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ON THE PRIMITIVE COHOMOLOGY OF SUBMANIFOLDS

BY

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1. Introduction

In this article, we study certain facts about the primitive cohomology of compact submanifolds of compact Kähler manifolds. Specifically, let $i\colon X \subset Y$ be the inclusion, where X is a compact submanifold of complex codimension q in Y, which is a compact Kähler manifold. Let $P^p(X)$ and $P^p(Y)$ denote the primitive cohomology of X and Y, respectively, with respect to some fixed Kähler metric on Y. Let π denote the projection of cohomology onto the primitive part. Then we shall show that

$$\pi i^* H^p(Y) \cap [RH^{p+2q-1}(Y - X) \cap P^p(X)] = [i^* H^p(X) \cap P^p(X)] \cap \pi RH^{p+2q-1}(Y - X) = 0,$$

where R is the Leray–Norguet residue operator. Here

$$RH^{p+2q-1}(Y - X) \cap P^p(X) \quad \text{and} \quad i^* H^p(Y) \cap P^p(X)$$

mean primitive p-forms which are residues and restrictions, respectively. If $n = \dim_{\mathbf{C}} X$, then we shall show that

$$\begin{aligned} P^n(X) &= \pi i^* H^n(Y) \oplus [RH^{n+2q-1}(Y - X) \cap P^n(X)] \\ &= [i^* H^n(Y) \cap P^n(Y)] \oplus \pi RH^{n+2q-1}(Y - X), \end{aligned}$$

and each summand is nondegenerate with respect to cup product, cf. (2.4) below for an example.

In homology if we let $F_p(X) = \mathrm{Hom}_{\mathbf{C}}(P^p(X), \mathbf{C}) \subset H_p(X)$ and θ be the projection of $H_p(X)$ onto $F_p(X)$, then the above result states that

$$\begin{aligned} F_p(X) &= [\ker i_* \cap F_p(X)] + \theta I H_{p+2q}(Y) \\ &= \theta \ker i_* + [IH_{p+2q}(Y) \cap F_p(Y)], \end{aligned}$$

where I is transverse intersection; and, when $p = n$, the sums are direct. This was proven for $n = 1$ and 2 by the author in [6].

In Chapter 3 we give some consequences of these results and indicate their application in studying the monodromy of compact Kähler manifolds.

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This paper was motivated by the author's attempt [7] to find a "nontranscendental" proof of the local invariant cycle problem, cf. Griffiths [10, p. 249]; also see Clemens [1] and Steenbrink [14]. These decompositions and the degeneracy of a certain spectral sequence are the essential facts from Kähler geometry that one needs to solve the problem. In Section 4, analogous results concerning this decomposition for schemes are conjectured to be true.

In this article, all homology and cohomology will be with coefficients in **C**. When we refer to cohomology classes as forms, we shall always mean the unique harmonic representative in the class. Finally, we shall identify $H_p(X) = \mathrm{Hom}_{\mathbf{C}}(H^p(X), \mathbf{C})$ with the duality given by integration: $\gamma\colon \omega \to \int_\gamma \omega$. Thus, for $0 \neq \omega \in H^p(X)$, we shall mean $\mathrm{Hom}(\omega) \in H_p(X)$ to be the unique homology class which has period one on ω, and zero periods on the orthogonal complement. Since Hom is self-dual, we shall also speak of $\mathrm{Hom}(\gamma)$ for $\gamma \in H_p(X)$.

Finally, the author would like to thank the referee for suggesting the present proofs in Chapter 2, which are a simpler exposition than our original proofs.

2. Main results

2.1. Let $i\colon X \subset Y$ be the inclusion of a compact Kähler submanifold of complex dimension n into a compact Kähler manifold of complex dimension $n+q$. Then choose a Kähler metric on Y which gives a Hodge decomposition of the cohomology of Y and X. Then for $p \leq n$, let $\pi\colon H^p(X) \to P^p(X)$ be the projective onto the primitive part, i.e., $H^p(X) = \bigoplus_{j\geq 0} L_X^j P^{p-2j}(X)$, where L_X is the Hodge operator on X. Let $P^p(Y)$ denote the primitive cohomology of Y.

2.1.1. Definition. We say $\gamma \in H_p(X)$ is *finite* iff if ω is a harmonic form such that $\int_\gamma \omega \neq 0$, then $\pi\omega \neq 0$. Let $F_p(X)$ denote the finite classes.

That is to say, if $H^p(X) \simeq \mathrm{Hom}_{\mathbf{C}}(H_p(X), \mathbf{C}) = H_p(X)^*$ considered as dual vector spaces via integration, $\omega\colon H_p(X) \to \mathbf{C}$ by $\omega(\gamma) = \int_\gamma \omega$, then γ is finite iff $\gamma^* \in P^p(X)$.

Equivalently, γ is finite iff $\gamma \cap \Omega_X = 0$ where Ω_X is the Kähler form and $\cap$ denotes cap product.

If X is a projective algebraic manifold, then γ is finite iff γ has a representative in the affine part of X, i.e., finite part of X.

Let $\theta\colon H_p(X) \to F_p(X)$ be the projection.

2.2. Theorem (2.2.1)

$$\pi i^* H^p(Y) \cap [RH^{p+2q-1}(Y-X) \cap P^p(X)] = [i^* H^p(Y) \cap P^p(X)] \cap \pi R H^{p+2q-1}(Y-X) = 0,$$

where R is the Leray–Norguet residue operator.

(2.2.2)

$$\begin{aligned} P^n(X) &= \pi i^* H^n(Y) \oplus [RH^{n+2q-1}(Y-X) \cap P^n(X)] \\ &= [i^* H^n(Y) \cap P^n(X)] \oplus \pi R H^{n+2q-1}(Y-X) \end{aligned}$$

and each summand is nondegenerate with respect to cup product.

(2.2.3)

$$F_p(X) = [\ker i_* \cap F_p(X)] + \theta I H_{p+2q}(Y)$$
$$= \theta[\ker i_* \cap H_p(X)] + [IH_{p+2q}(Y) \cap F_p(X)],$$

where $I\colon H_{p+2q}(Y) \to H_p(X)$ *is given by transverse intersection.*

(2.2.4) *If* $p = n$, *the sums in* (2.2.3) *are direct, and each summand is nondegenerate with respect to the intersection pairing.*

Proof of (2.2.1).

2.3.1. *If* $\omega = i^*\tilde{\omega}$ *and* $\eta = R(\tilde{\eta})$ *with* $\omega \in H^p(X)$ *and* $\eta \in H^{2n-p}(X)$, *then* $\int_X \omega \wedge \eta = 0$.

This follows because if TX is a tubular neighborhood of X in Y and τX is its boundary, then

$$\int_X \omega \wedge \eta = \int_{\tau X} \tilde{\omega} \wedge \tilde{\eta} = \int_{Y-TX} d(\tilde{\omega} \wedge \tilde{\eta}) = 0.$$

The last equality follows from Stokes' Theorem, while the first equality is the fundamental identity involving the residue operator.

