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A BANACH SPACE NOT CONTAINING l_1 WHOSE DUAL BALL IS NOT WEAK* SEQUENTIALLY COMPACT

BY

J. HAGLER AND E. ODELL[1]

The structure of Banach spaces with nonweak* sequentially compact dual balls was studied in [7], where it was proved that if X is separable and the unit ball of X^{**} is not weak* sequentially compact, then X^* contains a subspace isomorphic to $l_1(\Gamma)$ for some uncountable set Γ. Subsequently it was proved in [1] that if the unit ball of X^* is not weak* sequentially compact, then (a) either c_0 is a quotient of X or l_1 is isomorphic to a subspace of X, and (b) X has a separable subspace with nonseparable dual. In this note we give an example of a Banach space X whose dual ball is not weak* sequentially compact, but where X contains no subspace isomorphic to l_1. This answers a question posed by H. P. Rosenthal [7].

The example we construct draws on two ideas. First R. Haydon [2] exhibited a compact Hausdorff space K which is not sequentially compact such that $C(K)$ does not contain a subspace isomorphic to $l_1(\Gamma)$ for any uncountable set Γ. Central to this construction (and to ours) is the existence of a "thin" family of subsets of the integers which infinitely separates every infinite subset of the integers (see Lemma 1 below). Secondly, the space X we exhibit must be nonseparable. A key part of our construction is a nonseparable analogue of JT, the James tree (cf. [3] or [4]). This space has the property that JT^* is not separable, yet JT contains no isomorph of l_1. We recall the definition of JT below during the proof of Lemma 2.

Notation. If X is a Banach space and $(g_\alpha)_{\alpha \in I} \subseteq X$, then by $\langle g_\alpha \rangle_{\alpha \in I}$ we mean the linear span of the set $(g_\alpha)_{\alpha \in I}$, while $[g_\alpha]_{\alpha \in I}$ denotes the closure of $\langle g_\alpha \rangle_{\alpha \in I}$. Also if L and M are subsets of N, the set of natural numbers, then $|L|$ denotes the cardinality of L. $L \subset^a M$ means $|L \backslash M| < \infty$ and $L \cap M =^a \emptyset$ means $|L \cap M| < \infty$.

Other Banach space notation we use is standard and may be found in [5].

The definition of X.

LEMMA 1. *There is a well ordered set I, $<$ and a collection of infinite subsets of N, $(M_\alpha)_{\alpha \in I}$, such that:*

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(1) *If* $\alpha < \beta$ *then either* $M_\beta \subset^a M_\alpha$ *or* $M_\beta \cap M_\alpha =^a \emptyset$.

(2) *If* $M \subset N$, $|M| = \infty$ *then there is an* $\alpha \in I$ *such that* $|M \cap M_\alpha| = |M \backslash M_\alpha| = \infty$.

Proof. Let $(S_\alpha)_{\alpha \in J}$ be the collection of all infinite subsets of N and let $<$ be a well ordering of J. For each $\alpha \in J$ let M_α be an infinite subset of S_α with $|S_\alpha \backslash M_\alpha| = \infty$. We shall inductively choose $I \subset J$ so that (1) and (2) are satisfied. If α_1 is the first element of J, put α_1 in I. Let $\beta \in J$ and assume I has been defined for all $\alpha < \beta$. If there is an $\alpha < \beta$ such that

$$|S_\beta \cap M_\alpha| = |S_\beta \backslash M_\alpha| = \infty, \tag{3}$$

we "discard" β. If no $\alpha < \beta$ satisfies (3), we put β in I.

Clearly $(M_\alpha)_{\alpha \in I}$ satisfies (1) by construction. If $M \subset N$, $|M| = \infty$, then $M = S_\beta$ for some $\beta \in J$. If $\beta \in I$, then $|M \cap M_\beta| = |M \backslash M_\beta| = \infty$. If $\beta \notin I$ then there is an $\alpha < \beta$ so that (3) holds and thus (2) is proved, Q.E.D.

We wish to thank M. Wage for showing us the proof of Lemma 1 and for allowing us to reproduce here his argument.

Now define a *new* partial ordering $\leq$ on I as follows: $\alpha \leq \beta$ if $\alpha < \beta$ and $M_\beta \subset^a M_\alpha$. We note that $(I, \leq)$ is a tree (i.e., if $\beta \in I$, $\{\alpha \in I: \alpha \leq \beta\}$ is a well ordered set). Also every nonempty subset of $(I, \leq)$ has at least one minimal element.

Remarks. (1) The requirement that $\alpha < \beta$ in the definition of $\alpha \leq \beta$ is actually redundant. Indeed if $\alpha, \beta \in I$, $M_\beta \subset^a M_\alpha$ and $\alpha > \beta$ then $M_\alpha \subset^a M_\beta$ and so $M_\alpha =^a M_\beta$. But then $|S_\alpha \cap M_\beta| = |S_\alpha \backslash M_\beta| = \infty$ and this contradicts our definition of I in the proof of Lemma 1.

