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# INFINITE LOOP MAPS FROM $SF$ TO $BO_{\otimes}$ AT THE PRIME 2

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I. Atiyah, Bott, and Shapiro noticed in [4] that a stable Spin-bundle admits a canonical $kO$-orientation, which in turn gives rise to an $H$-map, $g$: BSpin $\to$ $B(SF; kO)$, where $B(SF; kO)$ is the classifying space for $kO$-oriented stable spherical fibrations (see [10]).

In [10] May showed that there exists a homotopy commutative diagram

$$(1.1)\quad \begin{array}{ccccccc} SF & \longrightarrow & SF/\mathrm{Spin} & \longrightarrow & \mathrm{BSpin} & \longrightarrow & BSF \\ \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| \\ SF & \xrightarrow[e]{} & BO_{\otimes} & \longrightarrow & B(SF, kO) & \longrightarrow & BSF, \end{array}$$

where the horizontal rows are fibration sequences of infinite loop spaces and infinite loop maps and $\chi$ is a sign reversing map. The map $f$ was first defined by Sullivan [14].

Madsen, Snaith, and Tornehave proved in [9] that $f$ is an infinite loop map. Using this and other results from [9] May observed in [10] that the above diagram is commutative in the category of infinite loop spaces and infinite loop maps, when localized away from 2.

In this note we shall see that this is true also when we localize at 2, so that (1.1) is globally commutative in the category of infinite loop spaces and infinite loop maps. For definition and properties of this category see [1] or [11].

The line of proof is to completely characterize the infinite loop maps from $SF_2$ to $BO_{2\otimes}$ up to homotopy of infinite loop maps. Especially we shall see that a map from $SF_2$ to $BO_{2\otimes}$ can be infinitely delooped in at most one way. Compare [9, Theorem 6.3], which gives the analogous result at odd primes.

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II. If $X$ is an infinite loop space we denote by $\mathbf{X}$ the spectrum of which it is the zeroth space. $X_2$ and $\mathbf{X}_2$ are their localization at 2; $\hat{X}_2$ and $\hat{\mathbf{X}}_2$ are their completions at 2. The reason we want to work with completed spaces and spectra is that there is the following very convenient isomorphism:

$$(2.1)\qquad [\mathbf{Y}, \hat{\mathbf{X}}_2] \xrightarrow{\cong} \varprojlim\, [Y(n), \hat{X}_2(n)],$$

where $[\mathbf{Y}, \mathbf{X}]$ denotes the homotopy classes of spectrum maps from $\mathbf{Y}$ to $\mathbf{X}$ and $X(n)$ means the $n$th space of the spectrum $\mathbf{X}$. The isomorphism is an immediate consequence of the canonical compact topology on $[Y, \hat{X}_2]$ for any completed

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space $\hat{X}_2$ and any $Y$, and the Milnor exact sequence

$$0 \to \varprojlim{}^1 [Y(n+1), \hat{X}_2(n)] \to [\mathbf{Y}, \hat{\mathbf{X}}_2] \to \varprojlim [Y(n), \hat{X}_2(n)] \to 0.$$

Let $BO \times Z$ be the classifying space for orthogonal $K$-theory, $KO$, and let $BO[m, \infty]$ denote its $(m-1)$ connected cover. From Bott periodicity we get the following four connective spectra (see [10] for example): $BO \times Z$, whose $8n$th space is $BO[8n, \infty]$; $BO$, whose $8n$th space is $BO[8n+1, \infty]$; $BSO$, whose $8n$th space is $BO[8n+2, \infty]$; BSpin, whose $8n$th space is $BO[8n+4, \infty]$.

We shall also use the following results about Eilenberg-Maclane complexes:

(2.2) $(KO)^{\sim *}(K(Z/2^n Z, i)) = 0$ if $i \geq 2$ and $n \geq 0$.
(2.3) $(KO)_2^{\sim \wedge *}(K(\hat{Z}_2, i)) = 0$ if $i \geq 3$.

