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A CHARACTERIZATION OF SOLUTIONS TO A PERTURBED LAPLACE EQUATION

BY

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1. Introduction

We consider the elliptic equation, in plane polar coordinates,

$$a(\rho)u_{\rho\rho} + b(\rho)u_\rho + u_{\theta\theta} = 0, \tag{1.1}$$

where a and b are defined for $0 < \rho \le 1$, and $\int_0^1 a(\rho)^{-1/2}\, d\rho = +\infty$ (hence $a(0) = 0$ and the equation has a singularity at the origin). Through the change of variable

$$r = \exp \int_1^\rho a^{-1/2} = f(\rho), \quad 0 < r \le 1,$$

equation (1.1) becomes

$$\Delta u + \frac{\varepsilon(r)}{r} u_r = 0, \tag{1.2}$$

where $\varepsilon(r) = a(\rho)^{-1/2}[b(\rho) - \frac{1}{2}a'(\rho)]$, $r = f(\rho)$.

Equation (1.2) is simply div (σ grad u) = 0 with $\varepsilon(r)/r = \sigma'/\sigma$, in the case that σ is a function of r alone. This equation in turn is the model equation for a number of physical situations, e.g., steady state temperature distribution without heat sources where σ is the coefficient of heat conduction of the medium; magnetic potential with σ the magnetic permeability of the medium; the potential of the electric field of a steady current where σ is the conductivity of the medium [3, p. 387].

We focus our attention on equation (1.2). It is the purpose of this note to characterize all (complex valued) solutions to (1.2) in the unit disk by identifying their generalized boundary values. The possibility of exploring such a task was suggested by Courant in the final paragraph of [4, p. 798]. This idea was vigorously pursued by Lions and Magenes in a series of papers culminating in [9] where they characterize the solutions to uniformly elliptic systems with analytic coefficients, defined in compact domains in R^n with analytic boundary. In [6] Johnson reproves this result for Laplace's equation in the unit disk, using different methods. He also obtains an integral representation for harmonic

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functions in terms of certain sequences of continuous functions on the boundary. Such a representation was obtained by Saylor [11] for the systems mentioned above.

The equation considered here differs from those above in that the coefficients are not assumed analytic. We show that there is a one-to-one correspondence between the set of solutions to (1.2) and the space $\mathscr{H}'$ of analytic functionals which also appear as the generalized boundary values of harmonic functions in the unit disk [6]. In a vague sense one can view this as a form of stability: by perturbing Laplace's equation in its lower order terms, one does not alter the class of solution boundary values.

Each solution is represented as a convolution of its boundary "function" with a certain kernel solution. The existence and verification of the crucial properties of this kernel function are obtained using classical stability theory. As an immediate consequence of the structure theorem for elements in $\mathscr{H}'$ [6, Theorem 6], we also obtain an integral representation theorem for solutions to (1.2). In the final section we briefly discuss generalizations to equations whose boundary values are not necessarily elements from $\mathscr{H}'$.

2. The kernel solution

Central in the study of harmonic functions is the Poisson kernel,

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2} = \sum_{k=-\infty}^{\infty} r^{|k|} e^{ik\theta}, \quad 0 \le r < 1.$$

It is itself harmonic, and through convolution integrals can represent every harmonic function in the unit disk. The few simple properties of the Poisson kernel needed to accomplish this rather broad representation can be simply listed in terms of the Fourier coefficients $R_k(r) = r^{|k|}$; they are symmetric in k, positive, increasing to 1 with r, and summable.

In this section we prove that equation (1.2) possesses a kernel solution

$$Q_r(\theta) = \sum_{k=-\infty}^{\infty} R_k(r) e^{ik\theta}, \quad 0 \le r < 1, \tag{2.1}$$

whose Fourier coefficients $R_k(r)$ are also symmetric in k, positive, increasing to 1 with r, and summable. In addition $Q_r(\theta)$ is real analytic in θ for each fixed r, which is the reason why the space of generalized boundary values is the same as for the harmonic functions.

Separating variables in (1.2) we obtain, as usual, the θ eigenfunctions $e^{ik\theta}$, $k = 0, \pm 1, \pm 2, \ldots$. The boundary value problem in r that must be solved is then

$$R'' + \frac{1 + \varepsilon(r)}{r} R' - \frac{k^2}{r^2} R = 0, \quad R_k(0) \text{ finite}, R_k(1) = 1. \tag{2.2}$$

For $k = 0$, the general solution to this equation is

$$R(r) = a + b \int_r^1 \frac{1}{t} \exp\left(\int_t^1 \frac{\varepsilon(s)}{s} ds\right) dt,$$

where a and b are arbitrary constants. If we assume $\int_0^1 (\varepsilon(r)/r)\, dr$ is convergent, then $R_0(r) \equiv 1$ is the unique solution to (2.2).

