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REPRESENTING HOMOLOGY CLASSES BY EMBEDDED CIRCLES ON A COMPACT SURFACE

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Introduction

It is well known that a one-dimensional homology class on a torus T^2, that winds around one standard generator m times and the other standard generator n times can be represented by one embedded circle if and only if m and n are relatively prime. However, the classification of one dimensional homology classes that can be represented by an embedded circle on an arbitrary compact surface has until now been only partially resolved. We define a homology class in $H_1(M^2)$ to be primitive if the induced class in $H_1(M^2)$/torsion is the zero class or is not a nontrivial multiple of any other class. In [6], [7], and [8] it is shown that primitive classes on orientable surfaces and primitive or twice primitive classes on nonorientable surfaces are precisely the classes representable by embedded circles.

Although there are algorithms (see [2] and [10]) for determining which elements of the fundamental group $\pi_1(M^2)$ can be represented by embedded circles, they are too complicated to deal with the above classification. For orientable compact surfaces we have developed an explicit algorithm for representing a primitive class by one embedded circle. When combined with the Classification Theorem for Oriented Surfaces, this algorithm becomes a useful tool for studying certain classification problems on surfaces, which we shall discuss later. We would like to thank the referee for his suggested improvements of the paper. We now outline the algorithm.

Section 1

Let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be the standard generators of $H_1(M_g, Z)$ where M_g denotes a surface of genus g. Let c_i be the standard circle which disconnects M_g between hole i and hole $i + 1$.

If $\gamma = \sum_{i=1}^{g} m_i \beta_i + n_i \alpha_i$, then we will let $[m_1, n_1], \ldots, [m_g, n_g]$ denote γ. If $\gamma = [m_1, n_1]$ is a primitive class on the torus, then γ can be represented by an embedded circle of "constant slope" n_1/m_1. Actually, for our purposes it will be better to represent $\gamma = [m_1, n_1]$ on a torus as follows: First represent γ by m_1

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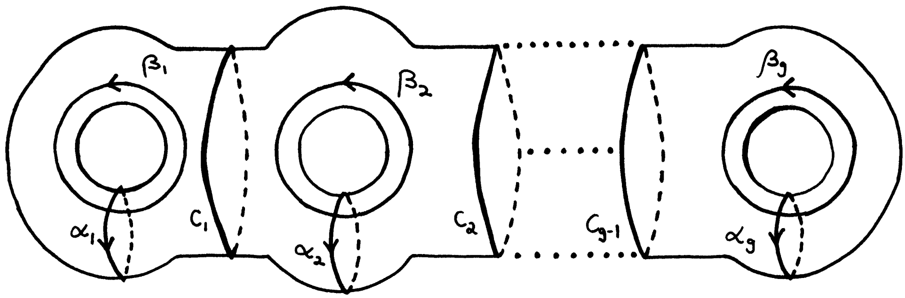

FIGURE 1

parallel copies of β_1 and n_1 parallel copies of α_1. By a simple homology in a small neighborhood of each point of intersection, we can represent γ by one embedded circle. This method of representing γ has the geometric advantage of achieving all the α_1 winding of γ in an arbitrarily small neighborhood of the standard curve representing α_1.

The construction of the algorithm, which we shall soon outline, is based on the following principle: Given a pair of nonbounding nonhomologous and nonintersecting Jordan curves on M_g, cutting M_g along them produces either one or two connected surfaces, with a total of four boundary components. Two surfaces occur only in the case that the sum of the two homology classes is zero, and so the left sides of the curves can be joined by a path in one of the components with opposite interaction signs at each end. If there is only one surface, the left sides of the curves can also be joined by a similar path.

To demonstrate the algorithm on a surface of genus two, we have chosen the following primitive class as a representative example. Let $\gamma = [h, k], [r, s]$ be a primitive class on M_2, with $(h, k) = m$, $(r, s) = n$, and $h > r > 0$ where (c, d) denotes the greatest common divisor of c and d.

Step 1. Represent γ in the standard way by m embedded circles $\eta_0, \ldots, \eta_{m-1}$, on the left, and by n embedded circles, $\delta_0, \ldots, \delta_{n-1}$, on the right.

For example, we can represent [3, 0], [2, 2] as indicated in Figure 2.