Here $\hat{Z}_2$ is the 2-adic integers and $(KO)_2^{\sim \wedge m}$ is the functor that sends a space $X$ to the completion at 2 of $(KO)^{\sim m}(X)$. (2.2) is a result of Anderson and Hodgkin [3] and has as an easy consequence (2.3) (see [9, Lemma 3.2]).

If we let $\mathbf{Spin}_2 = \Omega\mathbf{BSpin}_2$ then we have:

PROPOSITION 2.4. $[\mathbf{Spin}_2, \mathbf{B\hat{S}O}_2] = 0$.

*Proof.* $[\mathbf{Spin}_2, \mathbf{B\hat{S}O}] \cong \varprojlim [\Omega BO_2[8n+4, \infty], (BO)_2^{\wedge}[8n+2, \infty]]$ by (2.1). So we want to show

$$\varprojlim [\Omega BO_2[8n+4, \infty], (BO)_2^{\wedge}[8n+2, \infty]] = 0$$

or equivalently

$$\varprojlim [\Omega^8 BO_2[8n+4, \infty], \Omega^7 (BO)_2^{\wedge}[8n+2, \infty]] = 0$$

but, for $n > 1$,

$$\begin{aligned}
&[\Omega^8 BO_2[8n+4, \infty], \Omega^7 (BO)_2^{\wedge}[8n+2, \infty]] \\
&\qquad \cong [(BO)_2^{\wedge}[8(n-1)+4, \infty], \Omega^7 (BO)_2^{\wedge}[10, \infty]] \\
&\qquad \cong (KO)_2^{\sim \wedge 1}((BO)_2^{\wedge}[4, \infty]) \quad \text{by (2.2) and (2.3)} \\
&\qquad \cong (KO)_2^{\sim \wedge 1}(\mathrm{BSpin}) = 0 \quad \text{by Atiyah-Segal [5].}
\end{aligned}$$

*Remark.* The same sort of arguments give $[\mathbf{SO}_2, (\mathbf{BO})_2^{\wedge}] = 0$, with $\mathbf{SO}_2 = \Omega\mathbf{BSO}_2$. Compare [9, Proposition 3.12], which gives the analogous result at odd primes.

There is a fibration of spectra

(2.5) $$\mathbf{BSpin}_2 \xrightarrow{p} \mathbf{BSO}_2 \longrightarrow \mathbf{K}(\mathbf{Z}/2\mathbf{Z}, 2).$$

LEMMA 2.6. *$p$ induces an isomorphism*

$$p^*: [\mathbf{BSO}_2, (\mathbf{BSO})_2^{\wedge}] \to [\mathbf{BSpin}_2, (\mathbf{BSO})_2^{\wedge}].$$

*Proof.* From (2.5) we get an exact sequence

$$[\mathbf{K}(\mathbf{Z}/2\mathbf{Z}, 2), (\mathbf{BSO})_2^\wedge] \longrightarrow [\mathbf{BSO}_2, (\mathbf{BSO})_2^\wedge] \xrightarrow{p^*} [\mathbf{BSpin}_2, (\mathbf{BSO})_2^\wedge] \longrightarrow [\mathbf{K}(\mathbf{Z}/2\mathbf{Z}, 1), (\mathbf{BSO})_2^\wedge]$$

Here

$$\begin{aligned}[\mathbf{K}(\mathbf{Z}/2\mathbf{Z}, 2), (\mathbf{BSO})_2^\wedge] &\cong \underleftarrow{\lim}\ [K(Z/2Z, 8n+2), (BO)_2^\wedge[8n+2, \infty]] \\ &\cong \underleftarrow{\lim}\ [K(Z/2Z, 8n+2), (BO)_2^\wedge] = 0 \quad \text{by (2.2).}\end{aligned}$$

The same argument gives $[\mathbf{K}(\mathbf{Z}/2\mathbf{Z}, 1), (\mathbf{BO})_2^\wedge] = 0$, but this together with the fibration $\mathbf{K}(\mathbf{Z}/2\mathbf{Z}, 0) \to (\mathbf{BSO})_2^\wedge \to (\mathbf{BO})_2^\wedge$ gives $[\mathbf{K}(\mathbf{Z}/2\mathbf{Z}, 1), (\mathbf{BSO})_2^\wedge] = 0$. So the lemma is proved.

