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SOLVABLE GROUPS ADMITTING AN "ALMOST FIXED POINT FREE" AUTOMORPHISM OF PRIME ORDER

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1. Introduction and notation

In [9], Thompson has proved that a group admitting a fixed point free automorphism of prime order is necessarily nilpotent. In this paper, we relax somewhat the fixed point free hypothesis on the automorphism, but we do assume that the group in question is solvable. The specific hypothesis considered is the following:

Hypothesis 1.1. P is a group of prime order p, N is a solvable group and P acts on N as a group of automorphisms in such a way that for every prime divisor r of $|N|$, $[R, P] = R$ holds for every P-invariant Sylow r-subgroup R of N.

If p is not a Fermat prime (i.e., p is not of the form $1 + 2^s$) then the group N in the above hypothesis is necessarily nilpotent. This fact is a consequence of results appearing in a paper of E. Shult [8], although it is not explicitly stated there. A complete proof is given here.

The interesting case, occupying the bulk of this paper, is when p is a Fermat prime. In Section 4 we show that if $p \geq 17$, then N has a nilpotent normal 2-complement, equivalently, $N/\mathbf{F}(N)$ is a 2-group, where $\mathbf{F}(N)$ is the Fitting subgroup of N. For the remaining Fermat primes (3 and 5), $N/\mathbf{F}(N)$ need not be a 2-group, but some of its structure is determined. In particular, the possible prime divisors of the order of $N/\mathbf{F}(N)$ are determined (see Theorem 4.2(c)).

Whenever one group A acts on another group B as a group of automorphisms, the usual semidirect product AB may be constructed, and this idea is used implicitly throughout this paper. One frequent occurance of this is the case when B is an $F[A]$-module for some field F. Another obviously is $A = P$ and $B = N$ in the situation of hypothesis 1.1. Notice that this hypothesis is an example of a coprime action, as $|N|$ is necessarily prime to p.

The notation used throughout this paper is standard we hope, and we use [3] and [5] as general references for the standard group theoretical results needed. We also use [2] as a general reference for representation theory.

If G is a finite group, Irr (G) denotes the set of irreducible (complex) characters of G, and for $\chi \in$ Irr (G), let det $\chi \in$ Irr (G) denote the linear character

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defined by

$$(\det \chi)(g) = \det \mathfrak{X}(g)$$

where $\mathfrak{X}$ is any representation affording χ. If $N \lhd G$, we sometimes view characters of G/N as characters of G with N in their kernels (and similarly with Brauer characters).

2. Some preliminary lemmas and needed facts

LEMMA 2.1. *Let G be a group of the form PR where $|P| = p$ is a prime, and $R = [R, P]$ is a nontrivial r-group for some prime $r \neq p$. Assume $Z(R)$ is cyclic, P acts trivially on $Z(R)$, and that every characteristic abelian subgroup of R is contained in the center of R. Then R is extra special, and if r is odd,* $\exp(R) = r$.

Proof. See Lemma 1.2 of [4].

Let V be an irreducible $F[G]$-module over a field F of characteristic q. Then $\Delta = \mathrm{End}_{F[G]}(V)$ is a division ring finite dimensional over F, and if E is a maximal subfield of Δ (necessarily containing F) then V may be viewed as an irreducible $E[G]$-module. Since $\mathrm{End}_{E[G]}(V) = E \cdot 1_V$, the module V is absolutely irreducible as an $E[G]$-module. By a "Brauer character of V" we shall always mean a Brauer character of V viewed as an $E[G]$-module. It need not be uniquely determined, but this does not matter for our purposes. Notice that the degree of a Brauer character associated with V is $\leq \dim_F V$, with equality iff F is a splitting field for V.

Part (a) of the following lemma is implied by Theorem 3.1 and Corollary 3.2 of [8].

LEMMA 2.2. *Suppose PS is a group where P has prime order $p > 2$, and $S = [S, P]$ is a nontrivial s-group where s is a prime (different from p). Let V be an $F[PS]$-module where the characteristic of F is $q \neq s, p$. Finally assume $[V, P] = V$. Then:*

(a) *If $[V, S] \neq \{0\}$, then $s = 2$ and p is a Fermat prime.*
(b) *If PS is faithful and irreducible on V, then $\dim_F V = p - 1$, and S is an extra special 2-group of order $2(p-1)^2$. Moreover, the $F[PS]$-module V is unique up to isomorphism, and so is the group PS.*

Proof (Sketch). Notice that if $[V, S] \neq 0$ then $C_{PS}(U)$ is properly contained in S for any irreducible submodule U of V which is contained in $[V, S]$. Thus, the hypothesis of (b) are satisfied with PS replaced by $PS/\mathbf{C}_{PS}(U)$ and V replaced by U, and so (b) implies (a).

Assume now the hypothesis of (b) is satisfied. Let $E = \mathrm{End}_{F[PS]}(V)$ so that E is a finite field containing an isomorphic copy of F and view V as an $E[PS]$-

module. Let $U = V \otimes_E K$ where K is a finite extension of E such that K is a splitting field for all subgroups of PS.

By standard arguments (Clifford's Theorem and Mackey's Theorem) it is easy to establish that U_S is irreducible and U_{S_0} is homogeneous for all $S_0 \lhd PS$ with $S_0 \subseteq S$. By Lemma 2.1 then, S is extra special of order s^{2a+1} say.