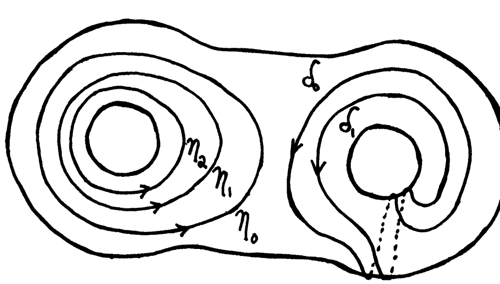

FIGURE 2

Step 2. Take connected sums of the corresponding circles η_i with δ_j until there are no more closed circles obstructing the hole on the right.

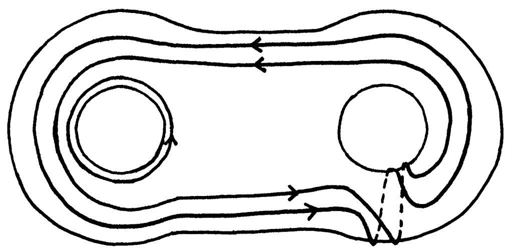

FIGURE 3

Step 3. Form a tube and take connected sums until there are no more circles obstructing the hole on the left.

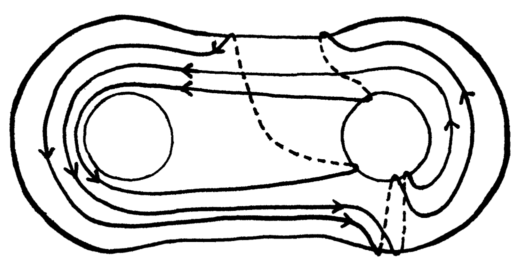

FIGURE 4

Step 4. Now take connected sums by forming a new tube at the lower center of M_2, and then passing through the tube we constructed in Step 3.

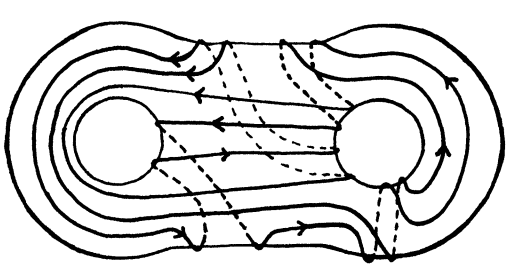

FIGURE 5

We now check that after $m + n - 1$ connections we have formed one embedded circle. Let $\bar{\sigma}$ refer to the connected sum of the embedded circle σ with other embedded circles.

Suppose that $n = km + r$ where $k > 0$ and $0 < r < m$. If $0 \leq i < r$ and $0 \leq j \leq k$, then at the $jm + i + 1$ connection, we get the connected sum

$$\bar{\eta}_i = \eta_i \# \delta_i \# \delta_{i+m} \# \cdots \# \delta_{(j-1)m+i} \# \delta_{jm+i}.$$

Hence, after $km + r$ connections, every δ_i appears in one of the connected sums $\bar{\eta}_0, \bar{\eta}_1, \ldots, \bar{\eta}_{m-1}$. At the $km + r + 1$ connection, we get

$$\begin{aligned}\bar{\eta}_r &\# \bar{\eta}_0 \\ &= (\eta_r \# \delta_r \# \delta_{r+m} \# \cdots \# \delta_{(k-1)m+r}) \# (\delta_0 \# \delta_m \# \cdots \# \delta_{km} \# \eta_0).\end{aligned}$$

Since r is a generator for Z_m, at the $m + n - 1$ connection there is one embedded circle corresponding to $\bar{\eta}_0 \# \bar{\eta}_1 \# \cdots \# \bar{\eta}_{m-1}$.

Remark. (1) If $\gamma = [h, k], [r, s]$ is primitive on M_2 with $r \geq h > 0$, then carry out the above procedure as faithfully as possible. Note that Step 3 and Step 4 will become mixed together. This method will represent γ by an embedded circle.

(2) Let $\gamma_2 = P\gamma$, where $\gamma = [h, k], [r, s]$ is primitive with $P > 0, h > 0, r > 0$, $m = (h, k)$, and $n = (r, s)$. Applying the algorithm described above yields P embedded circles after $P(m + n - 1)$ connections, each circle representing the class γ. A similar mod arithmetic calculation as above proves this fact.

