, $P = P'$ by definitions; thus let $p_1 = 1$, $m = 0$, and let $v: \Delta^1 \to X$ be a plot.

$$\begin{aligned}
\langle P'[w_1], v\rangle &= \langle \sigma\bar{I}[w_1], v\rangle \\
&= \langle \bar{I}[w_1], c(v)\rangle \\
&= -\left\langle \int w_1, c(v)\right\rangle \\
&= -\left(\int w_1\right)_{c(v)} \\
&= -\int_{\Delta^1} v^*w_1 \quad \text{since } c(v)I^0 = v \\
&= -\int_v w_1 \\
&= \langle \rho w_1, v\rangle \\
&= \langle P[w_1], v\rangle
\end{aligned}$$

as required.

Now let $r > 1$. If $m < 0$, P is zero by 3.2(ii) and P' is zero because $p_i = 0$ for at least one i. If $m = 0$ either $p_1 = \cdots = p_r = 1$, in which case the result is 3.2(iii), or some p_i is >1 and some $p_j = 0$; and then, both P and P' are zero, by 3.2(i).

This completes the proof.

3.3 PROPOSITION. *There is a natural morphism* $U: \mathbf{B}(\Lambda^*X) \to C^*(X)$ *such that* $U[\ ] = 1$ *and* $DU = P \cup U - U \cup P'$.

Apart from the fact that we are in the unaugmented theory, this means that P and P' a homotopic in the category DASH of [4]. Due to 3.21 we can define $U \mid \mathbf{B}_r^m(\Lambda^*X) = 0$ for $m \leq 0$ and $r > 0$.

We continue the construction by induction on r, and for each r by induction on m. The method is exactly that of [5]; once again, we use the fact that Λ^* is acyclic on models, and C^* corepresentable, cf., the proof of 2.1 above. We omit the details.

4. Augmentation and loop-spaces

We now return to the case of a differential space X with base-point $*$ already considered in Chapter 2. Again, $\Omega_S X \subset P_S X$ denotes the subspace of piecewise smooth loops at $*$; we use the notations preceding 1.21. By $C_{*0}(X) \subset C_* X$ we denote the singular complex generated by those smooth simplexes having all vertices at $*$; $C_0^*(X) = \text{Hom}\,(C_{*0}X, R)$ is the corresponding cochain-complex and

$$\bar{C}_0(X) = \ker \{\varepsilon\colon C_0^*(X) \to C_0^*(*)\}$$

the kernel of the augmentation; $j\colon C^*(X) \to C_0^*(X)$ is the restriction. We define the morphisms

$$P_0, P_0'\colon B^*(\Lambda^* X) \to C^{*+1}(X), \qquad U_0\colon B^*(\Lambda^* X) \to C^*(X)$$

by

$$P_0 = jPi \qquad \text{(cf., 2.1)}$$

$$P_0' = jP'i \qquad \text{(cf., 3.1)}$$

$$U_0 = jUi \qquad \text{(cf., 3.3)}$$

and obtain from 3.3 that

$$DU_0 = P_0 \cup U_0 - U_0 \cup P_0'. \tag{4.1}$$

It is also easily verified that the images of P_0, P_0', and U_0 are in $\bar{C}_0^{*+1}(X)$; since $\mathbf{B}^*(\Lambda^* X)$ contains negative-dimensional elements this is not entirely trivial. It follows that U_0 is a homotopy in DASH between P_0 and P_0', so that the maps $P_B, P_B'\colon B^*(\Lambda^* X) \to B^*(C^* X)$ are chain-homotopic; cf., 2.2 above and 3.2 in [4]. It follows that the proof of 2.3 can be based on P' instead of P: This is, essentially, Chen's proof. Now let $h\colon \Lambda^* P_S X \to \Lambda^* \Omega_S X$ be the restriction and suppose $W \in \Lambda^n P_S X$ is such that $hW = 0$. That means $W_\alpha = 0$ if $\alpha\colon U \to P_S X$ is a plot such that $\alpha(u)(0) = \alpha(u)(1) = *$ for all $u \in U$. Now let $v\colon \Delta^{n+1} \to x$ have all vertices $*$; then $c(v)\colon I^n \to PX$ satisfies $c(v)(u)(0) = c(v)(u)(1) = *$ for all $u \in I^n$ and hence $W_{c(v)} = 0$. Hence

$$\langle j\sigma W, v\rangle = \pm\langle W, cv\rangle = \pm\int_{I^n} W_{c(v)} = 0.$$

Thus $hW = 0$ implies $j\sigma W = 0$ and we can insert σ_0 in the commutative diagram

$$\begin{array}{ccc} \Lambda^* P_S X & \xrightarrow{\sigma} & C^{*+1}(X) \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle j} \\ \Lambda^* \Omega_S X & \xrightarrow[\sigma_0]{} & C_0^{*+1}(X). \end{array}$$

Now, $P' = \sigma\bar{I} + e$, cf., 3.1, and if $w \in \bar{\Lambda}^0 X$ then $w(*) = 0$, whence $jei = 0$. Hence

$$P_0' = j(\sigma\bar{I} + e)i = j\sigma\bar{I}i = \sigma_0 h\bar{I}i = \sigma_0\bar{I}_0$$

in the notation of 1.21.

Returning to the original notation we thus have:

4.2 PROPOSITION. *The formulas* $P_0'[\ ] = 0$,

$$\langle P_0'[w_1, \ldots, w_r], v\rangle = (-1)^{(n(n+3)/2)+r} \int_{I^n} \left(\int w_1 \cdots w_r\right)_{cv}$$

where $w_i \in \Lambda^{p_i}X$, $n = p_1 + \cdots + p_r - r$ *and* $v\colon \Delta^{n+1} \to X$ *is a plot with all vertices at* $*$, *define a map*

$$P_0'\colon B^*(\Lambda^*X) \to C_0^{*+1}(X)$$

of DASH *homotopic to that of* 2.1 *above*; *hence this map induces a homology isomorphism* $B^*(A^*) \to B^*(C_0^*(X))$ *in the situation of* 2.2 *above*.

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