(2) In [2], R. Haydon used Zorn's lemma to construct infinite subsets of the integers $(M_\alpha)_{\alpha \in I}$ satisfying (2) and such that if $\alpha \neq \beta$ then either $M_\beta \subset^a M_\alpha$, $M_\alpha \subset^a M_\beta$ or $M_\alpha \cap M_\beta =^a \emptyset$. The importance of Lemma 1 to us is the particular partial order it allows us to define.

By a *segment* B in I we shall mean a subset of I of the form

$$B = [\alpha, \beta] = \{\gamma \in I: \alpha \leq \gamma \leq \beta\},$$

where $\alpha, \beta \in I$. Let $(g_\alpha)_{\alpha \in I}$ be a linearly independent set of vectors in some vector space. If $(t_\alpha)_{\alpha \in I}$ is a finitely nonzero set of scalars, we define

$$(*) \quad \left\| \sum_{\alpha \in I} t_\alpha g_\alpha \right\| = \sup \left\{ \left[\sum_{i=1}^{k} \left(\sum_{\alpha \in B_i} t_\alpha \right)^2 \right]^{1/2} : B_1, \ldots, B_k \text{ are pairwise disjoint segments} \right\}.$$

Let Y be the completion of $\langle g_\alpha \rangle_{\alpha \in I}$ under this norm (Y is the nonseparable analogue of JT referred to above).

For each $\alpha \in I$, let 1_{M_α} be the indicator function of M_α in l_∞ and let

$$h_\alpha = (1_{M_\alpha}, g_\alpha) \in (l_\infty \oplus Y)_\infty.$$

Thus, for a finitely nonzero set of scalars $(t_\alpha)_{\alpha \in I}$,

$$\|\sum t_\alpha h_\alpha\| = \max\{\|\sum t_\alpha 1_{M_\alpha}\|_\infty, \|\sum t_\alpha g_\alpha\|\}.$$

Let X be the closed subspace of $(l_\infty \oplus Y)_\infty$ generated by $(h_\alpha)_{\alpha \in I}$ and let B_X^* denote the unit ball of X^*.

B_X^* is not weak sequentially compact.*

For $n \in N$, let $F_n(h_\alpha) = 1_{M_\alpha}(n)$ and extend F_n linearly to $\langle h_\alpha \rangle_{\alpha \in I}$. Then if $\sum t_\alpha h_\alpha \in \langle h_\alpha \rangle_{\alpha \in I}$,

$$|F_n(\sum t_\alpha h_\alpha)| = |\sum t_\alpha 1_{M_\alpha}(n)| \le \|\sum t_\alpha 1_{M_\alpha}\|_\infty \le \|\sum t_\alpha h_\alpha\|.$$

Thus F_n has a unique extension to a norm one element of X^* which we also denote by F_n. We claim that if M is an infinite subset of N, then $(F_n)_{n \in M}$ does not converge. Indeed by (2) there is an $\alpha \in I$ such that $|M \cap M_\alpha| = |M \backslash M_\alpha| = \infty$. Thus, $(F_n(h_\alpha))_{n \in M}$ does not converge.

X contains no subspace isomorphic to l_1.

LEMMA 2. *Every infinite dimensional subspace of Y contains an isomorph of l_2.*

Let us assume for the moment that Lemma 2 has been proved and that X contains an isomorph of l_1. Then there exists

$$e^n = (f^n, g^n) \in \langle h_\alpha \rangle_{\alpha \in I}$$

such that (e^n) is equivalent to the unit vector basis of l_1. By passing to a block basis of (e^n) if necessary we may assume that $\|g^n\| \to 0$ and (f^n) is equivalent to the unit vector basis of l_1. An easy application of Rosenthal's characterization of Banach spaces containing l_1 [6] yields a subsequence of (f^n) (which we continue to call (f^n)) and real numbers r and δ with $\delta > 0$ such that if

$$A_n = \{m \in N : f^n(m) > r + \delta\} \quad \text{and} \quad B_n = \{m \in N : f^n(m) < r\}$$

then $(A_n, B_n)_{n=1}^\infty$ is independent. This means that if F and G are disjoint finite subsets of N, then

$$\bigcap_{n \in F} A_n \cap \bigcap_{n \in G} B_n \neq \emptyset.$$

In particular $|A_n| = |B_n| = \infty$ for all n. We can also suppose that $r + \delta > 0$ (if not, multiply each e^n by -1) and fix n large enough so that $\|g^n\| < r + \delta$. We show that $|A_n| < \infty$, which is a contradiction.

Let

$$e^n = \sum_{\alpha \in D} t_\alpha h_\alpha = \left(\sum_{\alpha \in D} t_\alpha 1_{M_\alpha}, \sum_{\alpha \in D} t_\alpha g_\alpha \right),$$

where D is a finite subset of I. Choose a finite subset G of N and $\bar{M}_\alpha \subset M_\alpha$ for $\alpha \in D$ such that if $\alpha, \alpha' \in D$ are distinct, then

(i) $M_\alpha \cap M_{\alpha'} =^a \emptyset$ implies $\bar{M}_\alpha \cap \bar{M}_{\alpha'} = \emptyset$,
(ii) $M_\alpha \subset^a M_{\alpha'}$ implies $\bar{M}_\alpha \subset \bar{M}_{\alpha'}$,
(iii) $\bar{M}_\alpha \cap G = \emptyset$ and $M_\alpha \subset \bar{M}_\alpha \cup G$.