For $k > 0$, we first make a change in independent variable $t = \log(1/r)$, and then a change in dependent variable $v(t) = R(t) \exp\left(-\frac{1}{2}\int_0^t \varepsilon(e^{-s})\, ds\right)$. The equation in (2.2) reduces to

$$v'' - (k^2 + \phi(t))v = 0 \quad \text{where } \phi(t) = \tfrac{1}{4}\varepsilon(e^{-t})^2 + \tfrac{1}{2}e^{-t}\varepsilon'(e^{-t}).$$

If we assume $\varepsilon(r) \in C[0, 1]$, $\varepsilon(0) = 0$, and $\varepsilon'(r) = o(1/r)$ as $r \to 0$, then $\phi(t) \to 0$ as $t \to +\infty$, and hence according to Theorem 7 on page 44 of [1], the above equation has a solution $v(t)$ which satisfies the inequality

$$|v(t)| + |v'(t)| \le c \exp\left(-kt + d \int_0^t |\phi(s)|\, ds\right), \quad 0 \le t < \infty,$$

where c and d are positive constants independent of k. If $\int_0^\infty |\phi(s)|\, ds < \infty$, then this becomes $|v(t)| + |v'(t)| \le Ae^{-kt}$. Moreover, in this case Theorem 7 of [1] also guarantees the existence of an *unbounded* solution as well.

Returning to the variables $r = e^{-t}$ and $R = v \exp\left(\frac{1}{2}\int_0^t \varepsilon(e^{-s})\, ds\right)$ we find a pair of inequalities

$$|R(r)| \le Ar^k \exp\left(\frac{1}{2}\int_r^1 \frac{\varepsilon(x)}{x} dx\right); \qquad |R'(r)| \le Br^{k-1} \exp\left(\frac{1}{2}\int_r^1 \frac{\varepsilon(x)}{x} dx\right).$$

Now

$$\int_0^t |\phi(s)|\, ds \le \frac{1}{4}\int_r^1 \frac{\varepsilon(x)^2}{x} dx + \frac{1}{2}\int_r^1 |\varepsilon'(x)|\, dx,$$

which leads again to the hypothesis that

$$\int_0^1 \frac{\varepsilon(x)}{x} dx$$

be convergent, and also that $\int_0^1 |\varepsilon'(x)|\, dx$ be convergent.

A complete list of hypotheses imposed on the coefficient $\varepsilon(r)$ are as follows.

(2.3a) $\varepsilon(r)$ continuous for $0 \le r \le 1$, $\int_0^1 (\varepsilon(r)/r)\, dr$ convergent.

(2.3b) $\varepsilon'(r)$ Hölder continuous for $r_0 \le r \le 1$ for each $0 < r_0 < 1$, $\varepsilon'(r) = o(1/r)$ as $r \to 0$, $\int_0^1 |\varepsilon'(r)|\, dr$ convergent.

These conditions are independent, and (2.3a) implies $\varepsilon(0) = 0$. The two integral conditions relate to the density σ in the physical equation div $(\sigma \text{ grad } u) = 0$ since

$$\int_r^1 \frac{\varepsilon(x)}{x} dx = \log \frac{\sigma(1)}{\sigma(r)}.$$

Thus (2.3a) is satisfied, for example, if $\sigma \in C'[0, 1]$ and $\sigma(r) > 0$ on $[0, 1]$. Condition (2.3b) says that the oscillation of ε and hence of σ and σ' cannot be too great. These conditions do allow $\varepsilon(r)/r$ to be unbounded at $r = 0$. Throughout the rest of this paper we assume conditions (2.3a) and (2.3b) are satisfied.

We conclude from Theorem 7 of [1] that for each integer k, the boundary value problem (2.2) has a unique bounded solution $R_k(r)$ with the properties

$$|R_k(r)| \leq Ar^{|k|}, \quad |R_k'(r)| \leq Br^{|k|-1}, \quad 0 \leq r \leq 1, \tag{2.4}$$

where A and B can be chosen independent of k (see [12, chapter 4], where explicit formulas for A and B are found). Of course $R_{-k}(r) = R_k(r)$. From the differential equation we also have

$$|R_k''(r)| \leq Ck^2 r^{|k|-2}, \quad |k| \geq 2, 0 \leq r \leq 1. \tag{2.5}$$

Finally, for $|k|$ sufficiently large, we have more information. Since $\phi(t)$ is bounded, $0 \leq t < \infty$, setting $\tau = |k| t$, we get the equation