2.3.2. Lemma.

$$L_X(i^*H^p(Y)) \subset i^*H^{p+2}(Y) \quad \text{and} \quad L_X RH^{p+2q-1}(Y-X) \subset RH^{p+2q+1}(Y-X).$$

Proof of Lemma 2.3.2. The first part follows because $i^*L_Y = L_X$, while the second follows from the representation of $H^*(Y - X)$ by forms of the type $\phi \wedge K_X + \eta$ where ϕ and η are C^∞-forms on Y and K_X is the kernel associated to X, cf., Poly [13]. Then $R(\phi \wedge K_X + \eta) = i^*\phi$. Hence, if $\omega = R(\phi \wedge K_X + \eta)$, then $L_X\omega = R((L_Y\phi) \wedge K_X + L_Y\eta)$. This concludes the proof of Lemma 2.3.2.

Thus, suppose

$$\omega \in \pi i^*H^p(Y) \cap [RH^{p+2q-1}(Y - X) \cap P^p(X)].$$

Then, $\omega + L_X\phi \in i^*H^p(Y)$, for some ϕ. Since R is a real operator, we also have that $L_X^{n-p}\bar{\omega} \in RH^{2n+2q-p-1}(Y - X)$, by (2.3.2). Then by (2.3.1),

$$0 = \int_X (\omega \wedge L_X\phi) \wedge L_X^{n-p}\bar{\omega} = \int_X \omega \wedge L_X^{n-p}\bar{\omega},$$

since $L_X\phi \wedge L_X^{n-p}\bar{\omega} = L_X^{n-p+1}\bar{\omega} \wedge \phi = 0$ as $\bar{\omega} \in P^p(X)$ and $\dim_{\mathbf{C}} X = n$. But by Weil [16, p. 77], for $\omega \in P^p(X)$, $c_p \int_X \omega \wedge L_X^{n-p}\bar{\omega} > 0$ for some nonzero constant c_p. Thus, this is a contradiction, unless $\omega = 0$.

A similar proof works for $\omega \in [i^*H^p(Y) \cap P^p(X)] \cap \pi RH^{p+2q-1}(Y - X)$, which completes the proof of (2.2.1).

Proof of (2.2.2).

2.3.3. LEMMA. *If* $\omega \in H^p(X)$ *and* $\int_X \omega \wedge \eta = 0$ *for all* $\eta \in i^*H^{2n-p}(X)$, *then* $\omega \in RH^{p+2q-1}(Y - X)$.

Proof of Lemma 2.3.3. Thus, we assume

$$0 = \int_X \omega \wedge i^*\tilde{\eta} = \int_{(D_X)(i^*\tilde{\eta})} \omega$$

for all $\tilde{\eta} \in H^{2n-p}(Y)$, and where D_X is the Poincare' duality mapping $H^{2n-p}(X) \cong H_p(X)$. We also have the commutative diagram

$$\begin{array}{ccc} H^{2n-p}(X) & \xrightarrow{D_X} & H_p(X) \\ {\scriptstyle i^*}\uparrow & & \uparrow{\scriptstyle I} \\ H^{2n-p}(Y) & \xrightarrow{D_Y} & H_{p+2q}(Y) \end{array}$$

where I is transverse intersection and D_Y is Poincare' duality on Y. Hence $0 = \int_{ID_Y\tilde{\eta}} \omega$ for all $\tilde{\eta} \in H^{2n-p}(Y)$, i.e., ω has zero periods on $IH_{p+2q}(Y)$. Thus, $0 \neq \text{Hom } \omega \in \text{coker } I$. But the vector space transpose of I is the Gysin mapping G. Hence $G\omega = 0$. But from the exact sequence

$$H^{p+2q-1}(Y - X) \xrightarrow{R} H^p(X) \xrightarrow{G} H^{p+2q}(Y),$$

we get that $\omega \in RH^{p+2q-1}(Y - X)$. This concludes the proof of Lemma 2.3.3.

Thus, suppose $\omega \in P^n(X)$ and $\int_X \omega \wedge \eta = 0$ for all $\eta \in \pi i^*H^n(Y)$. If $\lambda \in i^*H^n(Y)$, then $\lambda = \pi\lambda + L_X\xi$ for some ξ. Then $\int_X \omega \wedge \lambda = \int_X \omega \wedge L_X\xi$ by the hypothesis on ω. But $\omega \wedge L_X\xi = L_X\omega \wedge \xi = 0$, since ω is primitive. Thus, by Lemma 2.3.3, this completes the proof of the first part of (2.2.2).

To prove the second decomposition, it suffices to show that if $\omega \in H^p(X)$ and $\int_X \omega \wedge \eta = 0$ for all $\eta \in RH^{2n-p+2q-1}(Y - X)$, then $\omega \in i^*H^p(Y)$. This is proven by a similar argument to (2.3.3) using the commutative diagram

$$\begin{array}{ccc} H^{2n-p}(X) & \xrightarrow{D_X} & H_p(X) \\ {\scriptstyle R}\uparrow & & \uparrow{\scriptstyle \partial_*} \\ H^{2n-p+2q-1}(Y - X) & \xrightarrow{D_{Y,X}} & H_{p+1}(Y, X) \end{array}$$

where $D_{Y,X}$ is the Poincare'-Lefschetz duality isomorphism.

Proof of (2.2.3) *and* (2.2.4). This is just a reformulation of the above results: π image $i^* \cap [\ker G \cap P^p(X)] = 0$ if and only if coker $i^* \cap P^p(X) + \pi$ coimage $G \simeq P^p(X)$. But the transpose of G is I. Similarly for the other decomposition.

This completes the proof of Theorem 2.2.

2.4. We note that the sum in (2.2.3) need not be direct for $p < n$. Equivalently,

$$\pi i^* P^p(Y) \oplus [RH^{p+2q-1}(Y - X) \cap P^p(X)] \subsetneq P^p(X).$$

For let $T \subset \mathbf{C}P_3$ be a torus and let $\sigma\colon Y \to \mathbf{C}P_3$ be the monoidal transform with center T. Let $X = \sigma^{-1}(T)$. Then by S.G.A. 5(vii),

$$H_1(Y) = 0, \quad H_1(X) = \mathbf{C} \oplus \mathbf{C} \simeq H_1(T) \quad \text{and} \quad i_*\colon H_3(X) \xrightarrow{\sim} H_3(Y).$$

Then ker $(i_* \mid H_1(X)) = H_1(X)$ and one easily shows that

$$\text{image}\ \{I\colon H_3(Y) \to H_1(X)\} = H_1(X).$$

Let $W = Y \times \mathbf{C}P_1$ and $V = X \times \mathbf{C}P_1$. Then $b_3(V) = 4$ and $b_3(W) = 2$ by the Künneth formula. The generators of $H^3(V)$ can be given by $\alpha_1, \alpha_2, \beta_1$, and β_2 where α_1 and β_1 correspond to a basis of $H^3(X)$. Then if α and β are generators of $H^1(X) \simeq H^1(V)$, then α_2 and β_2 will correspond to $\alpha \times h$ and $\beta \times h$ where h is a generator of $H^2(\mathbf{C}P^1)$. The cup product for this basis will then have as matrix (a_{ij}), where $a_{14} = a_{32} = 1$, $a_{41} = a_{23} = -1$ and the rest of the a_{ij} are zero. Then $L_V(\alpha) = \alpha_1 + \alpha_2$ and $L_V(\beta) = \beta_1 + \beta_2$, so that a basis of the primitive cohomology can be given by $\alpha_1 - \alpha_2$ and $\beta_1 - \beta_2$.