Let $\psi^3$ denote the Adams operation. As usual we then define $\mathbf{J}_2$ by the fibration of spectrum maps:

$$\mathbf{J}_2 \xrightarrow{\pi} \mathbf{BO}_2 \xrightarrow{\psi^3 - 1} \mathbf{BSpin}_2.$$

If we define $\tilde{\mathbf{J}}_2$ by the fibration

$$\tilde{\mathbf{J}}_2 \xrightarrow{\tilde{\pi}} \mathbf{BSO}_2 \xrightarrow{\psi^3 - 1} \mathbf{BSpin}_2, \tag{2.7}$$

then $\tilde{\mathbf{J}}_2$ is the 1-connected cover of $\mathbf{J}_2$.

THEOREM 2.8. $[\tilde{\mathbf{J}}_2, \mathbf{BSO}_2] = \hat{Z}_2$ *with generator* $\tilde{\pi}$.

*Proof.* Since all the homotopy groups of $\tilde{\mathbf{J}}_2$ are finite the completion

$$\Phi\colon \mathbf{BSO}_2 \to (\mathbf{BSO})_2^\wedge$$

induces an isomorphism

$$\Phi_*\colon [\tilde{\mathbf{J}}_2, \mathbf{BSO}_2] \xrightarrow{\cong} [\tilde{\mathbf{J}}_2, (\mathbf{BSO})_2^\wedge],$$

so we only have to show that $[\tilde{\mathbf{J}}_2, (\mathbf{BSO})_2^\wedge]$ is $\hat{Z}_2$ with generator $\Phi \circ \tilde{\pi}$.

From the fibration (2.7) and Proposition 2.4 we get an exact sequence

$$[\mathbf{BSpin}_2, (\mathbf{BSO})_2^\wedge] \xrightarrow{(\psi^3-1)^*} [\mathbf{BSO}_2, (\mathbf{BSO})_2^\wedge] \xrightarrow{\tilde{\pi}^*} [\tilde{\mathbf{J}}_2, (\mathbf{BSO})_2^\wedge] \longrightarrow 0.$$

Following [9] we let $\hat{A}_2^{SO}$ denote the homotopy classes of continuous $H$-maps from $(BSO)_2^\wedge$ to $(BSO)_2^\wedge$. Then there is an obvious map $\theta\colon \hat{Z}_2 \to \hat{A}_2^{SO}$ sending $a \in \hat{Z}_2$ to the Adams operation $\psi^a$. $\psi^1 = \psi^{-1}$ so $\theta\colon \hat{Z}_2^* \to \hat{A}_2^{SO}$ factors over $\hat{Z}_2^*/\{\pm 1\}$, where $\hat{Z}_2^*$ denotes the units in the 2-adic integers. But $\hat{Z}_2^*/\{\pm 1\}$ is monogenic, i.e., the powers $3^i$ are dense in $\hat{Z}_2^*/\{\pm 1\}$. So from [9, Section 2] it follows that the finite sums

$$a_0 \cdot 1 + \sum_{i=1}^{N} a_i \psi^{3^i}, \quad a_i \in \hat{Z}_2$$

are dense in $[\mathbf{BSO}_2, (\mathbf{BSO})_2^\wedge]$ and, as $\psi^{3^i} - 1 = (\psi^{3^{i-1}} + \cdots + \psi^3 + 1)(\psi^3 - 1)$, each $\psi^{3^i} - 1$ maps to zero in $[\tilde{\mathbf{J}}_2, (\mathbf{BSO})_2^\wedge]$ by Lemma 2.6. This proves the theorem. Compare [9, Proposition 6.8].

