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THE K-THEORY OF SOME MORE WELL-KNOWN SPACES

BY

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Introduction

In this paper we determine the real and complex K-theory of SG, the H-space of degree one maps in QS^0 under composition, and of the spaces known as the image and the cokernel of the J-homomorphism. The interest in these spaces stems from their role in (i) the general theory of infinite loopspaces, (ii) the theory of spherical fibrations.

The computations are accomplished by comparing the K-theory of SG and the image-of-J spaces with the K-theory of $Q_0 S^0$. Our main result (Theorem 2.5) states that the K-theory "Hopf algebras" are essentially isomorphic, the isomorphisms being induced by suitable d-invariants. There exist splittings (see Section 1) of the form $SG_p = \text{Cok } J_p \times J_p^{\otimes}$ which yield, as a corollary of Theorem 2.5, that Cok J_p is a K- and KO-theory point (Theorem 2.7). This result has turned out to be very useful [12], [17], [21]. In order to establish the isomorphisms it suffices to work with mod p K-homology. However the integral KO- and K-cohomology is more likely to interest homotopy theorists. This is determined in Theorem 2.10.

The paper is arranged as follows. In Section 1 we introduce all the infinite loopspaces which we will study and in Section 2 we state and establish the results. The latter proceeds according to the following program. We have to establish several $K_*(\ ;Z/p^r)$-isomorphisms. Using results of [9] on $K_*(QS^0; Z/p^r)$ we show that the d-invariant embeds this Hopf algebra into $K_*(BU^{\otimes} \times Z; Z/p^r)$. From this we derive all our isomorphisms by showing that the projection of $SG_p = \text{Cok } J_p \times J_p^{\otimes}$ onto $J_p^{\otimes}$ is injective in $K_*(\ ;Z/p^r)$ and hence is a K-theory isomorphism.

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1. The spaces

Let us briefly recall all the spaces with which we will be concerned. They are all H-spaces, in fact infinite loopspaces, and a general reference for further details is [13]. Let X_p denote the p-localization of X [19].

$BU^{\oplus} \times Z$ is the H-space representing $KU^0(\ )$. It is an E_∞-ring space in the

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manner described in [13, Chapter VIII, Section 1]. The $\oplus$ stands for the H-space structure induced from Whitney sum of bundles.

QS^0 is the limit of the suspension sequence

$$\cdots \to \Omega^n S^n \to \Omega^{n+1} S^{n+1} \to \cdots$$

and $Q_m S^0$ is the component consisting of maps of degree m. $Q_1 S^0$ is usually called SG. QS^0 is an E_∞-ring space in the manner described in [13, Chapter IV, Example 1.10]. There is, up to homotopy, exactly one map of E_∞-ring spaces $D\colon QS^0 \to BU^\oplus \times Z$ (Z is the integers) such that $D \mid S^0$ sends the basepoint into $BU^\oplus \times (0)$ and the other point into $BU^\oplus \times (1)$. This map is the d-invariant from stable cohomotopy to K-theory [1].

$BU^\otimes$ has the same underlying space as $BU^\oplus$ together with the H-space structure induced by tensor product in the multiplicative group $1 + \tilde{K}U^0(\ \ )$ (see [13, Chapter VI, Example 5.4; 14]). The d-invariant restricts to give an H-map $D\colon SG \to BU^\otimes$.

We may, of course, localize any of the above maps and spaces.

If p is an odd prime let q be a prime power generating the units of Z/p^2 and set $\psi = \psi^q$, the Adams operation. Define $J_p^\oplus \times Z$ by the fibring of infinite loopspaces

$$J_p^\oplus \times Z \longrightarrow BU_p^\oplus \times Z \xrightarrow{\psi - 1} BU_p^\oplus$$

where $(\psi - 1)(x) = \psi(x) - x$ for $x \in KU^0(\ \ )$.