Then α_1 and β_1 will generate $i^*H^3(W)$ and one computes that $RH^4(W - V) = iH^3(W)$. One can also see this equality by passing to homology and showing that $I(i_*\alpha_1^*) = \alpha_2^*$ and $I(i_*\beta_1^*) = \beta_2^*$ ($\alpha_i^* = \text{Hom}\ \alpha_i$). Thus, $P^3(V)$ is generated by $\pi\alpha_1$ and $\pi\beta_1$ $(\alpha_1 = \frac{1}{2}(\alpha_1 - \alpha_2) + \frac{1}{2}(\alpha_1 + \alpha_2))$ and $RH^4(W - V) \cap P^3(V) = 0$.

2.5. Theorem 2.2 is false for Y noncompact Kähler. For let

$$V = \{x^3 + y^3 + z^3 = 0\} \subset \mathbf{C}^3$$

and let π be the monoidal transform in $\mathbf{C}^3$ with center the origin. Let Y be the proper transform of V and $X = \pi^{-1}(0) \cap Y$. Then X is a torus and $i\colon X \subset Y$ induces an injection of $H_1(X)$ into $H_1(Y)$. But $H_3(Y) = 0$. However, if we compactify Y to $\bar{Y}$ so that $\bar{Y} - Y = X_1$ is also a torus, then we have that $I\colon H_3(\bar{Y}) \xrightarrow{\sim} H_1(X)$. In fact, if $i_1\colon X_1 \subset \bar{Y}$, then $(i_1)_* I_1 \gamma = i_* I \gamma$ for all $\gamma \in H_3(\bar{Y})$. This phenomenon is discussed below in (3.3.3).

2.6. In [7] we will actually need a slightly stronger version of Theorem 2.2. Suppose that X_j $(j = 1, \ldots, k)$ is a submanifold of complex codimension q in Y, a compact Kähler manifold of complex dimension $n + q$. Furthermore, suppose that the X_j are in general position so that $X = \bigcup_{j=1}^k X_j$ is a subvariety having normal crossings. Let $i\colon X \subset Y$ and $i_j\colon X_j \subset Y$ denote the inclusions. Then we are going to prove a result analogous to Theorem 2.2 for X.

So, if we compute the cohomology of X using the Maier–Vietoris sequence, we find that $H^p(X) = A^p \oplus B^p$ where $A^p \subset \bigoplus_j H^p(X_j)$ and B^p comes from the relative cocycles, i.e.,

$$0 \to B \to \bigoplus_{s=1}^{p} \bigoplus_{i_1 < \cdots < i_{s+1}} H^{p-s}(X_{i_1} \cap \cdots \cap X_{i_{s+1}}).$$

Let $\pi\colon A^p \to \bigoplus_{j=1}^k P^p(X_j)$ be the restriction to A^p of the projection mapping $\bigoplus \pi_j\colon \bigoplus_j H^p(X_j) \to \bigoplus_j P^p(X_j)$. We extend π to $H^p(X)$ by having $B^p \subset \ker \pi$.

2.6.1. DEFINITION. $P^p(X) = \pi A^p$ is a *primitive p-form of X*. $F_p(X) = \mathrm{Hom}_{\mathbf{C}}(P^p(X), \mathbf{C})$ is a *finite p-cycle of X*.

Let $\theta\colon H_p(X) \to F_p(X)$ be the projection.

Thus, $\omega \in P^p(X) \subset H^p(X)$ if and only if $\omega = \sum_{i \in I} \omega_i$ where $0 \neq \omega_i \in P^p(X_i)$ and $I \subset \{1, \ldots, k\}$.

In Gordon [4, Proposition 2.13], it is shown that there is an exact sequence

$$H_{p+2q}(Y) \xrightarrow{I} H_p(X)_\Delta \xrightarrow{\tau} H_{p+2q-1}(Y - X)$$

where I is geometric intersection and τ is the tube over cycles map, i.e., locally the product with the normal sphere. In [4, Corollary 2.8], it is shown that $H_p(X)_\Delta = \bigoplus_{s=1}^{p-1} H_{p-s+1}(\bar{M}_s)_\Delta$ and that

$$H_p(X)_\Delta \cap H_p(X) \subset H_p(\bar{M}_1)_\Delta \subset \bigoplus_{j=1}^{k} H_p(X_j).$$

In fact,

$$H_p(\bar{M}_1)_\Delta = \{(\gamma_1, \ldots, \gamma_k) \mid \gamma_j \in H_p(X_j) \text{ and } \gamma_i \cap X_j = \gamma_j \cap X_i, \text{ for all } i \neq j\}.$$

Also, see Gordon [5], especially Section 4.

If we consider the Leray spectral sequence of the inclusion map of $Y - X \subset Y$, then as Y is compact Kähler, this spectral sequence degenerates at E_{2q+1}, i.e., $E_{2q+1}^{r,s} \simeq E_\infty^{r,s}$, cf. Deligne [2]; see also Deligne, et al. [3]. Then in Gordon [8], degeneration is shown to be equivalent to the fact that

$$I(H_{p+2q}(Y)) \subseteq H_p(\bar{M}_1)_\Delta.$$

In [8, Lemma 2.4] it is shown that this fact is essentially equivalent to the fact that $B^p \cap \text{image } i^* = 0$. This last fact is a consequence of the principle of the 2 types, cf., [3]. Thus,

$$IH_{p+2q}(Y) \subseteq \bigoplus_j H_p(X_j),$$

so that $\theta I H_{p+2q}(Y)$ makes sense.

In cohomology, in Gordon [4, Chapter 5], it is shown that one has the following exact sequence (which is the vector space dual of the above sequence in homology):

$$H^{p+2q-1}(Y - X) \xrightarrow{R} H^p(X)_\Delta \xrightarrow{G} H^{p+2q}(Y)$$

where G is "essentially" the Gysin map, and R is the residue operator. Also

$$H^p(X)_\Delta = \bigoplus_{s=1}^{p+1} H^{p-s+1}(\bar{M}_s)_\Delta \quad \text{and} \quad H^p(\bar{M}_1)_\Delta \subset \bigoplus_{j=1}^{k} H^p(X_j).$$

Then, $H^p(X)_\Delta \cap H^p(X) \subset \bigoplus_{j=1}^{s} H^p(X_j)$. Thus,

$$RH^{p+2q-1}(Y-X) \cap H^p(X) \subset A^p,$$

so that $RH^{p+2q-1}(Y-X) \cap \pi A^p$ makes sense.