We now inductively generalize the algorithm A presented above for representing certain classes on M_2 by embedded circles. Let

$$\gamma = [m_1, n_1], \ldots, [m_g, n_g]$$

be a primitive class with $m_i > 0$ on the surface M_g.

(1) Represent γ in the standard way by embedded circles around each hole.

(2) Apply parts 2, 3, 4 of the algorithm A to the circles around hole 1 and hole 2. After $(m_1, n_1) + (m_2, n_2) - (m_1, n_1, m_2, n_2)$ connections we have (m_1, n_1, m_2, n_2) circles around the first two holes.

(3) Now apply the algorithm A to the circles around hole 2 and hole 3. After $(m_1, n_1, m_2, n_2) + (m_3, n_3) - (m_1, n_1, m_2, n_2, m_3, n_3)$ connections we are left with $(m_1, n_1, m_2, n_2, m_3, n_3)$ circles. Continue applying A to hole 3 and hole 4, hole 4 and hole 5, $\ldots$, hole $g - 1$ and hole g. Since $(m_1, n_1, \ldots, m_g, n_g) = 1$, we terminate this procedure with one embedded circle.

When $\gamma = [m_1, n_1], \ldots, [m_g, n_g]$ is primitive but the m_i are not necessarily positive, slight variations in the connections after step 1 are required to yield a general algorithm for representing γ. Least there be any uncertainty or ambiguity in the general case, we outline the completion of the algorithm. We encourage the reader to conceptualize the following in geometric terms so that he may carry out the algorithm directly.

Let $g\colon M_1 \to M_1$ be the order four diffeomorphism of a torus $M_1 = \mathbf{R}^2/Z^2$

induced by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let $f\colon M_1 \to M_1$ be an isotopy of g which changes g in a neighborhood of the two fixed points to be the identity function on disc neighborhoods of these points. The diffeomorphism $f\colon M_1 \to M_1$ induces a diffeomorphism $f_i\colon M_g \to M_g$ with (1)$f_{i*}(\alpha_i) = \beta_i$, (2)$f_{i*}(\beta_i) = -\alpha_i$ and (3)f_i is the identity outside of the region bounded by the curves c_{i-1} and c_i in Figure 1.

By application of appropriate least powers of the different f_i we get an $f\colon M \to M$ with

$$\gamma' = f_*(\gamma = [m_1, n_1], \ldots, [m_g, n_g]) = [h_1, k_1], \ldots, [h_g, k_g]$$

where $h_i > 0$ whenever m_i and n_i are not both zero. Apply the algorithm to γ' to get one embedded circle c. Now, $f^{-1}(c)$ is the circle representing γ by the algorithm.

We now consider the case of when M is an oriented compact surface with boundary.

THEOREM 1. *Let M be a compact oriented surface with boundary curves c_1, $c_2, \ldots, c_{n+1}$ oriented as the boundary. Let $\bar{M}$ be the surface obtained by placing disks on each boundary component of M and let $i\colon M \to \bar{M}$ denote the inclusion map. A one-dimensional homology class γ on M can be represented by an embedded circle if and only if $i_*(\gamma)$ is a nonzero primitive class or if $\gamma = \pm\sum_{i=1}^{n} K_i[c_i]$ where K_i is 0 or 1.*

Proof. If a homology class γ on M can be represented by an embedded circle α, then $i_*(\gamma)$ can be represented by an embedded circle. Hence $i_*(\gamma)$ is a primitive class on $\bar{M}$. Assume that α does not intersect the boundary of M and that $i_*(\gamma) = 0$. In this case α disconnects M into two pieces M_1 and M_2. If $c_{n+1} \subset M_2$, then there is a natural homology between $\pm\alpha$ and $\sum_{i=1}^{n} K_i[c_i]$ where K_i equals 0 or 1. This shows our conditions are necessary. We now prove they are sufficient.