Let $\mathbf{BO}_\otimes = \mathbf{BO} \times \{1\} \subset \mathbf{BO} \times \mathbf{Z}$; then $\mathbf{BO}_\otimes$ is a spectrum ([10] or [12]). Its 1-connected and 2-connected covers are denoted $\mathbf{BSO}_\otimes$ and $\mathbf{BSpin}_\otimes$ respectively.

According to [2] and [9] there is an equivalence of spectra

$$\rho_3\colon \mathbf{BSO}_2 \to \mathbf{BSO}_{2\otimes}. \tag{2.9}$$

As usual we also define $\mathbf{J}_{2\otimes}$ and its 1-connected cover, $\tilde{\mathbf{J}}_{2\otimes}$, by the fibrations

$$\mathbf{J}_{2\otimes} \xrightarrow{\pi_\otimes} \mathbf{BO}_{2\otimes} \xrightarrow{\psi^{3/1}} \mathbf{BSpin}_{2\otimes}$$
$$\tilde{\mathbf{J}}_{2\otimes} \xrightarrow{\pi_\otimes} \mathbf{BSO}_{2\otimes} \xrightarrow{\psi^{3/1}} \mathbf{BSpin}_{2\otimes}.$$

There is a commutative diagram of spectrum maps (see [9] and [10])

$$\begin{tikzcd} \mathbf{BSO}_2 \arrow[r, "\psi^{3}-1"] \arrow[d, "\simeq"', "\rho_3"] & \mathbf{BSpin}_2 \arrow[d, "\simeq"', "\tilde{\rho}_3"] \\ \mathbf{BSO}_{2\otimes} \arrow[r, "\psi^{3/1}"] & \mathbf{BSpin}_{2\otimes}. \end{tikzcd}$$

Hence there is an equivalence of spectra

$$\rho_3\colon \tilde{\mathbf{J}}_2 \xrightarrow{\simeq} \tilde{\mathbf{J}}_{2\otimes}. \tag{2.10}$$

The fibration

$$\mathbf{BSO}_{2\otimes} \xrightarrow{h} \mathbf{BO}_{2\otimes} \xrightarrow{w} \mathbf{K(Z/2Z,\ 1)} \tag{2.11}$$

admits a splitting $u\colon \mathbf{BO}_{2\otimes} \to \mathbf{BSO}_{2\otimes}$ such that $u \circ h = \mathrm{id}$ as spectrum maps (see [10, Lemma V 3.1]).

COROLLARY 2.12.

(i) $[\mathbf{J}_2, \mathbf{BO}_2] = \hat{Z}_2$ *with generator* $\pi$.

(ii) $[\mathbf{J}_{2\otimes}, \mathbf{BSO}_{2\otimes}] = \hat{Z}_2$ *with generator* $u \circ \pi_\otimes$.

(iii) $[\mathbf{J}_{2\otimes}, \mathbf{BO}_{2\otimes}] = [\mathbf{J}_{2\otimes}, \mathbf{BSO}_{2\otimes}] \oplus [\mathbf{J}_{2\otimes}, \mathbf{K(Z/2Z)}] = \hat{Z}_2 \oplus Z/2Z$ *with generators* $u \circ \pi_\otimes$ *and* $w \circ \pi_\otimes$.

*Proof.* (i) is immediate from the fibration sequence

$$\mathbf{K(Z/2Z,\ 0)} \to \tilde{\mathbf{J}}_2 \to \mathbf{J}_2 \to \mathbf{K(Z/2Z,\ 1)},$$

(2.2) and Theorem 2.8.

