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ON THE REGULARITY PROPERTIES OF THE VARIATION FUNCTION

BY

D. AHARONOV, A. ATZMON AND A. LEVIN

1. Preliminaries and statement of results

This paper is concerned with the relation between the regularity properties of a function and those of its variation function and with some applications of these relations to some problems about harmonic functions. Before we state our results we recall some definitions and known facts.

Let f be a bounded continuous function on an interval J. We denote by $\omega_1(f, h)$ the modulus of continuity of f and by $\omega_2(f, h)$ its modulus of smoothness. That is

$$\omega_1(f, h) = \sup_{\substack{x, x+t \in J \\ 0 \le t \le h}} |f(x+t) - f(x)|, \quad 0 < h \le 1$$

and

$$\omega_2(f, h) = \sup_{\substack{x-t, x, x+t \in J \\ 0 \le t \le h}} |f(x+t) + f(x-t) - 2f(x)|, \quad 0 < h \le 1.$$

As usual, we denote $\|f\|_\infty = \sup_{x \in J} |f(x)|$. The following relations exist between $\omega_1(f, h)$ and $\omega_2(f, h)$:

$$(1) \qquad \omega_1(f, h) \le c_f \sqrt{\omega_2(f, h)}$$

where c_f is a constant depending on f (see [5, p. 48]);

$$(2) \qquad \omega_2(f, h) \le 2\omega_1(f, h) \le 12h\left(\|f\|_\infty + \int_h^1 \frac{\omega_2(f, t)}{t^2}\,dt\right)$$

(see [5, p. 53]).

Given an increasing positive function ω on $(0, 1)$ such that $\lim_{h \to 0} \omega(h) = 0$, we denote by Λ_ω the set of all continuous functions f on $[0, 1]$ such that $\omega_1(f, h) = O(\omega(h))$, $h \to 0$, and by Λ_ω^* the set of all continuous functions f on $[0, 1]$ such that $\omega_2(f, h) = O(\omega(h))$, $h \to 0$.

It is well known (see [5] p. 50) that with respective norms

$$\|f\|_\omega = \|f\|_\infty + \sup_{0 < h \le 1} \frac{\omega_1(f, h)}{\omega(h)}, \qquad \|f\|_\omega^* = \|f\|_\infty + \sup_{0 < h \le 1} \frac{\omega_2(f, h)}{\omega(h)},$$

Λ_ω and Λ_ω^* are Banach spaces.

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The spaces Λ_{h^α}, $0 < \alpha \leq 1$, are of course the Lip α spaces on [0, 1]. The space Λ_h^* is known as the Zygmund class on [0, 1] and is usually denoted by Λ^*.

For a function f which is of bounded variation on the interval [0, 1], we denote for $0 \leq x \leq 1$ by $Vf(x)$ the total variation of f on the interval $[0, x]$ and call Vf the variation function of f.

Clearly $\omega_1(f, h) = O(h)$, $h \to 0$, implies that f is of bounded variation and $\omega_1(Vf, h) = O(h)$, $h \to 0$. On the other hand, given $0 < \alpha < 1$, one can construct a function f of bounded variation on [0, 1] such that $\omega_1(f, h) = O(h^\alpha)$, $h \to 0$ but $\omega_1(Vf, h) \neq O(h^\varepsilon)$, $h \to 0$ for every $\varepsilon > 0$. Such a construction appears in Shapiro [7, p. 272], and as remarked in the same paper, this fact was also noticed by J. P. Kahane.

Piranian (unpublished) proved a stronger result, namely, that there exists a function f of bounded variation on [0, 1] which satisfies

$$\omega_1(f, h) = O\left(h \ln \frac{1}{h}\right), \quad h \to 0,$$

but $\omega_1(Vf, h) \neq O(h^\varepsilon)$, $h \to 0$ for every $\varepsilon > 0$.

