LINEAR ALGEBRAIC GROUPS IN INFINITE DIMENSIONS

BY

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An important fact in the finite dimensional theory of Lie groups is that every closed subgroup of a Lie group is itself a Lie group. However, this is not always the case in infinite dimensions. In fact, it is known that there exists a nontrivial arcwise connected closed subgroup of a commutative Banach Lie group which does not contain any one-parameter subgroups (and hence cannot be a Banach Lie group).

In this note we show that algebraic subgroups of infinite dimensional general linear groups are Banach Lie groups and we give a sharp estimate on the size of the neighborhood of the identity which is covered by the exponential map. This estimate depends on the degree of the defining polynomial equations for the algebraic group. We obtain better estimates in many cases for the group of all bounded linear transformations leaving given multilinear mappings invariant and commuting with other given multilinear mappings, where the given mappings have bounded degree. We also show that the group of isometries of a complex Banach space X is a Lie group when the open unit ball of X is a homogeneous domain and that this does not hold without at least some restriction on the Banach space X.

Throughout, A denotes a real Banach algebra with identity e and $G(A)$ denotes the group of all invertible elements of A with the induced topology. A sub-semigroup of $G(A)$ is a subset G of $G(A)$ such that $e \in G$ and $xy \in G$ whenever $x, y \in G$.

DEFINITION. A subgroup (resp. sub-semigroup) G of $G(A)$ is called algebraic of degree $\leq n$ if there is a set Q of vector-valued continuous polynomials on $A \times A$ with degree $\leq n$ such that

$$G = \{x \in G(A): p(x, x^{-1}) = 0 \text{ for all } p \in Q\}.$$

Clearly any finite product and any intersection of algebraic groups (or semigroups) of degree $\leq n$ is algebraic of degree $\leq n$. Note that the definition includes the case where the defining equations for G are polynomial equations in x only. By the Hahn-Banach theorem, the polynomials in the definition can always be chosen to be complex valued.

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THEOREM 1. *If G is an algebraic subgroup of $G(A)$ of degree $\leq n$, then the set*

$$g = \{y \in A: \exp(ty) \in G \text{ for all } t \in \mathbf{R}\}$$

is a closed Lie subalgebra of A with the commutator product, and the principal determination of log *is a homeomorphism of the neighborhood of e in G consisting of all those $x \in G$ satisfying*

$$(1) \qquad \sigma(x) \subseteq \{z \in \mathbf{C}: |\arg z| < \pi/n\}$$

onto a neighborhood of 0 *in g. Thus G is a Banach Lie group in the norm topology and g is its Banach Lie algebra.*

COROLLARY 1. *If G is an algebraic subgroup of $G(A)$ of degree $\leq n$ and $x \in G$ satisfies* (1), *then $x = \exp y$ and $\exp(ty) \in G$ for all $t \in \mathbf{R}$, where $y = \log x$. In particular, each $x \in G$ satisfying* (1) *lies in a norm-continuous, one-parameter subgroup of G.*

Here, as throughout, the function log has its principal determination. If A is not a complex Banach algebra, then $\sigma(x)$ is by definition the spectrum of x in the complexification of A. Hence in particular, Theorem 1 and its corollary apply when (1) is replaced by

$$(2) \qquad \|e - x\| < \sin(\pi/n), \quad n > 1.$$

Notice that if G is the (not necessarily algebraic) group of all $x \in G(A)$ with $\|x\| = \|x^{-1}\| = 1$ and A is complex, then ig is the set of all Hermitian elements of A. (See [2, Section 5].) Also if A is the Banach algebra $L(X)$ of all bounded linear transformations on a Banach space X, then $G(A) = GL(X)$, the general linear group of X.

Example 1. The group $G = \{z \in \mathbf{C}\backslash\{0\}: \operatorname{Im}(z^n) = 0\}$ is algebraic of degree n and $g = \mathbf{R}$. Thus the domain in (1) and the constant in (2) are largest possible.