Similarly we may define $J_p^\otimes$ by a means of the fibring

$$J_p^\otimes \longrightarrow BU_p^\otimes \xrightarrow{\psi/1} BU_p^\otimes$$

where $(\psi/1)(y) = \psi(y)/y$ for $y \in 1 + \tilde{K}U^0(\ \ )$. For details see [13, Chapters V and VIII].

The J-theory D-invariant $D\colon QS_p^0 \to J_p^\oplus \times Z$ restricts to give an H-map $D\colon SG \to J_p^\oplus \times (1) = J_p^\otimes$. This in fact gives a split fibring of infinite loopspaces

$$\mathrm{Cok}\, J_p \longrightarrow SG_p \xrightarrow{D} J_p^\otimes$$

which defines $\mathrm{Cok}\, J_p$ and yields a splitting of infinite loopspaces

$$\mathrm{Cok}\, J_p \times J_p^\otimes = SG_p$$

when $p \neq 2$ [16], [20].

Now let us consider the case $p = 2$. Defined in the analogous manner we have H-spaces $BO^\oplus \times Z$ and $BO^\otimes$ and their coverings $BSO^\oplus$, $BSpin^\oplus$, $BSO^\otimes$ and $BSpin^\otimes$. Also there is a d-invariant $D\colon QS^0 \to BO^\oplus \times Z$ which restricts to an H-map $D\colon SG \to BO^\otimes$.

Set $\psi = \psi^3$ and define $(J_2')^\oplus \times Z$ by means of the fibring of infinite loopspaces

$$(J_2')^\oplus \times Z \longrightarrow BO_2^\oplus \times Z \xrightarrow{\psi - 1} BSpin_2^\oplus.$$

Define other J-spaces by means of the following fibrings of infinite loopspaces.

$$(J_2'')^\oplus \longrightarrow BSO_2^\oplus \xrightarrow{\psi - 1} BSpin_2^\oplus, \qquad J_2^\oplus \longrightarrow BSO_2^\oplus \xrightarrow{\psi - 1} BSO_2^\oplus.$$

For the $\otimes$-structures we have

$$J_2^{\otimes} \longrightarrow BSO_2^{\otimes} \xrightarrow{\psi/1} BSO_2^{\otimes},$$

and similar fibrings giving $(J_2')^{\otimes}$ and $(J_2'')^{\otimes}$.

The d-invariant

$$D\colon QS_2^0 \to (J_2')^{\oplus} \times Z$$

restricts to an H-map $D\colon SG_2 \to (J_2')^{\otimes}$.

Finally Sullivan's solution of the Adams conjecture [19] yields a commutative diagram in which the columns are split fibrings

$$\begin{tikzcd}
\mathrm{Cok}\, J_2 \arrow[r, phantom, "="] \arrow[d] & \mathrm{Cok}\, J_2 \arrow[d] \\
SG_2 \arrow[r] \arrow[d, "e'"'] & (G/O)_2 \arrow[d, "e"] \\
J_2^{\otimes} \arrow[r] & BSO_2^{\otimes}
\end{tikzcd}$$

In [12] this is shown to be a commutative diagram of infinite loopmaps. We will need the splittings

$$\mathrm{Cok}\, J_2 \times J_2^{\otimes} = SG_2 \quad \text{and} \quad \mathrm{Cok}\, J_2 \times BSO_2^{\otimes} = (G/0)_2.$$

2. The results

Let $K_*(\ ; Z/p^r)$ denote unitary K-homology with coefficients in Z/p^r. In this section (Theorem 2.5) we show that several of the infinite loop maps introduced in Section 1 induce isomorphisms in K-theory, real or complex. We also show (Theorem 2.7) that B^i Cok J_p is a K-theory point. This last fact has turned out to be quite useful [12], [17], [21].

The next result is a first approximation to what we want.

2.1. PROPOSITION. *Let D be the d-invariant of Section 1. Then for any prime, p, and integer $r \geq 1$,*

$$D_*\colon K_*(QS^0; Z/p^r) \to K_*(BU^{\oplus} \times Z; Z/p^r)$$

is injective.