$$v''(\tau) - \left(1 + \frac{1}{k^2}\phi(\tau/|k|)\right) v(\tau) = 0.$$

But then $(1/k^2)\phi(\tau/|k|)$ can be made *uniformly* small on $[0, \infty)$ and hence the asymptotic results of Theorem 7 [1] remain true for *all* $t \geq 0$ (see equation (5), p. 45). In particular, we have

$$R_k(r) \geq Dr^{|k|}, \quad R_k'(r) \geq 0, \quad 0 \leq r \leq 1, \tag{2.6}$$

for $|k|$ sufficiently large. We note, in this case, that $R_k(r)$ is *increasing* to 1 with r.

We now define the kernel function $Q_r(\theta)$ by (2.1). The series converges absolutely for $0 \leq r < 1$ and sub-uniformly (uniformly on compact subsets of the unit disk). Furthermore by (2.4) and (2.5) the series can be twice differentiated term by term, and hence $Q_r(\theta)$ is a solution to (1.2) in the unit disk. In addition (2.4) implies [7, p. 26] $Q_r(\theta)$ is *real analytic* in θ for each fixed r.

We summarize the results of this section in the following

THEOREM 2.1. *Equation* (1.2), *subject to conditions* (2.3a) *and* (2.3b), *possesses a solution $Q_r(\theta)$ in the open unit disk which is real analytic in θ for each r, and whose Fourier coefficients $R_k(r)$ are symmetric in k, summable, and for k sufficiently large are positive and increasing to* 1 *with r.*

The following result characterizes the solutions to (1.2) in terms of their Fourier series, subject of course to conditions (2.3a) and (2.3b).

THEOREM 2.2. *Every solution $u(r, \theta)$ of* (1.2) *in the punctured unit disk which is bounded in a neighborhood of the origin is continuous at the origin, is real analytic in θ for each* $0 < r < 1$, *and can be written*

$$u(r, \theta) = \sum_{k=-\infty}^{\infty} a_k R_k(r) e^{ik\theta}, \tag{2.7}$$

for some unique set of constants $\{a_k\}$ *satisfying*

$$\limsup_{|k|\to\infty} |a_k|^{1/|k|} \leq 1. \tag{2.8}$$

Conversely each sequence with property (2.8) *determines a solution through* (2.7).

By results of partial differential equations [2, p. 136], each solution $u(r, \theta)$ has Hölder continuous third partial derivatives in the punctured disk. Hence if we write

$$u(r, \theta) = \sum_{k=-\infty}^{\infty} \hat{u}_r(k)e^{ik\theta}, \quad 0 < r < 1,$$

$$Lu = \Delta u + \frac{\varepsilon(r)}{r} u_r \equiv 0 \equiv \sum_{k=-\infty}^{\infty} L(\hat{u}_r(k)e^{ik\theta})$$

$$= \sum_{k=-\infty}^{\infty} \left[\hat{u}_r'' + \frac{1+\varepsilon(r)}{r}\hat{u}_r' - \frac{k^2}{r^2}\hat{u}_r\right] e^{ik\theta}.$$

By the uniqueness of Fourier coefficients and the fact that u is bounded near the origin, we see that $\hat{u}_r(k)$ is a bounded solution of equation (2.2) and is therefore a unique constant multiple of $R_k(r)$. Condition (2.8) follows from (2.6), the root test, and the fact that the series (2.7) is absolutely convergent.

Condition (2.8) implies $\sum_{k=-\infty}^{\infty} |a_k| \rho^{|k|}$ converges for each $0 < \rho < 1$, and therefore if $0 < r < s < 1$, the Fourier coefficients of $u(r, \theta)$ satisfy

$$|a_k R_k(r)| \leq A|a_k| r^{|k|} = A|a_k|(r/s)^{|k|}s^{|k|} \leq Ms^{|k|},$$

and the analyticity follows.

For the converse, the series in (2.7) can be twice termwise differentiated because of (2.4) and (2.5). Since $R_k(r)e^{ik\theta}$ is a solution of (1.2), so is $u(r, \theta)$.

3. Boundary integral representation theorems

Setting $r = 1$ in (2.7) we see immediately that the boundary values of solutions to (1.2) must be the "Fourier" series $f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$ whose coefficients satisfy (2.8). Furthermore (2.7) says that each solution can be written as a convolution $u(r, \theta) = (Q_r * f)(\theta)$. The appropriate setting in which these statements have meaning and in which one can show that $u(r, \theta)$ "agrees" with $f(\theta)$ on the boundary, is the so-called space $\mathscr{H}'$ of *analytic functionals* or *hyperfunctions* [10], [8]. $\mathscr{H}'$ is the strong dual of a certain locally convex topological vector space $\mathscr{H}$ whose elements are the real analytic functions defined on the unit circle $|z| = 1$.