Example 2. Let l^∞ be the Banach algebra of all complex sequences with the sup norm. For each positive integer n, let $G_n = \{x \in G(l^\infty): x(n)^n = 1\}$ and put $G = \bigcap G_n$. Then G_n is algebraic of degree $\leq n$ and the group G is totally disconnected and not discrete, so G is not a Lie group. Thus intersections of algebraic groups with no restriction on the degree are not necessarily Lie groups. In particular, groups given by holomorphic equations are not in general Lie groups.

Proof of Theorem 1. Clearly g is closed since G is closed in $G(A)$. It follows from classical exponential formulae [3, p. 200] that $x + y$ and $[x, y] = xy - yx$ are in g whenever x and y are in g. (The formulae referred to can be established by straightforward Banach algebra techniques for the case we consider.) Thus g is a closed Lie subalgebra of A.

By passing to the complexification of A and applying the functional calculus (see [12, Chapter 1] and [7, Chapter 5]), we have that log is a bianalytic map of the open set U of all $x \in G(A)$ satisfying (1) onto the open set V of all $y \in A$ satisfying

$$\sigma(y) \subseteq \{z \in \mathbf{C}: |\mathrm{Im}\, z| < \pi/n\},$$

and $\exp = \log^{-1}$. Clearly exp maps $g \cap V$ into $G \cap U$. Thus the proof of Theorem 1 reduces to showing that log maps $G \cap U$ into g, and this follows from:

LEMMA 1. *Let $x \in G(A)$ satisfy* (1) *and suppose $p: A \times A \to \mathbf{C}$ is a polynomial of degree $\leq n$ with $p(x^k, x^{-k}) = 0$, $k = 0, 1, 2, \ldots$. Then $y = \log x$ satisfies p* $(\exp(ty), \exp(-ty)) = 0$ *for all $t \in \mathbf{R}$.*

Proof. Let B be the Banach algebra $A \times\!\!< A$ with, say, the max norm and put $u = (x, x^{-1})$ and $v = (y, -y)$. Then $u \in G(B)$, $v = \log u$, and $p(u^k) = 0$ for $k = 0, 1, 2, \ldots$. Hence it suffices to show that $p(\exp tv) = 0$ for all $t \in \mathbf{R}$. Now by [14, Section 4], p extends to a holomorphic polynomial on the complexification E of B and it can be verified directly that $\sigma_E(v) \subseteq \{z \in \mathbf{C}: |\mathrm{Im}\, z| < \pi/n\}$. Since p is continuous on E, there is a number $M > 0$ with $|p(w)| \leq M \max\{1, \|w\|^n\}$ for all $w \in E$. Consequently,

$$f(\lambda) = p(\exp \lambda v)$$

is an entire function of exponential type and, by the spectral mapping theorem,

$$\limsup_{t \to \infty} (1/t) \log |f(\pm it)| \leq \limsup_{k \to \infty} (n/k) \log^+ \|\exp \pm ikv)\|$$
$$\leq n \log^+ |\exp(\pm iv)|_\sigma < \pi,$$

where $|\ \ |_\sigma$ denotes the spectral radius. Also $f(k) = 0$ for all nonnegative integers k, so $f \equiv 0$ by Carlson's uniqueness theorem [1, p. 153]. Thus $p(\exp tv) = 0$ for all $t \in \mathbf{R}$, as desired.

PROPOSITION 1. *If G is an algebraic sub-semigroup of $G(A)$, then $x^{-1} \in G$ for all $x \in G$ satisfying* (1). *Moreover, the identity component of G is a group.*

COROLLARY 2. *Let M be a closed real subspace of a Banach space X and let α be a bounded linear transformation on X such that $\alpha(M) \subseteq M$ and $\sigma(\alpha)$ does not intersect the interval* $(-\infty, 0]$. *Then $\alpha(M) = M$.*