Proof. It is well known (for example [3]) that $K_*(BU^{\oplus})$ is the polynomial algebra on $K_*(BU(1))$, with any coefficients, and that

$$K_*(BU^{\oplus} \times Z) \cong K_*(BU^{\oplus})[u_0, u_0^{-1}].$$

In [9] (see also [18]) it is shown that

$$K_*(QS^0; Z/p^r) \cong Z/p^r[\theta_p, \theta_{p^2}, \ldots, \theta_{p^k}, \ldots, \theta_1, \theta_1^{-1}]$$

where $D_*(\theta_1) = u_0$. Now θ_{p^k} originates in Hom $(R(\Sigma_{p^k}), Z)$, as explained in [9], where $R(G)$ is the complex representation ring of G. If σ_{p^k} is an evaluation on a p^k-cycle we have

$$\sigma_{p^k} = \theta_1^{p^k} + p(\theta_p)^{p^{k-1}} + p^2(\theta_{p^2})^{p^{k-2}} + \cdots + p^k\theta_{p^k}. \tag{2.2}$$

Let $\psi^{p^t} \in K^0(BU^{\oplus} \times Z)$ be the Adams operation. By means of (2.2) we will show that

$$\langle D_*(\theta_{p^k}), \psi^{p^t}\rangle = \begin{cases} 0 & \text{if } t \geq k \\ -1 & \text{otherwise.} \end{cases} \tag{2.3}$$

Since ψ^{p^t} is primitive this shows that the $(D_*(\theta_{p^i}); i \geq 1)$ are linearly independent modulo decomposables. The injectivity of D_* follows at once. Here we have used the fact that

$$\langle xu_0^j, \psi^{p^t}\rangle = \langle x, \psi^{p^t}\rangle \quad \text{if } x \in \tilde{K}_*(BU^{\oplus}; Z/p^r) \text{ and } j \in Z.$$

The homomorphism

$$\text{Hom } (R(\Sigma_{p^k}), Z) \longrightarrow K_*(QS^0; Z/p^r) \xrightarrow{D_*} K_*(BU^{\oplus} \times Z; Z/p^r).$$

factors through the homomorphism induced by the inclusion of Σ_{p^k},

$$\text{Hom } (R(\Sigma_{p^k}), Z) \to \text{Hom } (R(U(p^k)), Z) \to K_*(BU(p^k) \times (p^k); Z/p^r).$$

Hence $\langle D_* \sigma_{p^k}, \psi^{p^t}\rangle$ is evaluation of ψ^{p^t} on a p^k-cycle. The inclusion $Z/p^k \subset U(p^k)$ is given by the regular representation $\sum_{j=0}^{p^k-1} y^j$ where y is the canonical one-dimensional representation. Hence the Kronecker pairing with ψ^{p^t} is given by the trace of a p^k-cycle on $\sum_{j=0}^{p^k-1} \psi^{p^t}(y^j) = \sum_{j=0}^{p^k-1} y^{jp^t}$. This is p^k if $t \geq k$ and zero otherwise.

Now $\langle(\theta_1)^{p^k}, \psi^{p^t}\rangle = p^k$ and $\langle a, \psi^{p^t}\rangle = 0$ for any of the decomposable terms in (2.2) so applying $\langle\ \ , \psi^{p^t}\rangle$ to (2.2) yields

$$p^k\langle D_* \theta_{p^k}, \psi^{p^t}\rangle + p^k = \begin{cases} p^k & \text{if } t \geq k \\ 0 & \text{otherwise.} \end{cases}$$

This equation holds in Z/p^s for any $s \geq 1$, which implies (2.3).