Let $m \in \bar{M}_\alpha$ for some $\alpha \in D$. Then there exists a unique sequence

$$\alpha_1 < \alpha_2 < \cdots < \alpha_k \quad \text{in } D$$

such that $m \in \bar{M}_{\alpha_i}$ for $1 \leq i \leq k$ and $m \notin \bar{M}_\alpha$ for $\alpha \in D \backslash \{\alpha_1, \ldots, \alpha_k\}$. Consider the segment $B = [\alpha_1, \alpha_k]$ in I. Then $f^n(m) = \sum_{i=1}^k t_{\alpha_i} = \sum_{\alpha \in B} t_\alpha$. Since $\|g^n\| < r + \delta$, $f^n(m) < r + \delta$ and so $m \notin A_n$. Thus $A_n \subset G$ is finite and we conclude that X does not contain l_1.

To prove the lemma, we present a simplified version of our original argument, as shown to us by Y. Benyamini.

Proof of Lemma 2. We show that every infinite dimensional subspace of Y contains an isomorph of an infinite dimensional subspace of JT and thus by [3] an isomorph of l_2. First let us recall the definition of JT. Let $T = \bigcup_{n=0}^\infty \{0, 1\}^n$ be a dyadic tree (i.e., if $\emptyset, \chi \in T$ with lengths n and m respectively then $\emptyset \leq \chi$ if $n \leq m$ and the first n terms of χ form $\emptyset$). If x is a finitely nonzero scalar-valued function on T let

$$\|x\| = \max \left[\sum_{i=1}^k \left(\sum_{\emptyset \in B_i} x(\emptyset)^2 \right) \right]^{1/2}$$

where the max is taken over all k and pairwise disjoint segments $B_1, \ldots, B_k$ in T. JT is the completion of the linear span of all such x with this norm.

Now let Z be an infinite dimensional subspace of Y. We can assume that Z is separable and that there is a countable set $I_0 \subset I$ such that $Z \subseteq [g_\alpha]_{\alpha \in I_0}$.

Let $I_1 = \{\alpha \in I_0 \colon \alpha \text{ is a minimal element of } I_0\}$ and for a countable ordinal β, set

$$I_\beta = \left\{ \alpha \in I_0 \colon \alpha \text{ is a minimal element of } I_0 \backslash \bigcup_{\gamma < \beta} I_\gamma \right\}.$$

Since I_0 is countable, there is a countable ordinal α_0 such that $I_0 = \bigcup_{\beta < \alpha_0} I_\beta$.

Now, let $\beta \leq \alpha_0$ be the smallest ordinal such that the restriction map to $\bigcup_{\alpha \leq \beta} I_\gamma$ is an isomorphism on an infinite dimensional subspace of Z. (This map is defined as follows: For a finitely nonzero set of scalars $(t_\alpha)_{\alpha \in I_0}$, define

$$R\left(\sum_{\alpha \in I_0} t_\alpha g_\alpha\right) = \sum_{\alpha \in \cup_{\gamma \leq \beta} I_\gamma} t_\alpha g_\alpha.$$

It is clear that $\|R\| \leq 1$.)

First, if the restriction map to I_β is an isomorphism on an infinite dimensional subspace of Z, then clearly l_2 imbeds in Z. (For instance, this case must occur if β is a successor.)

If not, then a standard gliding hump argument shows the existence of a subsequence $n_1 < n_2 < \cdots$ of N, and normalized basic sequences $(z_j) \subset Z$ and $(y_j) \subset Y$ such that

$$y_j \in \langle g_\alpha \rangle: \alpha \in \bigcup \{I_\delta: n_j \leq \delta < n_{j+1}\}$$

and $\|z_j - y_j\| < 2^{-j}$ for each $j \in N$. (Thus, by a standard perturbation argument, (z_j) is equivalent to (y_j).)

We claim that $[(y_j)]$ is isometric to a subspace of JT. Indeed let $y_j \in \langle g_\alpha \rangle_{\alpha \in D_j}$ where D_j is a finite subset of

$$\bigcup \{I_\delta: n_j \leq \delta < n_{j+1}\}$$

and choose an order preserving injection Q: $\bigcup_{j=1}^\infty D_j \to T$. Let $x_j \in JT$ be defined by $x_j(\emptyset) = y_j(Q^{-1}(\emptyset))$ for $\emptyset \in T$. Then (x_j) is isometrically equivalent to (y_j), Q.E.D.

Remark. If $(S, \leq)$ is any tree and $(g_\alpha)_{\alpha \in S}$ are linearly independent vectors in a vector space we may define a norm on $\langle g_\alpha \rangle_{\alpha \in S}$ by means of $(*)$ above. Lemma 2 remains valid for the resulting Banach space if every nonempty subset of S has a minimal element. In general it is false, however. For example if S is the set of rationals with the usual order then corresponding Banach space can be seen to contain c_0.

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Catholic University
Washington, D.C.
Yale University
New Haven, Connecticut