Proof. The first part of Proposition 1 is immediate from Lemma 1 with $t = -1$. To prove the second part, let $U = \{x \in G: x^{-1} \in G\}$ and put $r = \sin(\pi/n)$ or $r = 1$ according as $n > 1$ or $n = 1$. If $x \in U$, then U contains all $y \in G$ satisfying $\|x - y\| < r/\|x^{-1}\|$ by the first part. If y is the closure of U in G, then by the continuity of the map $x \to x^{-1}$ on $G(A)$, there is an $x \in U$

with $\|x^{-1}\| \, \|x - y\| < r$, so $y \in U$ by what we have just shown. Hence U is both an open and closed subset of G containing e, so U contains the identity component of G.

To deduce the corollary, take $A = L(X)$ and let Q be the set of all polynomials $p: A \to \mathbf{C}$ of the form $p(\beta) = \ell(\beta(x))$, where $x \in M$ and ℓ is a bounded real-linear functional on X with $\ell(M) = 0$. Then by the Hahn-Banach theorem, the semigroup $G = \{\beta \in G(A): \beta(M) \subseteq M\}$ is algebraic of degree ≤ 1. Thus by hypothesis and Proposition 1, $\alpha^{-1} \in G$.

Alternate proof of Theorem 1. To simplify matters, we consider only the case where A is a complex Banach algebra and the defining relations for G are holomorphic polynomials in x. One can deduce the theorem from this case using the techniques of the first proof.

Let P be the complex Banach space of all complex-valued holomorphic polynomials on A of degree $\leq n$ and define maps $\Phi: G(A) \to GL(P)$ and $\phi: A \to L(P)$ by

$$[\Phi(x)p]w = p(wx), \quad [\phi(x)p]w = Dp(w)(wx),$$

where $x, w \in A$ and $p \in P$. Since Φ is an analytic group homomorphism, by elementary Lie theory [3, p. 200], ϕ is a Lie homomorphism and $\Phi \circ \exp = \exp \circ \phi$. Let $Q = \{p \in P: p(G) = 0\}$ and note that Q is a closed complex subspace of P. Since G is algebraic of degree $\leq n$,

$$x \in G \Leftrightarrow \Phi(x)Q \subseteq Q$$

for all $x \in G(A)$, and therefore,

$$y \in g \Leftrightarrow \phi(y)Q \subseteq Q \tag{3}$$

for all $y \in A$. In particular, g is a closed Lie subalgebra of A.

Let $x \in G$ satisfy (1) and put $y = \log x$. As in the first proof, it suffices to show that $y \in g$. Let P_k be the space of all complex-valued homogeneous polynomials of degree k on A, i.e., $p \in P_k$ if and only if there is a continuous symmetric k-linear map $F: A^k \to \mathbf{C}$ such that

$$p(w) = F(w, \ldots, w) \quad \text{for all } w \in A.$$

(The map F is uniquely determined by p.) Then $P = P_0 \oplus \cdots \oplus P_n$, each P_k is an invariant subspace for $\phi(y)$, and $\phi(y)$ is given by

$$F(w, \ldots, w) \to kF(wy, w, \ldots, w)$$

on P_k. Since $\sigma(y) \subseteq \{z \in \mathbf{C}: |\mathrm{Im}\, z| < \pi/n\}$, it follows that $\sigma(\phi(y)) \subseteq \{z \in \mathbf{C}: |\mathrm{Im}\, z| < \pi\}$. Then since $\Phi(x) = \exp \phi(y)$, by the functional calculus and the spectral mapping theorem,

$$\phi(y) = \log \Phi(x) \quad \text{and} \quad \sigma(\Phi(x)) \subseteq \{z \in \mathbf{C}: |\arg z| < \pi\}.$$

By Runge's theorem, log is a uniform limit of polynomials on $\sigma(\Phi(x))$, so $\phi(y)$ is a limit of polynomials in $\Phi(x)$. Hence $\phi(y)Q \subseteq Q$ since $\Phi(x)Q \subseteq Q$, and therefore $y \in g$.