2.4. COROLLARY. *Let D be the J-theory d-invariant of Section 1 and let $\tilde{D}$ be its universal covering. Then for all $r \geq 1$, each of the homomorphisms below is injective.*

$$D_*: K_*(Q_0S_p^0; Z/p^r) \to K_*(J_p^{\oplus}; Z/p^r) \quad (p \neq 2),$$

$$D_*: K_*(SG_p; Z/p^r) \to K_*(J_p^{\otimes}; Z/p^r) \quad (p \neq 2),$$

$$D_*: K_*(Q_0S_2^0; Z/2^r) \to K_*((J_2')^{\oplus}; Z/2^r),$$

$$D_*: K_*(SG_2; Z/2^r) \to K_*((J_2')^{\otimes}; Z/2^r),$$

$$\tilde{D}_*: K_*(\tilde{Q}_0S_2^0; Z/2^r) \to K_*((J_2'')^{\oplus}; Z/2^r),$$

$$\tilde{D}_*: K_*(\tilde{S}G_2; Z/2^r) \to K_*((J_2'')^{\otimes}; Z/2^r).$$

Proof. By universality of the d-invariants the K-theory d-invariants factor through the J-theory D-invariants. This, together with the discussion of Section 1, accounts for the first four homomorphisms. $\tilde{D}_*$ is injective because, as spaces, $\tilde{Q}_0 S^0 = Q_0 S^0 \times RP^\infty$ and $J'_2 = J''_2 \times RP^\infty$.

2.5. THEOREM. *Let D and $\tilde{D}$ be as in Corollary 2.4. Then each of the maps*

$$\begin{aligned}
&D\colon Q_0 S^0_p \to J^\oplus_p \quad (p \neq 2),\\
&D\colon SG_p \to J^\otimes_p \quad (p \neq 2),\\
&D\colon Q_0 S^0_2 \to (J'_2)^\oplus,\\
&\tilde{D}\colon \tilde{Q}S^0_2 \to (J''_2)^\oplus\\
&D\colon SG_2 \to (J'_2)^\otimes,
\end{aligned}$$

and Sullivan's map (see Section 1) $e'\colon SG \to J^\otimes_2$ induces isomorphisms in real or complex K-cohomology.

Proof. When $p \neq 2$ we have $D\colon SG_p \to J^\otimes_p$ is a split surjection (of spaces) which induces an injection in $K_*(\ \ ; Z/p^r)$ by Corollary 2.4. Hence it induces an isomorphism on $K_*(\ \ ; Z/p^r)$. The same must be true of the d-invariant on each component of QS^0_p, in particular on $Q_0 S^0_p$. Now QS^0_p, SG_p, $J^\oplus_p$ and $J^\otimes_p$ have finite homotopy groups consisting of p-torsion. By an easy argument (see [18, Lemma 9.2]) using the K-theory universal coefficient theorem [2, p. 282 et seq.] and the Kunneth formula [5, p. 113] we see that D^* is an isomorphism on KU^*. The Bott sequence relating KO-theory and KU-theory [5, p. 207] now implies D^* is an isomorphism on KO^*.

Now let $p = 2$. By the above argument it suffices to demonstrate isomorphisms in $K_*(\ \ ; Z/2^r)$. Consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{SG}_2 \arrow[rr, "\tilde{D}"] \arrow[dr] \arrow[ddd, "\tilde{e}"'] & & (J''_2)^\otimes \arrow[r] & BSO^\otimes_2 \arrow[d] \\
& SG_2 \arrow[r, "D"] \arrow[dr] & (J'_2)^\otimes \arrow[r] & BO^\otimes_2 \arrow[dd, "h"] \\
& & (G/O)_2 \arrow[dr, "e"] & \\
(J''_2)^\otimes \arrow[rrr] & & & BSO^\otimes_2
\end{tikzcd}$$

Here h sends a vector bundle V to $V \otimes w_1 V$, $\tilde{e}'$ is the universal cover of the map e' of Section 1 and the remaining maps are the canonical ones.