2.6.2. COROLLARY. *With the above notation,*

(2.6.2.1)

$$\begin{aligned} 0 &= \pi i^* H^p(Y) \cap [RH^{p+2q-1}(Y-X) \cap P^p(X)] \\ &= [i^* H^p(Y) \cap P^p(X)] \cap \pi RH^{p+2q-1}(Y-X). \end{aligned}$$

(2.6.2.2)

$$\begin{aligned} F_p(X) &= [\ker i_* \cap F_p(X)] + \theta IH_{p+2q}(Y) \\ &= \theta(\ker i_* \cap H_p(X)_\Delta) + IH_{p+2q}(Y) \cap F_p(X). \end{aligned}$$

(2.6.2.3) *Furthermore for $p = n$, the sums in* (2.6.2.2) *are direct and $\pi i^* H^n(Y)$ and $RH^{n+2q-1}(Y-X) \cap P^n(X)$ generate $P^n(X)$. Similarly, $i^* H^n(Y) \cap P^n(X)$ and $\pi RH^{n+2q-1}(Y-X)$ generate $P^n(X)$.*

Proof of Corollary 2.6.2. Let $i_j\colon Y - X_j \subset Y - X$ be the inclusions. Then consider the diagram

$$\begin{tikzcd} & H^{p+2q-1}(Y-X) \arrow[dl, "R"'] \arrow[dr, phantom, "\square"] & \bigoplus_j H^{p+2q-1}(Y-X_j) \arrow[l, "\Sigma (i_j)_*"'] \arrow[d, "R_j"] \\ H^p(X)_\Delta \arrow[r, "f"] & A^p(X) \arrow[r, "g"] \arrow[dr] \arrow[dr, phantom, "\square", near end] & \bigoplus_j H^p(X_j) \arrow[d, "\pi_j"] \\ & & \bigoplus_j P^p(X_j) \end{tikzcd}$$

where R_j is the residue operator for X_j in Y, π_j the projection of $H^p(X_j)$ onto its primitive part, f is intersection with $H^p(X)$ (which by the above remarks lies in $A^p(X)$), and g is the inclusion mapping.

The bottom triangle commutes by the very definition of π, while the top diagram commutes by the constructions in Gordon [4, pp. 130–133], i.e., if $\gamma \in H_p(X)_\Delta \cap H_p(X)$, then

$$\gamma = (\gamma_1, \ldots, \gamma_k) \quad \text{for } \gamma_j \in H_p(X_j).$$

Then $\tau(\gamma) = \sum_{j=1}^{k} \tau_{1,j} \tilde{\gamma}_j$ and $\tau_{1,j}\gamma_j \in H_{p+2q-1}(Y - X_j)$. Hence,

$$\int_{\tau(\gamma)} \omega = \sum_{j=1}^{k} \int_{\gamma_j} R_j(\omega) = \int_{\gamma} R(\omega) \quad \text{for } \omega \in H^{p+2q-1}(Y-X).$$

But since g is an inclusion and the result is true for each of the $P^p(X_j)$ by (2.2.1), we have that (2.6.2.1) is true. By vector space duality, (2.6.2.2) follows. Similarly, since $P^p(X) = \pi A^p$, we get that (2.2.2) implies that $P^p(X)$ will be generated by the subspaces stated in (2.6.2.3). This proves Corollary 2.6.2.

3. Some consequences

3.1. Suppose i: $X \subset Y$, where Y is a compact Kähler manifold of complex dimension $n + q$ and X is a compact submanifold of codimension q.

Then $H^s(Y)$ has a natural complex inner product defined by

$$\langle \omega, \eta \rangle_Y = \int_Y \omega \wedge \bar{*}\, \eta$$

(where as usual, we take the unique harmonic form in each cohomology class) with $\bar{*}$ being the usual real Hodge star operator extended to be complex antilinear. With respect to this inner product, we let $\|\omega\|_Y = (\langle \omega, \omega \rangle_Y)^{1/2}$. Furthermore, if $\omega \in P^p(Y)$ and ω is orthogonal to η, then $L_Y^s \omega$ is orthogonal to $L_Y^s \eta$, cf. Weil [16, p. 22].

Let $K = \{\omega \in H^p(Y) \,|\, i^*\omega \cap P^p(X) = 0\}$. Let $K^\perp$ be the orthogonal complement to K with respect to $\langle \cdot, \cdot \rangle$ in $H^p(Y)$. Then $K^\perp \subset P^p(Y)$, because $L_Y H^{p-2}(Y) \subset K$ and $P^p(Y) \perp L_Y H^{p-2}(Y)$. Finally, let $\wedge_X$ denote the mapping of cup product with the Poincare′ dual class of $[X] \in H_{2n}(Y)$.

3.1.1. Proposition. *With the above notation, if* $0 \neq \omega \in K^\perp$, *then*

$$\wedge_X \omega = c_\omega (q!)^{-1} L_Y^q \omega \text{ modulo } L_Y^{q-s} \phi_s$$

for $n - p \geq s \geq 1$ *and* $\phi_s \in P^{p+2s}(Y)$. *Also,* $c_\omega = (\|\omega\|_Y)^2 (\|i^*\omega\|)^{-2}$, *i.e.,* c_ω *depends only on the Kähler metric.*

3.1.2. Since $H_s(Y) = \mathrm{Hom}_{\mathbf{C}}\,(H^s(Y), \mathbf{C})$, $H_s(Y)$ inherits the complex inner product from $H^s(Y)$. Also for $p \leq n$, L_Y^q: $H^p(Y) \to H^{p+2q}(Y)$ is an injection, and this also induces an injection $H_p(Y) \to H_{p+2q}(Y)$ by the identifications via Hom. We also denote this map on homology as L_Y^q.

3.2. *Proof of Proposition* 3.1.1. Let ω_j be an orthonormal basis of $K^\perp$.

(3.2.1) dim $i^*K^\perp = \dim K^\perp$.

For if $\sum_j a_j i^*\omega_j = 0$, then $\sum_j a_j \omega_j \in K$.

3.2.2. Lemma. *Let* W *be a compact, complex manifold,* D_W *be the Poincare′ duality isomorphism, and, with respect to some hermitian metric, let* $\bar{*}$ *denote the real star operator extended to be complex antilinear. Then for* $0 \neq \omega \in H^*(W)$, Hom $D_W(\omega) = (\|\omega\|_W)^{-2}(\bar{*}\omega)$.

Proof of Lemma 3.2.2. Let ω_i be an orthonormal basis of $H^p(W)$. Then $\delta_{ij} = \int_W \omega_i \wedge \bar{*}\, \omega_j = \int_{D_W(\bar{*}\omega_j)} \omega_i$ by the definition of the Poincare′ duality isomorphism, i.e., Poincare' duality states that $(\alpha \cup \beta)[W]$ is a nondegenerate pairing for $\alpha \in H^p(W)$, $\beta \in H^{2n-p}(W)$. This proves Lemma 3.2.2.