Note that the first part of the above proof holds when A is any real Banach algebra and Q is any closed subspace of complex-valued polynomials on A of degree $\leq n$ such that Q contains the polynomial $w \to p(wx)$ whenever both Q contains p and $x \in G$. Thus by (3), the Lie algebra of

$$G = \{x \in G(A): p(x) = 0 \text{ for all } p \in Q\}$$

is given by

$$g = \{y \in A: Dp(e)y = 0 \text{ for all } p \in Q\}.$$

A number of interesting examples of algebraic groups of degree $\leq n$ are included under the following:

PROPOSITION 2. *Let X and Y be real Banach spaces and let F be a continuous n-linear map on X with values in Y. Let $m = 0$ or $m = 1$, and suppose $Y = X$ when $m = 1$. Then the set G of all $\alpha \in GL(X)$ satisfying*

$$(4) \qquad F(\alpha x_1, \dots, \alpha x_n) = \alpha^m F(x_1, \dots, x_n) \quad \text{for all } x_1, \dots, x_n \in X,$$

is a Banach Lie group whose Banach Lie algebra is the set g of all $\delta \in L(X)$ satisfying

$$(5) \quad F(\delta x_1, x_2, \dots, x_n) + \dots + F(x_1, \dots, x_{n-1}, \delta x_n) = m\delta F(x_1, \dots, (x_n)$$

for all $x_1, \dots, x_n \in X$, and g is as in Theorem 1. Also if $\alpha \in G$ and

$$(6) \quad \sigma(\alpha) \subseteq \{z \in \mathbf{C}: |\arg z| < 2\pi/(n + m)\}, \quad n + m \neq 1,$$

then $\delta = \log \alpha$ is in g.

Here α^m and $m\delta$ denote the identity and zero transformations on Y, respectively, when $m = 0$.

Clearly, any group which is an intersection of groups of the above type with $n \leq N$ is a Banach Lie group with Banach Lie algebra equal to the intersection of the corresponding Banach Lie algebras, and (6) holds with n replaced by N.

Note that the group G of Proposition 2 is algebraic of degree $\leq (n + m)/2$ when $n + m$ is even and algebraic of degree $\leq (n + m + 1)/2$ when $n + m$ is odd. However, in the second case (unlike the first) this is not enough to obtain Proposition 2 directly from Theorem 1.

Example 3. If A is a not necessarily associative Banach algebra (e.g., a Banach Lie algebra or a Banach Jordan algebra), then Proposition 2 applies with $n = 2$ and $m = 1$ to the group

$$G = \{\rho \in GL(A): \rho(xy) = \rho(x)\rho(y) \text{ for all } x, y \in A\}$$

of all automorphisms of A and

$$g = \{\delta \in L(A): \delta(xy) = (\delta x)y + x(\delta y) \text{ for all } x, y \in A\}$$

is the Lie algebra of all derivations of A. This contains results given in [4, p. 314], [9, p. 420], and [15]. An example given in [10, p. 269] shows that the constant $2\pi/3$ in (6) is largest possible.

Example 4. If H is a complex Hilbert space and $J \in L(H)$, then Proposition 2 applies with $n = 2$ and $m = 0$ to the group

$$G = \{A \in GL(H)\colon A^{\tau}JA = J\}$$

and

$$g = \{B \in L(H)\colon B^{\tau}J + JB = 0\},$$

where τ may be either the adjoint map or a transpose map. Clearly this example includes the infinite dimensional unitary, orthogonal, and symplectic groups. (See [5, 11.4].)