Most routes in (2.6) are obviously seen to commute. The triangle involving D, e and h is easily seen to commute once one knows the definition of e. The argument is given, for example, in [11, Appendix]. The map $\widetilde{SG}_2 \to BO^\otimes_2$ is injective in $K_*(\ ;Z/2^r)$ because $S\tilde{G}_2 \to SG_2$ is split and the rest of the composite is the KO-theory D-invariant which, by Proposition 2.1, induces an injection since its composition with complexification $BO_2^{\otimes} \to BU_2^{\otimes}$ does. Since the right hand column is an equivalence we see that $(\tilde{e}')_*$ is injective. Hence $(\tilde{e}')_*$ is an isomorphism since $\tilde{e}'$ is a split surjection of spaces (see Section 1). Since

$$e' = \tilde{e}' \times 1_{RP^\infty}: SG_2 = S\tilde{G}_2 \times RP^\infty \to (J_2'')^{\otimes} \times RP^\infty = J_2^{\otimes}$$

we have proved the assertion about Sullivan's map. From diagram (2.6) it is now simple to show $\tilde{D}$ and hence D induce $K_*(\ ;Z/2^r)$—isomorphisms and the proof is complete.

The following result was first proved in [10] in the case $p \neq 2$ and was first proved in general in [18, Theorem 9.3 and 9.9].

2.7. THEOREM. *Let p be any prime. Then the space B^i Cok J_p $(i \geq 0)$ defined in Section 1, has the same real or complex K-theory as a point.*

Proof. In each of the splittings $SG_p = \mathrm{Cok}\, J_p \times J_p^{\otimes}$, described in Section 1, we have shown that the right projection is a $K_*(\ ;Z/p^r)$-isomorphism. By the Kunneth formula $\tilde{K}_*(\mathrm{Cok}\, J_p; Z/p^r) = 0$. The Rothenberg–Steenrod type of spectral sequences [4], [15]

$$E_*^2 = \mathrm{Tor}^*_{K_*(X;Z/p)}(Z/p, Z/p) \Rightarrow K_*(BX; Z/p)$$

for $X = B^i \mathrm{Cok}\, J_p$ shows that $\tilde{K}_*(B^{i+1} \mathrm{Cok}\, J_p; Z/p) = 0$ for all $i \geq 0$. The proof is completed by a universal coefficient theorem argument similar to the one used in the proof of Theorem 2.5.

2.8. K- and KO-cohomology

In order to render Theorem 2.5 of practical use the homotopy theorist needs to know $KO^\alpha(\ )$ and $KU^\alpha(\ )$ of the spaces involved. This data may be derived from [6], [7], [8].

First we recall that $\rho_k: J_p^{\oplus} \to J_p^{\otimes}$ is an equivalence of infinite loopspaces for any prime p and any integer k which is prime to p [8, III, Section 4.4], [12]. Also when $p = 2$, ρ_3 induces an equivalence of infinite loopspaces $\rho_3: (J_2'')^{\oplus} \to (J_2'')^{\otimes}$. Furthermore as infinite loopspaces $SG_2 = S\tilde{G}_2 \times RP^\infty$ and $(J_2')^{\otimes} = RP^\infty \times (J_2'')^{\otimes}$ where $RP^\infty = K(Z/2; 1)$. Hence the K- and KO-groups of all the spaces in Theorem 2.5 are equal to those of $Q_0 S_p^0$ or $\tilde{Q}_0 S_p^0$ for suitable choices of p. As explained in [9] if Σ_∞ is the union of the symmetric groups Σ_n, then QS^0 and $B\Sigma_\infty \times Z$ have the same K-theory. $B\Sigma_\infty$ is a torsion space so, by [6, Lemma 4.6], its representable K-theory and its inverse limit K-theory coincide and from [7, Theorem 2.1] we obtain

$$K^1(B\Sigma_\infty) = 0, \qquad K^0(B\Sigma_\infty) \cong \varprojlim_n R(\Sigma_n)^\wedge.$$