Suppose that $i_*(\gamma)$ is a nonzero primitive class on M. Then by applying the algorithm $i_*(\gamma)$ can be represented by an embedded circle α in $M \subset \bar{M}$. Note that $\gamma - [\alpha] = \sum_{i=1}^{n} m_i[c_i]$. Following the technique in the algorithm we take the connected sum of α with m_1 copies of c_1. Thus we may assume that $m_1 = 0$. An inductive argument shows that we can represent γ by an embedded circle. If $\gamma = \pm\sum_{i=1}^{n} K_i[c_i]$ with K_i equal to 0 or 1 then γ can be directly represented by an embedded circle. □

Section 2

We now apply the algorithm to give a geometric proof that the diffeomorphisms on a surface M_g induce the automorphisms of $H_1(M_g, Z)$ which preserve intersection-pairing on homology; equivalently, with respect to the standard basis of $H_1(M_g, Z)$, the automorphisms induced by the diffeomorphisms generate the group S_g of $2g \times 2g$ integer symplectic matrices.

Generators for S_g were first found by Clebsch and Gordan in 1866. In 1890, H. Burkhardt [1] showed that these generators are induced by homeomorphisms of M_g. See [5] for further discussion.

THEOREM 2. *If $L: H_1(M_g, Z) \to H_1(M_g Z)$ preserves intersection-pairing on homology, then L is induced by a diffeomorphism of M_g.*

Proof. Let $\overline{\alpha_i}$, $\overline{\beta_i}$ denote the standard representatives for the standard generators α_i, β_i of $H_1(M_g, Z)$. Suppose $L(\alpha_i) = \eta_i$ and $L(\beta_i) = \delta_i$.

Claim. If there exists a diffeomorphism $f_k: M_g \to M_g$ with $f_{k*}(\eta_i) = \alpha_i$ and $f_{k*}(\delta_i) = \beta_i$ for all $i < k$ where $k \le g$, then there exists a diffeomorphism $f_{k+1}: M_g \to M_g$ such that $f_{k+1*}(\eta_i) = \alpha_i$ and $f_{k+1*}(\delta_i) = \beta_i$ for all $i < k + 1$.

Proof of Claim. First cut M into left- and right-hand pieces N_1 and N_2 along the curve c_{k-1} in Figure 1 so that

$$A = \{\overline{\alpha_i}, \overline{\beta_i} \,|\, i < k\} \subset N_1 \quad \text{and} \quad B = \{\overline{\alpha_i}, \overline{\beta_i} \,|\, i < k\} \subset N_2.$$

Since $f_{k*}(\eta_k) \cap \gamma = 0$ for all $\gamma \in A$, the constructive nature of the algorithm allows us to represent $f_{k*}(\eta_k)$ by an embedded circle $\sigma_1: S^1 \to N_2$. By the Classification Theorem for Oriented Surfaces with Boundary, there is a diffeomorphism $I: N_2 - \sigma_1 \to N_2 - \bar{\alpha}_k$ inducing a diffeomorphism $g: N_2 \to N_2$ interchanging σ_1 and $\bar{\alpha}_k$ such that $g|_{\partial N_2} = \mathrm{id}_{\partial N_2}$. Note that $g_*(f_{k*}(\eta_k)) = \alpha_k$.

Since g preserved intersection pairing on homology,

$$\rho = g_*(f_{k*}(\delta_k)) = r\alpha_k + \beta_k + \sum_{i=k+1}^{g} (m_i\alpha_i + n_i\beta_i).$$

By Step 1 of the algorithm ρ can be represented by a system of circles ρ_1, $\rho_2, \ldots, \rho_s$ with the property that ρ_1 intersects $\bar{\alpha}_k$ in one point and is the only circle appearing between c_{k-1} and c_k in Figure 1. Application of the algorithm to the circles $\rho_1, \ldots, \rho_s$ yields one embedded circle $\sigma_2: S^1 \to N_2$ such that σ_2 geometrically intersects $\bar{\alpha}_k$ in one point. Since $N_2 - (\overline{\alpha_k} \cup \overline{\beta_k})$ and $N_2 - (\overline{\alpha_k} \cup \sigma_2)$ are diffeomorphic, after making proper identifications, we obtain a diffeomorphism $h: N_2 \to N_2$ with

$$h_*(\alpha_k) = \alpha_k, \quad h_*(g_*(f_{k*}(\delta_k))) = \beta_k \quad \text{and} \quad h|_{\partial N_2} = \mathrm{id}_{\partial N_2}.$$

Extend the diffeomorphisms g and h to M_g by the identity on N_1. Now define $f_{k+1}: M_g \to M_g$ by $f_{k+1} = h \circ g \circ f_k$. This proves the claim.