From (2.9), (2.10) and Theorem 2.8 we know $[\tilde{\mathbf{J}}_{2\otimes}, \mathbf{BSO}_{2\otimes}] = \hat{Z}_2$ on generator $\tilde{\pi}_\otimes$ and from the fibration

$$\tilde{\mathbf{J}}_{2\otimes} \xrightarrow{q} \mathbf{J}_{2\otimes} \longrightarrow \mathbf{K(Z/2Z,\ 1)}$$

we get an exact sequence

$$[\mathbf{K(Z/2Z,\ 1)}, \mathbf{BSO}_{2\otimes}] \longrightarrow [\mathbf{J}_{2\otimes}, \mathbf{BSO}_{2\otimes}] \xrightarrow{q^*} [\tilde{\mathbf{J}}_{2\otimes}, \mathbf{BSO}_{2\otimes}].$$

$q^*$ is a monomorphism since $[\mathbf{K(Z/2Z,\ 1)}, \mathbf{BSO}_{2\otimes}] = 0$ (see proof of Lemma 2.6). But $q^*$ is also a morphism of $\hat{Z}_2$-modules and since it sends $u \circ \pi_\otimes$ so $\tilde{\pi}_\otimes$ it is actually an isomorphism so (ii) follows.

(iii) follows from (2.11) and (ii).

Let $SF$ denote the 1-component of $\Omega^\infty S^\infty$. Then $SF$ is the zeroth space of a spectrum $\mathbf{SF}$ ([6] or [10]). Following [10] we have a spectrum map $\varepsilon_2 \colon \mathbf{SF}_2 \to \mathbf{J}_{2\otimes}$ the fiber of which is denoted $\mathbf{Coker\ J}_{2\otimes}$. There is a commutative diagram

$$\begin{tikzcd} \mathbf{Coker\ J}_{2\otimes} \arrow[r] & \mathbf{SF}_2 \arrow[r, "\varepsilon_2"] \arrow[d, "e"] & \mathbf{J}_{2\otimes} \arrow[dl, "\pi_\otimes"] \\ & \mathbf{BO}_{2\otimes}. & \end{tikzcd}$$

COROLLARY 2.13.

(i) $[\mathbf{SF}_2, \mathbf{BSO}_{2\otimes}] = \hat{Z}_2$ *with generator* $u \circ e$.
(ii) $[\mathbf{SF}_2, \mathbf{BO}_{2\otimes}] = \hat{Z}_2 \oplus Z/2Z$ *with generators* $u \circ e$ *and* $w \circ e$.

*Proof.* From [8] and [13] we know $[\mathbf{Coker\ J}_{2\otimes}, \mathbf{BSO}_{2\otimes}]$ and $[\mathbf{BCoker\ J}_{2\otimes}, \mathbf{BSO}_{2\otimes}]$ are trivial so it is only a rewriting of Corollary 2.12 (ii) and (iii).

COROLLARY 2.14. *The reduction* $[\mathbf{SF}_2, \mathbf{BO}_{2\otimes}] \to [SF_2, BO_{2\otimes}]$ *is a monomorphism; that is, a map* $SF_2 \to BO_{2\otimes}$ *can be infinitely delooped in at most one way.*

*Proof.* This follows from Corollary 2.13 since $[SF_2, BSO_{2\otimes}]$ is torsion-free (see [7]) and the reduction $[\mathbf{SF}_2, \mathbf{K}(\mathbf{Z}/2\mathbf{Z}, 1)] \to [SF_2, K(Z/2Z, 1)]$ is an isomorphism.

THEOREM 2.15. *The diagram* (1.1) *is a commutative diagram of infinite loop spaces and infinite loop maps.*

*Proof.* We only have to prove the result when we localize at 2 (see [10]). The square to the left must commute by Corollary 2.14 since all maps are infinite loop maps. Using standard Barratt-Puppe sequence arguments and the fact that $g$ is basically uniquely determined by the diagram it follows that $g$ is an infinite loop map and that the whole diagram is commutative in the category of infinite loop spaces and infinite loop maps. Compare [10, Theorem V 7.9].

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