Example 5. If $\mathfrak{A}$ is a J^*-algebra [6], then Proposition 2 applies with $n = 3$ and $m = 1$ to the group

$$G = \{\rho \in GL(\mathfrak{A})\colon \rho(AB^*A) = \rho(A)\rho(B)^*\rho(A) \text{ for all } A, B \in \mathfrak{A}\}$$

of all J^*-isomorphisms and

$$g = \{\delta \in L(\mathfrak{A})\colon \delta(AB^*A) = \delta(A)B^*A + A\delta(B)^*A + AB^*\delta(A) \text{ for all } A, B \in \mathfrak{A}\}.$$

It is known that G is the group of all isometries of $\mathfrak{A}$ onto itself and hence ig is the space of all Hermitian operators on $\mathfrak{A}$. For further results for the case where $\mathfrak{A}$ is a C^*-algebra, see [8] and [13].

Proof of Proposition 2. Our proof will be independent of Lemma 1. By complexification, we may assume that X and Y are complex Banach spaces and that all linear and multilinear mappings mentioned are complex linear in each variable. Obviously, g as defined above is a closed Lie subalgebra of $L(X)$. We first show that g is as in Theorem 1. Suppose $\delta \in L(X)$ and $\exp(t\delta) \in G$ for all $t \in \mathbf{R}$. Replacing α by $\exp(t\delta)$ in (4) and differentiating at $t = 0$, we see that δ satisfies (5) so $\delta \in g$. Now suppose $\delta \in g$, and let W be the Banach space of all continuous n-linear mappings F' on X with values in Y. Define $B_0, \ldots, B_n \in L(W)$ by

$$B_0(F')(x_1, \ldots, x_n) = m\delta F'(x_1, \ldots, x_n),$$

$$B_k(F')(x_1, \ldots, x_n) = F'(x_1, \ldots, x_{k-1}, \delta x_k, x_{k-1}, \ldots, x_n), \quad 1 \leq k \leq n,$$

and put $B = B_1 + \ldots + B_n - B_0$. Then $B(F) = 0$ and $B_0, \ldots, B_n$ commute, so

$$(\exp B_1) \cdots (\exp B_n)F = (\exp B_0)\,(\exp B)F = (\exp \delta)^m F,$$

which shows that $\exp \delta \in G$. Hence $\exp(t\delta) \in G$ for all $t \in \mathbf{R}$ since g is a linear space.

A slight extension of the argument given in [3, p. 209], proves the last assertion of the proposition. Thus, as before, G is a Banach Lie group with Banach Lie algebra g.

THEOREM 2. *Let X be a complex Banach space and suppose the open unit ball B of X is homogeneous (i.e., the group G of biholomorphic mappings of B acts transitively on B). Then the group K of all invertible linear isometries of X is an* *algebraic subgroup of $GL(X)$ of degree ≤ 2. In particular, K is a Banach Lie group and every $\alpha \in K$ with $\|I - \alpha\| < \sqrt{2}$ lies in a norm-continuous, one-parameter subgroup of K.*

Proof. Let $\mathscr{P}$ denote the complex Banach space of all polynomial vector fields of degree ≤ 2 on X. The elements of $\mathscr{P}$ are written in the form $h(z)(\partial/\partial z)$ where $h: X \to X$ is a polynomial of degree ≤ 2. In [11] it has been shown that the set $\mathfrak{g}$ of all complete holomorphic vector fields on B is a closed real-linear subspace of $\mathscr{P}$ and that $\mathfrak{g}$ is a Banach Lie algebra in the induced topology. Furthermore, $\mathfrak{g}$ admits (as a topological vector space) a direct sum decomposition of the form

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\mathfrak{k} = \{\lambda(z)(\partial/\partial z): \lambda \in L(X) \text{ and } \exp(t\lambda) \in K \text{ for all } t \in \mathbf{R}\}$$

and

$$\mathfrak{p} = \{(a - q_a(z))(\partial/\partial z): a \in X\}.$$

Here $a \to q_a$ is a certain continuous, injective, conjugate-linear mapping from X into the Banach space of all homogeneous quadratic mappings of X into X. For every $\alpha \in GL(X)$, define a map $Ad(\alpha) \in GL(\mathscr{P})$ by