In fact, it is easy to prove the following more general result:

THEOREM 1. *Let ω be an increasing continuous function on* [0, 1] *which satisfies $\omega(0) = 0$ and*

$$\omega(h_1 + h_2) \leq \omega(h_1) + \omega(h_2), \quad \textit{for } 0 \leq h_1 + h_2 \leq 1, \tag{3}$$

$$\lim_{h \to 0} \frac{\omega(h)}{h} = \infty. \tag{4}$$

Then, given any positive function ρ on [0, 1] *such that $\lim_{h \to 0} \rho(h) = 0$, one can construct a continuous function f, of bounded variation on* [0, 1], *which satisfies $\omega_1(f, h) = O(\omega(h))$, $h \to 0$, but $\omega_1(Vf, h) \neq O(\rho(h))$, $h \to 0$.*

The results for $\omega_2(f, h)$ are more complicated. We have the following:

THEOREM 2. *Let ω be a positive increasing function on* [0, 1] *such that*

$$\alpha\omega(h) \leq \omega(2h) \leq \beta\omega(h), \quad 0 < h \leq 1/2, \quad 1 < \alpha < \beta < 4, \tag{5}$$

and assume that

$$\int_0^1 \frac{\omega^2(t)}{t^3} dt = \infty. \tag{6}$$

Then, given any positive function ρ on [0, 1] *such that $\lim_{h \to 0} \rho(h) = 0$, one can construct a continuous function f of bounded variation on* [0, 1], *such that $\omega_2(f, h) = O(\omega(h))$, $h \to 0$, but $\omega_1(Vf, h) \neq O(\rho(h))$, $h \to 0$.*

Remarks. (1) It follows from (1) that $\omega_1(Vf, h)$ can be replaced by $\omega_2(Vf, h)$ in the conclusion of Theorem 2.

(2) It is clear that Theorem 2 is equivalent to the assertion that if ω and ρ satisfy the hypothesis of the theorem, then one can construct a continuous function f of bounded variation on $[0, 1]$, such that $\omega_2(f, h) = O(\omega(h))$, $h \to 0$, but f does not admit any representation as a difference of two increasing functions f_1, f_2 which satisfy

$$\omega_1(f_j, h) = O(\rho(h)), \quad h \to 0, \quad j = 1, 2.$$

(3) Concerning the integral appearing in (6) we refer to [8].

In the positive direction we have the following result:

THEOREM 3. *Let f be a continuous function on* $[0, 1]$ *such that*

$$\int_0^1 \frac{(\omega_2(f, t))^2}{t^3} dt < \infty. \tag{7}$$

Then f is of bounded variation and

$$\omega_2(Vf, h) = O\left\{h \left(\int_0^h t^{-3}(\omega_2(f, t))^2 \, dt\right)^{1/2}\right\}, \quad h \to 0, \tag{8}$$

$$\omega_1(Vf, h) = O\left(\omega_2(Vf, h) + h\left(\int_h^1 t^{-2}\omega_2(f, t)\, dt + 1\right)\right), \quad h \to 0. \tag{9}$$

The following result is an application of Theorems 2 and 3.

COROLLARY 1. (a) *Given any positive function ρ on* $[0, 1]$ *such that* $\lim_{h \to 0} \rho(h) = 0$, *there exists a continuous function f of bounded variation on* $[0, 1]$ *such that*

$$\omega_2(f, h) = O\left(h \ln^{-1/2} \frac{1}{h}\right), \quad h \to 0,$$

but

$$\omega_1(Vf, h) \neq O(\rho(h)), \quad h \to 0.$$

(b) *Let* $1/2 < \lambda < 1$. *Then every continuous function f on* $[0, 1]$ *which satisfies*

$$\omega_2(f, h) = O\left(h \ln^{-\lambda} \frac{1}{h}\right), \quad h \to 0,$$

is of bounded variation on $[0, 1]$ *and*

$$\omega_2(Vf, h) = O\left(h \ln^{1/2-\lambda} \frac{1}{h}\right), \quad h \to 0,$$

$$\omega_1(Vf, h) = O\left(h \ln^{1-\lambda} \frac{1}{h}\right), \quad h \to 0.$$