We list here the pertinent properties of $\mathscr{H}$ and $\mathscr{H}'$. For details see [8] and [6]. For each integer $n = 1, 2, \ldots$, denote by $\mathscr{H}_n$ the Banach space, with sup norm, of all 2π-periodic functions $\phi(\theta)$ which are continuous in the closed strip $|\text{Im } \theta| \leq 1/n$, analytic in the open strip $|\text{Im } \theta| < 1/n$. The topological space

$\mathscr{H} = \bigcup_{n=1}^{\infty} \mathscr{H}_n$ is the topological inductive limit of the $\mathscr{H}_n$'s. The topology on $\mathscr{H}$ is the finest locally convex topology for which each natural imbedding of $\mathscr{H}_n \to \mathscr{H}$ is continuous. In terms of Fourier series, the *elements* of $\mathscr{H}$ may be described by

$$\mathscr{H} = \left\{\phi \colon \phi(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}, \limsup_{|k| \to \infty} |c_k|^{1/|k|} < 1\right\}, \tag{3.1}$$

which is equivalent to saying, there exist constants $0 < r < 1$ and $M > 0$ such that

$$|c_k| = |\hat{\phi}(k)| \leq M r^{|k|}, \tag{3.2}$$

for all integers k.

$\mathscr{H}$ is a nonmetrizable, complete Montel space. The strong dual of $\mathscr{H}$ is $\mathscr{H}'$, which is also a Frechet space, as well as a Montel space, whose strong dual, in turn, is $\mathscr{H}$. A weakly convergent sequence in $\mathscr{H}$ or $\mathscr{H}'$ is also strongly convergent to the same limit [8, p. 370].

The Fourier coefficients of an element f of $\mathscr{H}'$ can be defined by

$$\hat{f}(k) = \frac{1}{2\pi} f(e^{-ik\theta}). \tag{3.3}$$

The Fourier coefficients (3.3) satisfy (2.8) and the Fourier series $\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ik\theta}$ converges in $\mathscr{H}'$ to f. Conversely any sequence $\{a_k\}$ satisfying (2.8) is the sequence of Fourier coefficients of a unique f in $\mathscr{H}'$. In addition, boundedness in $\mathscr{H}'$ and convergence of the Fourier coefficients imply strong convergence in $\mathscr{H}'$.

Convolution may be defined by

$$(f * g)(\theta) = \sum_{k=-\infty}^{\infty} a_k b_k e^{ik\theta}, \tag{3.4}$$

where $f = \sum a_k e^{ik\theta}$ and $g = \sum b_k e^{ik\theta}$. The mapping $f \to f * g$ is continuous for each $g \in \mathscr{H}'$.

Differentiation may be defined as is usually done for distributions, and it commutes with convolution.

We can now interpret Theorem 2.2 as a generalized Poisson integral representation.

THEOREM 3.1. *A function $u(r, \theta)$ in the unit disk is a solution to (1.2) if and only if there is a generalized function f in $\mathscr{H}'$ such that*

$$u(r, \theta) = u_r(\theta) = (Q_r * f)(\theta) \tag{3.5}$$

for each $0 < r < 1$. Furthermore $u_r \to f$ in $\mathscr{H}'$ as $r \to 1$ and consequently f is uniquely determined.

Except for the convergence at the boundary, this theorem is simply a restatement of Theorem 2.2, where $f(\theta) = \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}$, the a_k being determined by

(2.7). Condition (2.8) implies $f \in \mathscr{H}'$. Since $R_k(r)$ is increasing to 1 with r, Proposition 10 of [6] implies that the family of functions $Q_r \in \mathscr{H} \subset \mathscr{H}'$ is *bounded* in $\mathscr{H}'$. Hence $Q_r \to \sum_{k=-\infty}^{\infty} e^{ik\theta} = \delta$ in $\mathscr{H}'$ as $r \to 1$, where δ denotes the Dirac delta function (see the remarks preceding (3.4)). Hence $u_r = Q_r * f \to \delta * f = f$ in $\mathscr{H}'$, by the continuity of convolution. This completes the proof.

In the case f is a continuous function on the boundary, it is not, at first glance, clear that the solution $u_r = Q_r * f$ is continuous on the *closed* disk and equal to f on the boundary. That this is in fact true can be shown as a corollary of Theorem 3.1. Let $v(r, \theta) = v_r(\theta)$ denote the unique classical solution which agrees with f on the boundary [5, p. 176]. Then $v_r \to f$ uniformly as $r \to 1$ and therefore $v_r \to f$ in $\mathscr{H}'$. By the uniqueness part of Theorem 3.1, $v \equiv u$.