Here $R(\Sigma_n)$ is the complex representation ring and $(\ )^\wedge$ denotes $IR(\Sigma_n)$-adic completion. Similar results are true for the universal cover, BA_∞, where A_∞ is the union of the alternating groups $A_n \subset \Sigma_n$. We therefore obtain

$$K^1(BA_\infty) = 0, \qquad K^0(BA_\infty) \cong \varprojlim R(A_n)^\wedge.$$

Disjoint union of permutations endows these K-rings with "Hopf algebra" structures

$$\Delta\colon K^*(B\Sigma_\infty) \to K^*(B\Sigma_\infty) \hat{\otimes} K^*(B\Sigma_\infty)$$

and

$$\Delta\colon K^*(BA_\infty) \to K^*(BA_\infty) \hat{\otimes} K^*(BA_\infty)$$

where $\hat{\otimes}$ is completed tensor product. If

$$\det\colon R(\Sigma_2)^\wedge \to \varprojlim_n R(\Sigma_n)^\wedge$$

is the homomorphism induced by the sign homomorphism it is easy to see that the Hopf algebra $K^*(BA_\infty)$ is the quotient of $K^*(B\Sigma_\infty)$ by the ideal $\langle \mathrm{im}\ (\det)\rangle$.

From [5, p. 172, Corollary 1.5] we know that all the generators of the $R(\Sigma_n)$ are Burnside representations and hence real. Thus complexification, $c\colon RO(\Sigma_n) \to R(\Sigma_n)$, is an isomorphism. This together with the results of [7, Section 8] and the Bott sequence [5, p. 207] easily yield isomorphisms of $KO^*(*)$-Hopf algebras

$$KO^*(B\Sigma_\infty) \cong \left(\varprojlim_n RO(\Sigma_n)^\wedge\right) \otimes KO^*(*), \tag{2.9}$$

$$KO^*(BA_\infty) \cong \left(\varprojlim_n RO(A_n)^\wedge\right) \otimes KO^*(*).$$

Here $KO^*(*)$ is the KO-theory of a point. We remark that

$$\varprojlim_n RO(A_n)^\wedge \cong \varprojlim_n R(A_n)^\wedge$$

the isomorphism being induced by complexification. Note that $\otimes$-product and inverse limits do not in general commute so that the isomorphisms of (2.9) have to be verified with care. One may, for example, start with the isomorphism in degree zero given by [7, Section 8] and then manipulate the Bott sequence.

To summarize the discussion of Section 2.8, we have the following isomorphisms. We write $K_{\mathbf{C}}^*$, $K_{\mathbf{R}}^*$ for representable unitary, orthogonal K-theory respectively and $(\ \ )_p$ denotes p-localization.

2.10. THEOREM. *Let Λ be $\mathbf{R}$ or $\mathbf{C}$ and let p be a prime.*

(a) *Let W denote one of the infinite loopspaces $Q_0 S_p^0$, SG_p, $J_p^\oplus$, $J_p^\otimes$ if $p \neq 2$ or $Q_0 S_2^0$, $(J_2')^\oplus$. Then, as Hopf algebras,*

$$K_\Lambda^*(W)_p \cong \left(\left(\varprojlim_n R(\Sigma_n)^\wedge\right) \otimes K_\Lambda^*(*)\right)_p.$$

(b) *Let X denote one of the infinite loopspaces $\tilde{Q}_0 S_2^0$, $S\tilde{G}_2$, $(J_2'')^{\oplus}$, $(J_2'')^{\otimes}$. Then, as Hopf algebras,*

$$K_\Lambda^*(X)_2 \cong \left(\left(\varprojlim_n R(A_n)^\wedge\right) \otimes K_\Lambda^*(*)\right)_2.$$

(c) *Let Y denote one of the infinite loopspaces SG_2, $J_2^{\oplus}$, $J_2^{\otimes}$, $(J_2')^{\otimes}$. Then, as Hopf algebras,*

$$K_\Lambda^*(Y)_2 \cong \left(\left(\left(\varprojlim_n R(A_n)^\wedge\right) \hat{\otimes} R(\Sigma_2)^\wedge\right) \otimes K_\Lambda^*(*)\right)_2.$$

Here $R(\Sigma_2)^\wedge$ is the Hopf algebra $K_\Lambda^0(RP^\infty)$.

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