(c) *Let ω be an increasing function on* [0, 1] *such that for some constant $l > 2$,*

$$l\omega(h) \leq \omega(2h), \quad 0 \leq h \leq 1/2. \tag{10}$$

Then every continuous function f on [0, 1] *which satisfies $\omega_2(f, h) = O(\omega(h))$, $h \to 0$, is of bounded variation and $\omega_2(Vf, h) = O(\omega(h))$, $h \to 0$.*

(Examples of functions which satisfy (10) are

$$\omega(h) = h^\alpha \left(1 + \ln^{-\lambda} \frac{1}{h}\right),$$

for $\alpha > 1$, $\lambda \geq 0$.)

Part (a) clearly follows from Theorem 2 and part (b) follows from Theorem 3. Part (c) follows from Theorem 3 and the fact that (10) implies that

$$\int_0^1 \frac{\omega^2(t)}{t^3} dt < \infty \quad \text{and} \quad h\left(\int_0^h t^{-3}\omega^2(t)\, dt\right)^{1/2} = O(\omega(h)), \quad h \to 0.$$

2. Application to harmonic functions

The above mentioned results are closely related to problems of representing harmonic functions, which satisfy certain conditions, as a difference of two positive harmonic functions which satisfy similar conditions. More precisely, assume that a function u is harmonic in the open unit disc

$$\{re^{i\theta}, 0 \leq \theta \leq 2\pi, 0 \leq r < 1\},$$

and belongs to the class h^1, that is, u admits a Stieltjes integral representation:

$$u(r, \theta) = \int_0^{2\pi} P(r, \theta - t)\, df(t)$$

where $P(r, \theta)$ is the Poisson kernel and f is a function of bounded variation on $[0, 2\pi]$. It is then possible to represent u as a difference of two positive harmonic functions u_1, u_2, and each such representation corresponds to a representation of f as a difference of two nondecreasing functions f_1, f_2. Clearly the representations are not unique.

Assume now, that in addition of being in h^1, u also satisfies some growth conditions. The question arises whether or not u can be represented as a difference of two positive harmonic functions, each of them satisfying some growth condition depending on that of u.

The connection between this problem and the problems concerning the regularity problems of the variation function discussed before, is furnished by the following generalized version of a theorem of Zygmund [9].

THEOREM 4. *Let ω be a decreasing function on* [0, 1] *which satisfies* (5). *Assume that u is an h^1 function,*

$$u(r, \theta) = \int_0^{2\pi} P(r, \theta - t)\, df(t),$$

where f is of bounded variation in $[0, 2\pi]$. Then u satisfies the condition

$$\frac{\partial u}{\partial \theta}(r, \theta) = O((1-r)^{-2}\omega(1-r)), \quad r \to 1,$$

uniformly in $\theta \in [0, 2\pi]$ if and only if $\omega_2(f, h) = O(\omega(h))$, $h \to 0$.

For $\omega(t) = t$, Theorem 4 is an equivalent form (see [2]) of a Theorem of Zygmund [9]. The general form of Theorem 4 is obtained by minor modifications of Zygmund's proof and the proof in [2, pp. 249–250], by using the estimates

$$\int_\delta^1 t^{-n}\omega(t)\,dt = O(\delta^{1-n}\omega(\delta)), \quad \delta \to 0, \qquad (n = 3, 4),$$

$$\int_0^1 d\delta \int_\delta^1 t^{-2}\omega(t)\,dt < \infty,$$

$$\int_0^\delta t^{-1}\omega(t)\,dt = O(\omega(\delta)), \quad \delta \to 0,$$

which follow easily from the properties of ω. We omit the details.