Finally, Johnson's structure theorem [6, Theorem 6] for elements in $\mathscr{H}'$ allows us to prove the following representation theorem.

THEOREM 3.2. *A function $u(r, \theta)$ in the unit disk is a solution to* (1.2) *if and only if there is a sequence $\{g_n\}$ of continuous functions on the unit circle such that*

$$\lim_{n\to\infty} (n! \, \|g_n\|_\infty)^{1/n} = 0 \tag{3.6}$$

and

$$u(r, \theta) = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} Q_r^{(n)}(\theta - t) g_n(t) \, dt. \tag{3.7}$$

The convergence in (3.7) is *subuniform* (uniform on compact subsets of the unit disk). If f is in $\mathscr{H}'$, then the above mentioned theorem states that $f = \sum_{n=0}^{\infty} g_n^{(n)}$ for a sequence of continuous functions on the unit circle satisfying (3.6), the convergence being, of course, in $\mathscr{H}'$, and the differentiation in the sense of distributions. Thus

$$u_r = Q_r * f = \sum_{n=0}^{\infty} Q_r * g_n^{(n)} = \sum_{n=0}^{\infty} Q_r^{(n)} * g_n,$$

and (3.7) is established with convergence in $\mathscr{H}'$. To show the convergence in (3.7) is subuniform, we need only show the right-hand side converges subuniformly, therefore converges in $\mathscr{H}'$, and then invoke the uniqueness of convergence in $\mathscr{H}'$.

If $0 < r < 1$ is fixed, then simple estimates based on (2.4) yield that the right-hand side of (3.7) is dominated in absolute value by a constant times $\sum_{k=0}^{\infty} (\sum_{n=0}^{\infty} \|g_n\|_\infty k^n) r^k$, in the disk $|z| \le r < 1$. But this series converges, for if one chooses $\varepsilon < \ln (1/r)$, then, except for a constant factor, (3.6) implies that this series is dominated by

$$\sum_{k=0}^{\infty} \left(\sum_{n=0}^{\infty} \frac{(\varepsilon k)^n}{n!} \right) r^k = \sum_{k=0}^{\infty} (e^{\varepsilon} r)^k < \infty,$$

since $e^{\varepsilon} r < 1$.

4. Generalizations

We conclude with some observations on our methods. Given an elliptic equation $Lu = 0$, one tries to characterize all solutions in the unit disk by first finding a kernel solution $Q_r(\theta)$ whose Fourier coefficients $R_k(r)$ behave essentially like the Fourier coefficients of the Poisson kernel. Depending on the nature of L, it is possible that $Q_r(\theta)$ will *not* be real analytic in θ for each r. A description of solutions through their distributional boundary values can then no longer be facilitated by means of $\mathscr{H}$ and $\mathscr{H}'$. What is needed is a new test space, call it $\mathscr{G}$, and its corresponding dual $\mathscr{G}'$. Using as motivation the characterization of elements in $\mathscr{H}$ given by (3.2), we adopt the following:

DEFINITION 4.1. Let $\mathscr{G}$ be the set of all complex valued integrable functions ϕ on the unit circle for which there exist constants $0 < r < 1$ and $M > 0$ such that

$$|\hat{\phi}(k)| \leq M\,|R_k(r)|, \quad \text{for all integers } k. \tag{4.1}$$

A locally convex topology can be given to $\mathscr{G}$, along the same lines as was done for $\mathscr{H}$. Then $\mathscr{G}$, and its strong dual $\mathscr{G}'$, can be shown to have at least all the properties of $\mathscr{H}$ and $\mathscr{H}'$ listed in Section 3. Theorems 2.2 and 3.1 can then be proved for solutions to the equation $Lu = 0$. A technical point that arises in this development is that $R_k(r)$ must satisfy some multiplicative property analogous to $r^k s^k = (rs)^k$, which is satisfied by the Fourier coefficients $R_k(r) = r^{|k|}$ of the Poisson kernel. The exact form of this property that is used is: *for each pair* $0 \leq a < b < 1$, *there exist* $0 < c < 1$ *and* $M > 0$ *such that*

$$|R_k(a)| \leq M\,|R_k(b)R_k(c)|, \quad \textit{for all integers } k. \tag{4.2}$$

For details the reader is referred to [12].

An example of an operator L for which the corresponding $\mathscr{G}$ does not reduce to $\mathscr{H}$ will appear in a subsequent paper.

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