Now, consider the diagram

$$
\begin{tikzcd}
H^{2n-p}(X) \arrow[d, "\mathrm{Hom}"'] \arrow[dd, bend right=60, "\bar{*}_X"'] & H^{2n-p}(Y) \arrow[l, "i_*"'] \arrow[d, "\mathrm{Hom}"] \arrow[dd, bend left=60, "\bar{*}_Y"] \\
H_{2n-p}(X) \arrow[r, "i_*"] \arrow[dr, phantom, "\Box"] & H_{2n-p}(Y) \arrow[d, "(D_Y)^{-1}"] \\
H^p(X) \arrow[u, "D_X"] \arrow[r, "G"] & H^{p+2q}(Y) \\
H^p(Y) \arrow[u, "i^*"] \arrow[ur, "\wedge X"'] \arrow[u, phantom, shift left=6, "\Box"] &
\end{tikzcd}
$$

where $\Box$ means the diagram commutes and G is the Gysin mapping of the normal sphere bundle of X in Y.

Let $\omega_j \in K^\perp$. Then $i^*\omega_j \in P^p(X)$, and, by (3.2.2),

$$
\begin{aligned}
D_X i^*\omega_j &= c_{j,X} \operatorname{Hom} (\bar{*}_X)^{-1} i^*\omega_j \\
&= c_{j,X} \operatorname{Hom} (-1)^{p(p+3)/2}((n-p)!)^{-1} \sum_r A_r L_X^{n-p} i^*\bar{\omega}_{j,r},
\end{aligned}
$$

where $c_{j,X} = (\|i^*\omega_j\|_X)^{-2}$, $A_r = -(\sqrt{-1})^{2r-p}$ and $\omega_{j,r}$ is the $(r, p-r)$ part of ω_j. This follows from Weil [16, p. 22] and the fact that $(\bar{*}_X)^{-1} = (-1)^p \bar{*}_X$.

Thus, this says that

$$i_* D_X i^*\omega_j = c_{j,X} L_Y^{n-p} \operatorname{Hom} (A\bar{\omega}_j) + \gamma$$

where

$$\langle L_Y^{n-p} \operatorname{Hom} A\bar{\omega}_j, \gamma\rangle = 0 \quad \text{and} \quad A = (-1)^{p(p+3)/2}((n-p)!)^{-1} A_r$$

on the $(r, p-r)$ part of ω_j. In other words we are using the notation of (3.1.2) and we can always write $i_* D_X i^*\omega_j$ as a sum of a multiple of $\operatorname{Hom} L_Y^{n-p}\bar{\omega}$ and something in its orthogonal complement. The coefficient $c_{j,X}A$ follows from the calculation in the previous paragraph.

Now, we can assume that $\gamma = \sum_{s \geq 0} L_Y^{n-p-s}\gamma_s$ for $\gamma_s \in F_{p+2s}(Y)$. This is because if $s < 0$ and $\phi_s = \operatorname{Hom} \gamma_s$, then $i^*(L_Y^{n-p-s}\phi_s) = L_X^{n-p-s} i^*\phi_s$ is orthogonal to $L_X^{n-p}P^p(X)$. Hence for $s < 0$, $L_Y^{n-p-s}\gamma_s$ could not be in the image of $i_*[\operatorname{Hom} L_X^{n-p}P^p(X)]$.

Thus, to complete the proof of the proposition, it suffices to show that $L_Y^{n-p}\gamma_0 = 0$, because $\bar{*}_Y L_Y^{n-p}(A\bar{\omega}_j) = (q!)^{-1} L_Y^q \omega_j$.

We let $\phi_0 = \operatorname{Hom} \gamma_0 \in P^p(Y)$. Then $0 = \langle L_Y^{n-p}\omega_j, L_Y^{n-p}\phi_0\rangle = \langle \omega_j, \phi_0\rangle$. Now, $\phi_0 = \alpha_1 + \alpha_2$ with $\alpha_1 \in K$ and $\alpha_2 \in K^\perp$ such that $\langle \omega_j, \alpha_i\rangle = 0$ for $i = 1$ and 2, i.e.,

$$(K \oplus K^\perp) \cap (\omega_j)^\perp = [K \cap (\omega_j)^\perp] \oplus [K^\perp \cap (\omega_j)^\perp].$$

3.2.3. LEMMA. $\alpha_2 = 0$.

Proof of Lemma 3.2.3. We claim that

$$\langle L_Y^{n-p} \operatorname{Hom} \alpha_2, i_* D_X i^* \omega_j \rangle = 0.$$

By hypothesis, $\alpha_2 = \sum_{k \neq j} d_k L_Y^{n-p} \omega_k$. Then

$$0 \neq \langle L_Y^{n-p} \operatorname{Hom} \alpha_2, i_* D_X i^* \omega_j \rangle$$

implies that $0 \neq \int_{D_X i^* \omega_j} i^* L_Y^{n-p} \alpha_2$. But $\operatorname{Hom} D_X i^* \omega_j$ has nonzero projection onto $i^*(L_Y^{n-p} \omega_j)$. Thus, this implies that

$$i^* L_Y^{n-p} \alpha_2 = \sum_k a_k L_X^{n-p} i^* \omega_k \quad \text{with } a_j \neq 0.$$

However, $i^* L_Y^{n-p} \alpha_2 = \sum_{k \neq j} d_k L_X^{n-p} i^* \omega_k$ which contradicts (3.2.1).

Hence, $L_Y^{n-p} \operatorname{Hom} \omega_j$, $i_* D_X i^* \omega_j$, $L_Y^{n-p} \operatorname{Hom} \alpha_1$, and $\sum_{s \geq 1} L_Y^{n-p-s} \gamma_s$ all belong to $(L_Y^{n-p} \operatorname{Hom} \alpha_2)^\perp$. But

$$\operatorname{Hom} L_Y^{n-p} \alpha_2 = \left(i_* D_X i^* \omega_j - L_Y^{n-p} \operatorname{Hom} \alpha_1 - \sum_{s \geq 1} L_Y^{n-p-s} \gamma_s - c_{j,X} L_Y^{n-p} \operatorname{Hom} A\bar{\omega}_j \right) \in (L_Y^{n-p} \operatorname{Hom} \alpha_2)^\perp.$$

This proves Lemma 3.2.3.

3.2.4. LEMMA. $\alpha_1 = 0$.

Proof of Lemma 3.2.4. It suffices to show that

$$\langle L_Y^{n-p} \operatorname{Hom} \alpha_1, i_* D_X i^* \omega_j \rangle = 0$$

by the same argument in (3.2.3). If

$$0 \neq a_j = \langle L_Y^{n-p} \operatorname{Hom} \alpha_1, i_* D_X i^* \omega_j \rangle,$$

then this implies that the projection of $i^* \alpha_1$ onto the primitive part of X is nonzero. But as $\alpha_1 \in K$ this implies that $i^* \alpha_1 = a_j i^* \omega_j + \xi + L_X \phi$ for some $\xi \in P^p(X)$ and $\phi \neq 0$. Thus, $(\alpha_1 - a_j \omega_j) \in K$, i.e., $\omega_j \in K \cap K^\perp$. This proves Lemma 3.2.4 and Proposition 3.1.1.