Using Theorem 4 and Remark 2 which follows the statement of Theorem 2, we see that Corollary 1 implies the following:

THEOREM 5. (a) *For every $\varepsilon > 0$ there exists a function u in h^1 which satisfies*

$$\frac{\partial u}{\partial \theta}(r, \theta) = O\left((1-r)^{-1} \ln^{-1/2} \frac{1}{1-r}\right), \quad r \to 1,$$

uniformly in $\theta \in [0, 2\pi]$, which admits no representation of the form $u = u_1 - u_2$, with u_1, u_2 positive harmonic functions such that

$$\frac{\partial u_j}{\partial \theta}(r, \theta) = O((1-r)^{-2+\varepsilon}), \quad r \to 1,$$

uniformly in $\theta \in [0, 2\pi]$, for $j = 1, 2$.

(b) *Let $1/2 < \lambda < 1$. Then every function u in h^1 which satisfies*

$$\frac{\partial u}{\partial \theta}(r, \theta) = O\left((1-r)^{-1} \ln^{-\lambda} \frac{1}{1-r}\right), \quad r \to 1,$$

uniformly in $\theta \in [0, 2\pi]$, can be represented as a difference of two positive harmonic functions u_1, u_2 which satisfy

$$\frac{\partial u_j}{\partial \theta}(r, \theta) = O\left((1-r)^{-1} \ln^{1/2-\lambda} \frac{1}{1-r}\right), \quad r \to 1,$$

uniformly in $\theta \in [0, 2\pi]$.

3. Proofs of Theorems 1, 2, 3

In order to prove Theorems 1 and 2 we shall need the following simple result.

LEMMA 1. *Let B be a Banach space of continuous functions on* $[0, 1]$ *with norm* $\|\cdot\|$. *Suppose that for every interval* $I \subset [0, 1]$ *there exists a function f in B which is of bounded variation on* $[0, 1]$ *such that* (i) $\operatorname{supp} f \subset I$, (ii) $\|f\| < 1$, (iii) $V_I f \geq 1$ (*where $V_I f$ denotes the variation of f on the interval I*).

Then given any positive function ρ on $[0, 1]$ *such that* $\lim_{h \to 0} \rho(h) = 0$ *there exists a function f_o in B which is of bounded variation on* $[0, 1]$ *and*

$$\omega_1(Vf_0, h) \neq O(\rho(h)), \quad h \to 0.$$

Proof. Let $I_n = [\alpha_n, \beta_n]$ be a sequence of mutually disjoint intervals in $[0, 1]$ such that $\rho(\beta_n - \alpha_n) < n^{-3}$, $n = 1, 2, \ldots$. By the hypothesis on B, there exists for every $n = 1, 2, \ldots$, a function f_n in B with support in I_n such that $\|f_n\| < n^{-2}$ and $V_{I_n}(f_n) = n^{-2}$. Clearly the function $f_o = \sum f_n$ is in B and is of bounded variation on $[0, 1]$. However,

$$\frac{Vf_o(\beta_n) - Vf_o(\alpha_n)}{\rho(\beta_n - \alpha_n)} = \frac{V_{I_n}(f_n)}{\rho(\beta_n - \alpha_n)} > n$$

so that $\omega_1(Vf_o, h) \neq O(\rho(h))$, $h \to 0$, and the lemma is proved.

Proof of Theorem 1. Theorem 1 will be proved by showing that for ω which satisfies the conditions of Theorem 1, the Banach space Λ_ω satisfies the hypothesis of Lemma 1.