By induction, the claim yields a diffeomorphism $f: M_g \to M_g$ such that $f_*(\eta_i) = \alpha_i$ and $f_*(\delta_i) = \beta_i$ for all i, $1 \le i \le g$. Hence, $(f^{-1})_* = L$. □

The next corollary is a new result and its proof follows directly from the statement and proof of Theorem 2.

COROLLARY. *Suppose $\gamma_1, \ldots, \gamma_k \in H_1(M_g, Z)$ are rationally independent classes expressed in vector notation in terms of the standard basis of $H_1(M_g, Z)$. Then these classes can be represented by pairwise disjoint embedded circles iff they appear as the first k column vectors of a $2g \times 2g$ integer symplectic matrix.*

Example. If $\gamma_1 = [1, 0], [0, 0]$ and $\gamma_2 = [1, 0], [3, 0]$ on M_2, then $\gamma_1 \cap \gamma_2 = 0$ on homology, but any two embedded circles representing these classes geometrically intersect.

Section 3

In this section we shall apply the algorithm to generalize a result of Papakyriakopoulos. In [9] he proved a special case of the following theorem for homotopy 3-spheres. It seems that our theorem applied to a homology 3-sphere of genus 2 should give a counterexample to his group theory conjecture 2 in [9].

Note. We will call a system $\{X_i, Y_i \mid i = 1, 2, \ldots, g\}$ of circles on a surface N of genus g canonical if $M - \bigcup_{i=1}^{g} \{X_i \cup Y_i\}$ is a sphere with g holes.

THEOREM 3. *Suppose M^3 is a homology 3-sphere and T and T' are solid g-holed handlebodies giving rise to a Heegard splitting of M^3. If $N = \partial T = \partial T'$ then there exist two canonical systems*

$$\{X_i, Y_i \mid i = 1, 2, \ldots, g\} \quad \textit{and} \quad \{X_i', Y_i' \mid i = 1, 2, \ldots, g\}$$

for N with the following properties.

(i) *X_i is contractible in T.*
(ii) *X_i' is contractible in T'.*
(iii) *X_i is homologous to Y_i'.*
(iv) *Y_i is homologous to X_i'.*

For simplicity we will give the proof of the above theorem for the case when $g = 2$. The proof of the general case can be carried out in an inductive manner.

LEMMA. *Let γ_1 and γ_2 be independent homology classes on $N = \partial T$ which are zero homology classes on T. Then γ_1 and γ_2 can be represented by disjoint embedded circles contractible in T if and only if γ_1 and γ_2 expressed in terms of the usual basis for N appear as the first two columns of a 4×4 symplectic matrix.*

Proof. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the generators of the usual homology basis for $N = \partial T$. Since the homology of T is generated by β_1 and β_2, γ_1 can be represented by k copies of α_1 and l copies of α_2. Since the α_i bound disks and the connected sum of circles bounding disks again bound disks, the algorithm demonstrates that γ_1 can be represented by a circle δ_1 bounding a disk in T.

For convenience, assume that $\delta_1 = \alpha_1$ on N. Since $\gamma_2 = 0$ on T and γ_1 and γ_2 appear as the first two columns of a symplectic matrix, then γ_2 is homologous to $\pm(k\alpha_1 + \alpha_2)$. By applying the algorithm to γ_2, we can easily represent γ_2 by a circle δ_2 which is contractible in T with δ_1 and δ_2 free of intersections.

Proof of Theorem 3. By Theorem (32.3) in [9], there exists a canonical system $\{X_1, X_2, Y_1, Y_2\}$ for N with X_i contractible in T and Y_i homologous to zero in T'. By the corollary to Theorem 2, we can assume that $[Y_1]$ and $[Y_2]$ appear as the first 2 columns of a symplectic matrix for N considered to be bounding T'. By the above Lemma $[Y_1]$ and $[Y_2]$ can be represented by embedded circles X'_1 and X'_2 which are contractible in T'. By arguments similar to the proof of Theorem 2, X'_1 and X'_2 can be completed to give a canonical basis X'_1, X'_2, Y'_1, Y'_2 of N with Y'_1 homologous to X_1 and Y'_2 homologous to X'_1. This completes the proof of our theorem. □

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