3.3. Using the notation of (3.1.2), we let

$$J = \{\gamma \in L_Y^q H_p(Y) \mid I\gamma \cap F_p(X) = 0\}$$

where $I\colon H_{p+2q}(Y) \to H_p(X)$ is the intersection mapping. Let $J^\perp$ be the orthogonal complement to J in $L_Y^q H_p(Y)$.

3.3.1. COROLLARY. *Suppose that* $0 \neq \gamma \in J^\perp$. *Then*

$$i_* I\gamma = c_\gamma (q!)(L_Y^q)^{-1} \gamma \quad \textit{where } c_\gamma = (\|I\gamma\|_X)^2 (\|\gamma\|_Y)^{-2}.$$

Corollary 3.3.1 is equivalent to Proposition 3.1.1. This follows immediately from the fact that I is the vector space dual to G and from the following lemma:

3.3.2. LEMMA. *$J^{\perp} \subset L_Y^q F_q(Y)$ and $J \simeq \mathrm{Hom}\, L_Y^q K$.*

The proof of the lemma is straightforward.

3.3.3. COROLLARY. *Let $X_j, j = 1, \ldots, k$ be submanifolds of Y of codimension q. Let $J = \{\gamma \in L_Y^q H_p(Y) \mid I_j\gamma \cap F_p(X_j) = 0$ for some $j\}$, where I_j is intersection with X_j. Let $J^{\perp}$ be the orthogonal complement to J in $L_Y^q H_p(Y)$. Let $0 \neq \gamma \in J^{\perp}$. Then for all $1 \leq i, j \leq k$,*

$$0 \neq (\|I_i\gamma\|_{X_i})^{-2}(i_i)_*(I_i\gamma) = (\|I_j\gamma\|_{X_j})^{-2}(i_j)_*(I_j\gamma).$$

3.3.4. We note that Corollary 3.3.3 has applications in studying the monodromy of compact Kähler manifolds. Namely, suppose that

$$\pi\colon W \to \{t \mid |t| \leq 1\} \subset \mathbf{C}$$

is a proper map such that for $t \neq 0$, $\pi^{-1}(t) = V_t$ is a compact Kähler n-dimensional manifold. Let $j\colon V_t \subset W$. If T denotes the action on $H_*(V_t)$ induced by $\pi_1(\{t \mid 0 < |t| \leq 1\})$, then the local invariant cycle problem says that $\ker j_* = \text{image}\,(T - I)$ where I is induced from the identity mapping. It is easy to show that image $(T - I) \subset \ker j_*$. We can suppose that T is unipotent and that $\pi^{-1}(0) = V_0$ has normal crossings, so that $V_0 = \bigcup_i X_i$. We set $\bar{M}_k = \cup (X_{i_1} \cap \cdots \cap X_{i_k})$, the k-points. Now, because T and I commute with L_{V_t}, it suffices to consider $\ker j_* \cap F_p(V_t)$. Then one can show that $\ker j_* \cap F_p(V_t)$ is generated by

$$F_{p-k+1}(\bar{M}_k) = \sum_{i_1 < \cdots < i_k} F_{p-k+1}(X_{i_1} \cap \cdots \cap X_{i_k}) \quad \text{for } 2 \leq k \leq p,$$

i.e., there exists a map $g\colon \bigoplus_{k=2}^{p} F_{p-k+1}(\bar{M}_k) \to F_p(V_t)$ such that image $g = \ker j_*$. Furthermore, the relations among the $gF_{p-k+1}(\bar{M}_k)$ are given by

$$I_{k-1}\colon H_{p-k+3}(\bar{M}_{k-1}) \to H_{p-k+1}(\bar{M}_k),$$

which is induced by intersection on each component, i.e., if

$$\gamma \in H_{p-k+3}(X_{i_1} \cap \cdots \cap X_{i_{k-1}})$$

and I_{i_k} denotes intersection with $X_{i_1} \cap \cdots \cap X_{i_{k-1}} \cap X_{i_k}$, then $I_{k-1}(\gamma) = \sum_{i_k} I_{i_k}(\gamma)$ implies that $gI_{k-1}\gamma = \sum_{i_k} gI_{i_k}(\gamma) = 0$ in $H_p(V_t)$.

Finally, suppose $\gamma \in F_{p-k+1}(\bar{M}_k)$. Then to show that $g(\gamma) \in$ image $T - I$, it can be shown that it suffices to show that γ belongs to the kernel of the inclusion mapping $\bar{M}_k \subset \bar{M}_{k-1}$. So if γ is not in the kernel of $\bar{M}_k \subset \bar{M}_{k-1}$, then by Corollary 2.6.2, $\gamma = I_{k-1}\alpha$ where $Y = X_{i_1} \cap \cdots \cap X_{i_{k-1}}$. Then by Corollary 3.3.3, each of the $I_{i_k}\alpha$ is homologous in Y, which will imply that γ is in the kernel of $\bar{M}_k \subset \bar{M}_{k-1}$. Hence $I_{i_k}\alpha = 0$ except for at most one k, which implies that $0 = gI_{k-1}\alpha = g\gamma$.

For details of these arguments for $p = n = 2$, see Gordon [9] and Todorov [15].

3.4. Let us consider some example of (3.3.3).

3.4.1. Let $Y = \mathbf{C}P_2 \times \mathbf{C}P_1$, $X_1 = \mathbf{C}P_1 \times \mathbf{C}P_1$, and $X_2 = T \times \mathbf{C}P_1$, where T is the nonsingular curve of degree 3 in $\mathbf{C}P_2$. Let Y have the Kähler metric induced from the Segre′ embedding. Then $H^2(Y)$ is generated by h_1 and h_2 for

$$h_1 \in H^2(\mathbf{C}P_2 \times \{o\}) \quad \text{and} \quad h_2 \in H^2(\{o\} \times \mathbf{C}P_1).$$

Let $0 \neq 1 \in H^0(Y)$. Then $\Omega_Y = L_Y(1) = h_1 + h_2$ and $L_Y^2(1) = h_1^2 + 2h_1h_2$, where

$$\text{Hom } h_1^2 = [\mathbf{C}P_2 \times \{o\}] \quad \text{and} \quad \text{Hom } h_1h_2 = [\mathbf{C}P_1 \times \mathbf{C}P_1].$$

Hence $h_1 - 2h_2 \in P^2(Y)$ and $L_Y(h_1 - 2h_2) = h_1^2 - h_1h_2$.