Let $I = [a, b] \subset [0, 1]$ and set $\delta = b - a$. Since $\lim_{h \to 0} \omega(h)/h = \infty$ we can find a positive integer n such that

$$\delta\omega\left(\frac{\delta}{2n}\right) > \frac{\delta}{2n}. \tag{11}$$

Consider the function θ on $(-\infty, \infty)$ defined by $\theta(x) = \omega(x)$ for $0 \leq x < \delta/2n$, $\theta(x) = \omega(\delta/n - x)$ for $\delta/2n \leq x \leq \delta/n$ and $\theta(x) = 0$ elsewhere. Let f be the function on $[0, 1]$ defined by

$$f(x) = \sum_{j=0}^{n-1} \theta\left(x - a - \frac{j\delta}{n}\right), \quad 0 \leq x \leq 1.$$

It is clear that $\operatorname{supp} f \subset I$ and that $V_I f = 2n\omega(\delta/2n)$; hence by (11) we have $V_I f > 1$. It is also easily verified (see [5, p. 45]) that $f \in \Lambda_\omega$ and $\|f\|_\omega \leq 1$. Thus Λ_ω satisfies the hypothesis of Lemma 1.

Proof of Theorem 2. Again, we shall show that for ω which satisfies conditions (5) and (6) of Theorem 2, the Banach Space Λ_ω^* satisfies the hypothesis of Lemma 1.

Let $D_n = \sum_{k=1}^{n} \omega(2^{-k})$ and consider the sequence of trigonometric polynomials

$$p_n(x) = D_n^{-1} \sum_{k=1}^{n} \omega(2^{-k}) \cos 2^k x, \quad -\infty < x < \infty.$$

We claim that

$$\|p_n\|_\infty \le 1, \quad n = 1, 2, \ldots, \tag{12}$$

$$\omega_2(p_n, h) \le c\omega(h), \quad 0 \le h \le 1, \quad n = 1, 2, \ldots, \tag{13}$$

where c is a constant independent of n, and

$$\lim_{n\to\infty} \overset{\pi}{\underset{-\pi}{V}} (p_n) = \infty. \tag{14}$$

It is clear that (12) holds, and (13) is proved by a standard argument which uses (5) (see [1, p. 317]).

We turn to the proof of (14). Since ω is increasing, we have

$$\int_{2^{-k-1}}^{2^{-k}} \frac{(\omega(t))^2}{t^3} dt \le 2 \cdot 2^{2k}(\omega(2^{-k}))^2 \tag{15}$$

and therefore it follows from (4) that

$$\sum_{k=0}^{\infty} (\omega(2^{-k}))^2 2^{2k} = \infty. \tag{16}$$

Let $M_n = D_n^{-1}(\sum_{k=1}^{n} (\omega(2^{-k}))^2 2^{2k})^{1/2}$. It follows easily from (5), (16) that

$$\lim_{n\to\infty} M_n = \infty. \tag{17}$$

Applying a lemma of Sidon (see [4, p. 108]) and Parseval's identity to the trigonometric polynomials p_n' we obtain

$$\int_{-\pi}^{\pi} |p_n'(x)|\, dx \ge c\left(\int_{-\pi}^{\pi} |p_n'(x)|^2\, dx\right)^{1/2} = cM_n, \quad n = 1, 2, \ldots, \tag{18}$$

where c is an absolute positive constant.

Since

$$\overset{\pi}{\underset{-\pi}{V}} p_n = \int_{-\pi}^{\pi} |p_n'(x)|\, dx,$$

we obtain (14) from (17) and (18).

We construct now a sequence of continuous functions g_n, $n = 1, 2, \ldots$, with compact supports which have the same properties as the trigonometric polynomials p_n, $n = 1, 2, \ldots$.

Let g be a C^2 function on $(-\infty, \infty)$ such that $g(x) = 1$ for $|x| \le \pi$ and $g(x) = 0$ for $|x| \ge 4$. Consider the functions $g_n = g \cdot p_n$, $n = 1, 2, \ldots$. We claim that

$$\sup_n \|g_n\|_\infty < \infty, \tag{19}$$

$$\omega_2(g_n, h) \le K_1\omega(h), \quad 0 < h \le 1, \tag{20}$$

$n = 1, 2, \ldots$, where K_1 is a positive constant independent of n, and

$$\lim_{n\to\infty} \overset{4}{\underset{-4}{V}} g_n = \infty. \tag{21}$$

It is clear that (19) follows from (12) and (21) follows from (14). Since g is a C^2 function of compact support, $\omega_1(g, h) = O(h)$, $h \to 0$, and $\omega_2(g, h) = O(h^2)$, $h \to 0$. Therefore using the estimate

$$\omega_2(gp_n, h) \le \|g\|_\infty \omega_2(p_n, h) + \|p_n\|_\infty \omega_2(g, h) + \omega_1(g, 2h)\omega_1(p_n, h), \quad 0 \le h \le 1$$