$$h(z)(\partial/\partial z) \to \alpha h(\alpha^{-1}z)(\partial/\partial z).$$

Then $\mathfrak{p}$ is Ad (α)-invariant for every $\alpha \in K$, i.e., $\alpha N = N\alpha$, and therefore $G = NK$, where N is the subgroup of G generated by exp $(\mathfrak{p})$. On the other hand, if $\alpha \in GL(X)$ and Ad $(\alpha)\mathfrak{p} = \mathfrak{p}$, then $\alpha \in K$ since

$$\alpha(B) = \alpha G(0) = \alpha N(0) = N\alpha(0) = N(0) = B.$$

Therefore,

$$\begin{aligned} K &= \{\alpha \in GL(X): \operatorname{Ad}(\alpha)\mathfrak{p} = \mathfrak{p}\} \\ &= \{\alpha \in GL(X): q_a(\alpha z) = \alpha q_{\alpha^{-1}a}(z) \text{ for all } a, z \in X\}, \end{aligned}$$

and K is algebraic of degree ≤ 2. Every $\alpha \in K$ with $\|I - \alpha\| < \sqrt{2}$ satisfies Re $\sigma(\alpha) > 0$ and therefore lies on a norm-continuous, one-parameter subgroup of K by Corollary 1.

Example 6. We construct a complex Banach space X such that the group G of invertible isometries of X is not a Lie group in the norm topology. Thus some restriction on the Banach space X in Theorem 2 is necessary.

Given a positive integer $n > 1$, let $E_n = \bigcup_{k=1}^n E_{n,k}$, where

$$E_{n,k} = \{(\lambda \cos k\pi/n, \lambda \sin k\pi/n): |\lambda| = 1, \lambda \in \mathbf{C}\},$$

and let K_n be the convex hull of E_n. Then K_n is a closed, bounded, balanced convex subset of $\mathbf{C}^2$ with nonempty interior, so K_n is the closed unit ball of $\mathbf{C}^2$ with respect to some norm. Call this normed space X_n. Note that E_n is the set of extreme points of the closed unit ball of X_n and that $E_{n1}, \ldots, E_{nn}$ are the components of E_n.

Let $X = \times_{n=2}^{\infty} X_n$ and give X the sup norm. For each n, define a continuous map $f_n\colon G \to X_n$ by $f_n(\alpha) = \pi_n(\alpha(e))$, where π_n is the projection of X onto the nth coordinate and e is the constant sequence in X with each term $e_1 = (1, 0)$. Let E be the set of extreme points of the closed unit ball of X and note that $x \in E$ if and only if $\pi_n x \in E_n$ for all n. Then $f_n(G) \subseteq E_n$ since $\alpha(E) \subseteq E$ and $e \in E$. Hence if α is in the identity component of G, it follows that $f_n(\alpha) \in E_{nn}$ for all n since $e_1 \in E_{nn}$.

Now given a positive integer $n > 1$, let

$$\gamma = \begin{bmatrix} \cos\dfrac{\pi}{n} & -\sin\dfrac{\pi}{n} \\ \\ \sin\dfrac{\pi}{n} & \cos\dfrac{\pi}{n} \end{bmatrix}$$

and define a map α on X by $\alpha(\{x_k\}) = \{y_k\}$, where $y_k = x_k$ for $k \neq n$ and $y_n = \gamma x_n$. Then $\alpha \in G$ since both γ and γ^{-1} map K_n into itself. However, $f_n(\alpha) = \gamma e_1 \in E_{n1}$, so α is not in the identity component of G. Thus we have obtained a sequence of elements of G which converge to the identity but which are not in the identity component of G, so G cannot be a Lie group.

The example mentioned in the introduction can be found in the lecture notes: K. H. Hofmann, Theorie directe des groupes de Lie I-IV, Seminaire Dubreil (1973/74), p. 2-08. The authors would like to thank John Duncan for some comments which led us to Lemma 1. See his paper with M. J. Crabb, *Some inequalities for norm unitaries in Banach algebras*, to appear in Proc. Edinburgh Math. Soc.

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