Let a, $b \in H_2(X_1)$ with $a = [\mathbf{C}P_1 \times \{o\}]$ and $b = [\{o\} \times \mathbf{C}P_1]$; and let c, $d \in H_2(X_2]$ with $c = [T \times \{o\}]$ and $d = [\{o\} \times \mathbf{C}P_1]$. Hence, $(i_1)_* a = \frac{1}{3}(i_2)_* c$ and $(i_1)_* b = (i_2)_* d$. Let $a^* = \text{Hom } a$, etc., so that

$$(i_1)^* L_Y(1) = a^* + b^*, \quad (i_2)^* L_Y(1) = \tfrac{1}{3}c^* + d^*.$$

Hence $a - b \in F_2(X_1)$ and $c - 3d \in F_2(X_2)$.

Then J is generated by

$$\text{Hom } L_Y^2(1) = [\mathbf{C}P_2 \times \{o\}] + 2[\mathbf{C}P_1 \times \mathbf{C}P_1],$$

and $J^\perp$ is generated by $[\mathbf{C}P_2 \times \{o\}] - [\mathbf{C}P_1 \times \mathbf{C}P_1] = \alpha$, i.e., $I_1\alpha = a - b$, $I_2(\alpha) = c - 3d$. Furthermore,

$$(i_1)_* I_1 \alpha = (i_1)_*(a - b) = \tfrac{1}{3}(i_2)_*(c - 3d) = \tfrac{1}{3}(i_2)_* I_2 \alpha$$

and $6 = (\|I_2\alpha\|_{X_2})^2$, $2 = (\|I_1\alpha\|_{X_1})^2$.

3.4.2. As another example, let $Y = \mathbf{C}P_1 \times \mathbf{C}P_1 \times \mathbf{C}P_1 = a \times b \times c$ with $X_1 = a \times c$ and $X_2 = b \times c$. Suppose Y has the product metric. So, if $a^* = \text{Hom } a$, etc., then $\Omega_Y = L_Y(1) = a^* + b^* + c^*$, and $P^2(Y)$ is generated by $a^* - b^*$ and $b^* - c^*$. If $\alpha = a \times b$, then $(i_1)_* I_1 \alpha = a \neq b = (i_2)_* I_2 \alpha$. But in this case, $J = H_4(Y)$, i.e.,

$$\begin{aligned} &I_1[L_Y(a - b) - L_Y(b - c)] \\ &= (L_{X_1}(1))^* \quad \text{and} \quad I_2[2L_Y(a - b) + L_Y(b - c)] = (L_{X_2}(1))^*. \end{aligned}$$

3.5. In fact, for $p = 2$ we can actually prove a stronger result than 3.3.1.

3.5.1. Corollary. *Let X_j be submanifolds of Y of codimension q for $j = 1, \ldots, k$. Let*

$$J_\theta = \{\gamma \in L_Y^q H_2(Y) \,|\, \theta_j I_j \gamma = 0 \text{ for some } j\}$$

where θ_j is the projection onto the finite part of X_j. Let $J_\theta^\perp$ be the orthogonal complement to J_θ in $L_Y^q H_2(Y)$. Suppose $0 \neq \gamma \in J_\theta^\perp$. Then for all $1 \leq i, j \leq k$, we have

$$(\|\theta_j I_j \gamma\|_{X_j})^{-2}(i_j)_*(\theta_j I_j \gamma) = (\|\theta_i I_i \gamma\|_{X_i})^{-2}(i_i)_*(\theta_i I_i \gamma).$$

The proof is the same as for (3.1.1) with the obvious modifications. The essential fact one needs is that $I_j J_\theta$ is still orthogonal to $I_j J_\theta^\perp$ in $H_p(X_j)$.

Note that in 3.4.2, $J_\theta = H_4(Y)$ as well.

3.5.2. However, 3.5.1 is false for $p \geq 3$. For example for the case $p = 3$, we start with $T \subset \mathbf{C}P_4$, where T denotes the torus in $\mathbf{C}P_2 \subset \mathbf{C}P_4$. Let $\pi_1\colon Y' \to \mathbf{C}P_4$ be the monoidal transform with center T and let $(\pi_1)^{-1}(T) = X'$. Then consider

$$X' = X' \times \{o\} \subset Y' \times \mathbf{C}P_1,$$

and let $\pi_2\colon Y \to Y' \times \mathbf{C}P_1$ be the monoidal transform with center X'. Let $X_1 = (\pi_2)^{-1}(X')$ and let $X_2 = Y' \subset Y$.

Then $b_1(X') = 2 = b_3(X') = b_3(Y')$ and $b_1(Y') = 0$. This gives $P^3(Y') = H^3(Y')$. Also, $b_1(X_1) = 2$, $b_3(X_1) = 4$, $b_1(Y) = 0$ and $b_3(Y) = 4$; and so again $P^3(Y) = H^3(Y)$.

Let α and β be an orthonormal basis for $H_1(T)$, which are considered as a basis for $H_1(X')$ and $H_1(X_1)$ as well. Let $\alpha_1 = L_{X'}(\alpha)$ and $\beta_1 = L_{X'}(\beta)$, which are an orthogonal basis for $H_3(X')$. We let $\alpha_2 = -(\pi_2)^{-1}\alpha$ and $\beta_2 = -(\pi_2)^{-1}\beta$, which completes α_1 and β_1 to a basis of $H_3(X_1)$. Then $L_{X_1}(\alpha) = \alpha_1 + \alpha_2$ and $L_{X_1}(\beta) = \beta_1 + \beta_2$.

3.5.2.1. LEMMA. *$\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ are an orthogonal basis of $H_3(X_1)$.*

Proof of Lemma 3.5.2.1. It is clear that $\langle \alpha_i, \beta_j \rangle = 0$ since $\langle \alpha, \beta \rangle = 0$ in T. It suffices to show that $\langle \alpha_1, \alpha_2 \rangle = 0$. But Hom $D_{X_1}(\alpha_2) = -(\pi_1)^{-1}\beta$, i.e., $\pi_1\colon X' \to T$ so that $(\pi_1)^{-1}(\beta)$ defines a class in $H_5(X')$; and in the inclusion of $X' \subset X_1$, we identify $(\pi_1)^{-1}\beta$ with its image. It is the dual to α_2, since $\alpha_2 = -(\pi_1)^{-1}\alpha$ so that the dual of α_2 will be the dual of α in X', the base of the bundle. But

$$\alpha_1 \cdot (\pi_1)^{-1}\beta = L_{X'}(\alpha) \cdot (\pi_1)^{-1}\beta = 0.$$

This is because in the fibre F_Q over $Q = \alpha \cdot \beta$, we have that

$$F_Q = \mathbf{C}P_2 \times \mathbf{C}P_1 \quad \text{and} \quad L_{X'}(\alpha) \cap F_Q = \mathbf{C}P_1 \times \{o\}$$

while $(\pi_1)^{-1}(\beta) \cap F_Q = \mathbf{C}P_2 \times \{o\}$. But $(\mathbf{C}P_1 \times \{o\}) \cdot (\mathbf{C}P_2 \times \{o\}) = 0$ in $\mathbf{C}P_2 \times \mathbf{C}P_1$. This concludes the proof of Lemma 3.5.2.1.

Thus, $\tilde{\alpha} = \alpha_1 - \alpha_2$ and $\tilde{\beta} = \beta_1 - \beta_2$ are a basis for $F_3(X_1)$.