(which follows from the definitions of ω_1 and ω_2) we obtain by virtue of (12) and the properties of g that $\omega_2(gp_n, h) \le K_2(\omega(h) + h^2 + h\omega_1(p_n, h))$, $n = 1, 2, \ldots$, where K_2 is a constant independent of n. Therefore using (2), (12), and (13) we have

$$\omega_2(gp_n, h) \le K_3 \left[\omega(h) + h^2\left(\int_h^1 t^{-2}\omega(t)\, dt\right)\right], \quad n = 1, 2, \ldots,$$

where K_3 is another constant independent of n. But (5) easily implies that

$$h^2\left(1 + \int_h^1 t^{-2}\omega(t)\, dt\right) = O(\omega(h)), \quad h \to 0;$$

hence (20) is proved.

To complete the proof of the theorem, consider an interval $I = [a, b] \subset [0, 1]$ and set $c = (a + b)/2$, $\delta = (b - a)/8$. For every $n = 1, 2, \ldots$, let f_n be the function on $[0, 1]$ defined by

$$f_n(x) = g_n\left(\frac{x - c}{\delta}\right), \quad 0 \le x \le 1.$$

Since $\operatorname{supp} g_n \subset [-4, 4]$ it is clear that $\operatorname{supp} f_n \subset I$, and (19), (20), (5) imply that $\sup_n \|f_n\|_\omega^* < \infty$. Since $V_I f_n = V_{-4}^4 g_n$, it follows from (21) that $\lim_{n\to\infty} V_I f_n = \infty$. Consequently, Λ_ω^* satisfies the hypothesis of Lemma 1, and the proof of Theorem 2 is complete.

We turn now to the proof of Theorem 3. We shall need the following lemmas.

LEMMA 2. *Let f be a periodic continuous function on $(-\infty, \infty)$ with period T and assume that $\int_0^1 t^{-3}(\omega_2(f, t))^2\, dt < \infty$. Then f is absolutely continuous and*

$$\int_{-T}^{T} (f'(x))^2\, dx \le c \int_0^1 t^{-3}(\omega_2(f, t))^2\, dt$$

where c is a constant which depends only on T.

This lemma was proved by Stein and Zygmund [8] in a somewhat different context. For a simple proof due to L. Carleson the reader is referred to [7].

LEMMA 3. *Let g be a continuous function on $[-1, 1]$ and assume that*

$$\int_0^1 \frac{(\omega_2(g, t))^2}{t^3} dt < \infty \tag{22}$$

Then

$$\overset{1}{\underset{-1}{V}} g \le c\left[\left(\int_0^1 t^{-3}(\omega_2(g, t))^2\, dt\right)^{1/2} + |g(1) - g(-1)|\right] \tag{23}$$

and

$$\left|\overset{1}{\underset{0}{V}} g - \overset{0}{\underset{-1}{V}} g\right| \le c\left(\int_0^1 t^{-3}(\omega_2(g, t))^2\, dt\right)^{1/2} \tag{24}$$

where c is an absolute constant.

Proof. Consider the linear function

$$L(x) = g(-1) + \frac{x+1}{L}(g(1) - g(-1)), \quad \infty < x < \infty$$

and let G be the periodic function on $(-\infty, \infty)$ with period 4 such that $G(x) = f(x) - L(x)$ for $-1 \le x \le 1$ and $G(x) = G(2 - x)$ for $1 \le x \le 3$. It is easy to check that $\omega_2(G, h) \le 3\omega_2(g, h)$, $0 < h \le 1$ (see [5, p. 51]) and therefore using Schwartz's inequality we obtain from Lemma 2 and (22) that

$$\overset{1}{\underset{-1}{V}} G \le \overset{2}{\underset{-2}{V}} G = \int_{-2}^2 |G'(t)|\, dt \le c\left(\int_0^1 t^{-3}(\omega_2(g, t))^2\, dt\right)^{1/2} \tag{25}$$

Hence from the definition of G it follows that

$$\overset{1}{\underset{-1}{V}} g \le \overset{1}{\underset{-1}{V}} G + \overset{1}{\underset{-1}{V}} L \le c\left[\left(\int_0^1 t^{-3}(\omega_2(g, t))^2\, dt\right)^{1/2} + |g(1) - g(-1)|\right].$$

This proves (23).

The estimate (24) follows from (25) by adding the obvious inequalities

$$\pm\left(\overset{1}{\underset{0}{V}} G - \overset{1}{\underset{0}{V}} L\right) \le \overset{1}{\underset{0}{V}} g \le \overset{1}{\underset{0}{V}} G + \overset{1}{\underset{0}{V}} L$$

$$-\overset{0}{\underset{-1}{V}} G - \overset{0}{\underset{-1}{V}} L \le -\overset{0}{\underset{-1}{V}} g \le \pm\left(\overset{0}{\underset{-1}{V}} G - \overset{0}{\underset{-1}{V}} L\right)$$

and using the fact that $V_0^1 L = V_{-1}^0 L$.

We are now in position to complete the proof of Theorem 3.

Let f be a function which satisfies the hypothesis of the theorem and assume that $0 \le x - h \le x \le x + h \le 1$.

We have to estimate

$$\Delta_h^2 Vf(x) = Vf(x+h) + Vf(x-h) - 2Vf(x) = \overset{x+h}{\underset{0}{V}} f - \overset{x}{\underset{x-h}{V}} f.$$

Consider the function

$$g(t) = f(x + ht), \quad -1 \le t \le 1.$$

It is clear that

$$\omega_2(g, \delta) \le \omega_2(f, h\delta), 0 \le \delta \le 1 \quad \text{and} \quad \Delta_h^2 Vf(x) = \overset{1}{\underset{0}{V}} g - \overset{0}{\underset{-1}{V}} g.$$

Therefore using (7) and the estimate (24) of Lemma 3 we obtain

$$|\Delta_h^2 Vf(x)| \le c \left(\int_0^1 \delta^{-3} (\omega_2(f, \delta h))^2 \, d\delta \right)^{1/2} = ch \left(\int_0^h \delta^{-3} (\omega_2(f, \delta))^2 \, d\delta \right)^{1/2}.$$

This proves (8).

To prove (9), one proceeds similarly using (2) and (23). We omit the details.

We conclude with some remarks.

(1) In the estimates (8) and (9) of Theorem 3, one cannot in general replace "O" by "o". This can be easily seen by considering the function $f(x) = x^\alpha$ for some $\alpha > 1$.

(2) In the investigation of the regularity properties of the variation function one may consider moduli of smoothness of order higher than 2, that is $\omega_k(f, h)$ for $k > 2$ (see [5, p. 47] for the definition). However, there are no essential differences between the results for ω_2 and those for ω_k, $k > 2$. By replacing ω_2 by ω_k, for some $k > 2$, in the hypotheses of Theorems 2 and 3, it follows that the conclusion of Theorem 2 remains unchanged, and the estimates (8) and (9) of Theorem 3 are replaced by similar estimates. The proofs are the same as for ω_2.

(3) Analogous problems to those treated in this paper may be asked for the Lebesgue decomposition of a function of bounded variation f into its absolutely continuous part f_a and its singular part f_s; that is, one may study the relation between the regularity properties of f and those of f_a and f_s. These problems are investigated in [3], [6], and [7].

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HAIFA, ISRAEL