Let $a = (i_1)_* \alpha_1$ and $b = (i_1)_* \alpha_2$. Then a and b are orthogonal, as α_1 is orthogonal to α_2 in X_1. Then, up to positive constants, $I_1 L_Y a = L_{X_1} \alpha + \tilde{\alpha}$ and $I_2 L_Y b = L_{X_1} \alpha - \tilde{\alpha}$, while $0 \neq I_2 L_Y a \in F_3(X_2)$ and $I_2 L_Y b = 0$, since $Y' = X_2$.

Thus, $L_Y(i_1)_* \alpha_2$ and $L_Y(i_2)_* \beta_2$ generate J_θ, and a basis of $J_\theta^\perp$ is given by $L_Y(i_1)_* \alpha_1$ and $L_Y(i_2)_* \beta_1$. But, up to positive constants,

$$(i_1)_* \theta_1 I_1 L_Y a = (i_1)_* \tilde{\alpha} = (i_1)_*(\alpha_1 - \alpha_2)$$

while

$$(i_2)_* \theta_2 I_2 L_Y a = (i_2)_* \alpha_2.$$

However $(i_1)_*(\alpha_1 - \alpha_2)$ and $(i_2)_* \alpha_2$ are linearly independent in $H_3(Y)$.

The reason the proof breaks down is that $\theta_1 I_1 L_Y a$ is a multiple of $\theta_1 I_1 L_Y b$, i.e., $\theta_1 J_\theta$ is not orthogonal to $\theta_1 J_\theta^\perp$ in $F_3(X_1)$.

4. Some conjectures about schemes

Suppose that Y is an integral algebraic k-scheme, where k is an arbitrary algebraically closed field of any characteristic. We assume that Y is a smooth subscheme of projective space $P_N(k)$ and the dimension of Y is $n + q$. Suppose further that X_j for $j = 1, \ldots, s$ are smooth subschemes of Y of codimension q and that i_j: $X_j \subset Y$ are the inclusions.

Let $\mathscr{H}$ denote a Weil cohomology theory, cf. Kleiman [11]. We also suppose that $\mathscr{H}$ satisfies the second Lefschetz theorem, which states that L_Z^i: $\mathscr{H}^{n-i}(Z) \to \mathscr{H}^{n+i}(Z)$ is an isomorphism for all i where Z is a smooth subscheme of $P_N(k)$ of dimension n, cf. Katz–Messing [10]. Let

$$P^{n-i}(Z) = \ker L_Z^{i+1} | \mathscr{H}^{n-i}(Z),$$

and call $P^{n-i}(Z)$ the primitive cohomology. Let π: $\mathscr{H}^{n-i}(Z) \to P^{n-i}(Z)$ denote the projection. If D_Z denotes Poincare′ duality, viewed as an isomorphism $\mathscr{H}^{n-i}(Z) \to \mathscr{H}^{n+i}(Z)$, then D_Z induces a nondegenerate pairing on $\mathscr{H}^i(Z)$ for all i by $\langle a, b\rangle = a \cdot D_Z b$ and $\cdot$ is the Kronecker pairing.

4.1. Definition. Suppose we tensor the coefficient field of $\mathscr{H}$ with the complex numbers. Then we shall say that Z has property $I(Z)$ if for all $a \in P^{n-i-2j}(Z)$, $D_Z(L_Z^j a) = cL_Z^{i+j}\bar{a}$ where c is a nonzero constant depending on a, i, n, and j, and the overbar denotes complex conjugation.

Then for $j = 1, \ldots, s$ we define G_j: $\mathscr{H}^p(X_j) \to \mathscr{H}^{p+2q}(Y)$, the Gysin homomorphism, by $G_j = (D_Y)^{-1} \circ (i_j)_* \circ D_{X_j}$.

4.2. *Conjecture.* If $I(X_j)$ and $I(Y)$ are true, then

(i) $\pi(i_j)^*\mathscr{H}^p(Y) \cap [\ker G_j \cap P^p(X_j)]$

$$= [(i_j)^*\mathscr{H}^p(Y) \cap P^p(X_j)] \cap \pi(\ker G_j \cap \mathscr{H}^p(X_j)) = 0.$$

(ii) $P^n(X_j) = \pi(i_j)^*\mathscr{H}^n(Y) \oplus [\ker G_j \cap P^n(X_j)]$

$$= [(i_j)^*\mathscr{H}^n(Y) \cap P^n(X_j)] \oplus \pi(\ker G_j \cap \mathscr{H}^n(X_j))$$

Let $X = \bigcup_{j=1}^s X_j$ and suppose that i: $X \subset Y$ has normal crossings in Y. Let f: $\mathscr{H}^p(X) \to \bigoplus_{j=1}^s P^p(X_j)$ be defined by

$$a \to \bigoplus_{j=1}^{s} \pi_j(j_j)^* a$$

where $\pi_j\colon \mathscr{H}^p(X_j) \to P^p(X_j)$ is the projection map and $j_j\colon X_j \subset X$ is the inclusion. Define $P^p(X)$ to be the orthogonal complement with respect to $\langle \cdot, \cdot \rangle$ of $\ker f$. Let $\pi\colon \mathscr{H}^p(X) \to P^p(X)$ be the projection.

4.3. *Conjecture.* Suppose that the spectral sequence of the inclusion mapping $Y - X \subset Y$ degenerates at E_{2q+1} and suppose that $I(Y)$ and $I(X_j)$ are true for all j. Then for $p \leq n$, $\ker\,(\pi i^* \,|\, \mathscr{H}^p(Y)) = \ker\,(\Omega_X \,|\, \mathscr{H}^p(Y))$ where Ω_X is cup product with $\gamma_X \in \mathscr{H}^{2q}(Y)$ (γ_X being the "dual" class to the algebraic cycle X).

Suppose $A \subset \{1, \ldots, s\}$ and let

$$J(A) = \{\omega \in \mathscr{H}^p(Y) \,|\, (i_j)^*\omega \cap P^p(X_j) = 0 \text{ for some } j \in A\}.$$

Let $J^\perp(A)$ be the orthogonal complement in $\mathscr{H}^p(Y)$ to J with respect to $\langle \cdot, \cdot \rangle$. Let $\| \cdot \|$ denote the norm with respect to $\langle \cdot, \cdot \rangle$. Finally, let Ω_{X_j} be cup product with the dual class to the algebraic cycle X_j.

4.4. *Conjecture.* Suppose that $I(Y)$ and $I(X_j)$ are true for all $j \in A$. If $0 \neq \omega \in J^\perp(A)$, then for all $j \in A$,

$$\Omega_{X_j}(\omega) = \|\omega\|_Y)^2(\|(i_j)^*\omega\|_{X_j})^{-2}(q!)^{-1}L_Y^q\omega \quad \text{modulo } L_Y^{q-s}\phi_s$$

for $n - p \geq s \geq 1$ and $\phi_s \in P^{p+2s}(Y)$.

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