If ϕ is the Brauer character of U then $\phi \in \mathrm{Irr}\,(PS)$ as $q \nmid |PS|$. Also $[U, P] = U$ implies $(\phi_P, 1_P)_P = 0$. Now the characters of PS may be computed (see Satz 17.13 on p. 574 of [5]) and in particular, $\phi_P = m\rho + \delta\mu$ where ρ is the regular character of P, $\mu \in \mathrm{Irr}\,(P)$ and $\delta = \pm 1$. Since $(\phi_P, 1_P)_P = 0$, it follows that $m = 1$, $\mu = 1_P$ and $\delta = -1$. Thus $\phi_P = \rho - 1_P$.

Hence, $s^a = \phi(1) = p - 1$. Since $p > 2$, $p - 1$ is even, forcing $s = 2$ and $p = 1 + 2^a$, a Fermat prime. Also $|S| = 2^{2a+1} = 2(p-1)^2$.

Again, from the character theory of PS, ϕ must be rational valued. Hence $\mathrm{tr}\,(x_U) \in GF(q) \subseteq F$, where $x_U\colon U \to U$ is the linear transformation determined by $x \in PS$. All Schur indices for finite fields are trivial, so $F = E$ and $\dim_K U = \dim_F V = p - 1$, and V is unique up to isomorphism. Finally, there are two extra special groups of order $2(p-1)^2$ up to isomorphism, but only one of these admits a group of automorphisms of order p. A Sylow p-subgroup of the full automorphism group of this group has order p, and it readily follows that the group PS is unique up to isomorphism.

COROLLARY 2.3. *Let PS be a group where P has prime order $p > 2$, and $S = [S, P]$ is a nontrivial s-group for some prime $s \neq p$. Assume PS acts faithfully on a finite abelian group A having order prime to ps and which satisfies $[A, P] = A$. Then $s = 2$, p is a Fermat prime and S is a subdirect product of isomorphic extra special 2-groups, each having order $2(p-1)^2$.*

Proof. Since the order of A is prime to ps, PS acts faithfully on $A/\phi(A)$ and $[A/\phi(A), P] = A/\phi(A)$. Clearly, we may replace A by $A/\phi(A)$ so as to assume $\phi(A) = 1$. Then, A is a completely reducible abelian group under PS and we may write $A = V_1 \dotplus V_2 \dotplus \cdots \dotplus V_k$ where each V_i is an irreducible $GF(q_i)[PS]$-module for some prime q_i different from p and s. Then, $[V_i, P] = V_i$ for all i, and PS is faithful on $[A, S]$. It follows that S is a subdirect product of the groups $S/\mathbf{C}_{PS}(V_i)$ where i ranges over all indices for which $[V_i, S] = V_i$, and we are done by Lemma 2.2(b).

LEMMA 2.4. *Let P be a cyclic group of prime order p, and let P act on a group N satisfying $O^{q,r}(N) < O^q(N) < N$. Assume*

$$[N/O^q(N), P] = N/O^q(N) \quad \textit{and} \quad [O^q(N)/O^{q,r}(N), P] = O^q(N)/O^{q,r}(N).$$

If q is odd, or if p is not a Fermat prime, then

$$O^{q,r}(N) = O^q(N) \cap O^r(N).$$

Proof. In general, $O^{q,r}(N) \leq O^q(N) \cap O^r(N)$, and the lemma is unaffected if $O^{q,r}(N)$ is factored out. Thus, we may assume $O^{q,r}(N) = 1$. Let $R = O^q(N)$ and let Q be a P-invariant Sylow q-subgroup of N. Then R is the unique Sylow r-subgroup of N. If $\phi(R) \neq 1$, then by induction, $Q\phi(R)/\phi(R) \lhd N/\phi(R)$. Hence $[Q, R] \leq \phi(R)$, and so $[Q, R] = 1$, proving the lemma. Thus, we may assume $\phi(R) = 1$ so that R is a vector space over $GF(r)$. For odd p, the hypotheses of Lemma 2 are satisfied with $S = Q$ and $R = V$. Thus, if $[Q, R] \neq 1$ then $q = 2$ and p is a Fermat prime, a contradiction, and we are finished in this case. Thus, we may assume $p = 2$. If $O_q(N) > 1$, then by induction $Q/O_q(N)$ is normal in $N/O_q(N)$ and hence $Q \lhd N$. Thus, we may assume $O_q(N) = 1$. Hence, PQ is faithful on R. Now, P inverts some element $x \neq 1$ of Q and so P acts without fixed points on $\langle x \rangle R$. By Thompson's theorem [9], $\langle x \rangle R$ is nilpotent, which contradicts that $\langle x \rangle$ is faithful on R. (Actually, since $p = 2$, a more elementary argument can be used to prove directly that $\langle x \rangle R$ is abelian.)

3. Representation theory

This section contains some technical results from representation theory which will be useful for the next section.

LEMMA 3.1. *Let G be a group of the form PR where P has prime order p and $R = [R, P]$ is a nontrivial r-group for some prime r (necessarily different from p). Let U be an $F[PR]$-module where F is a finite field of characteristic r. Then, $[RU, P] = RU$ if and only if* $\hom_{F[PR]}(U, F) = \{0\}$.

Proof. In general, we have $[RU, P] = [R, P][R, P, U][U, P]$. Since $[R, P] = R$, this simplifies to $[RU, P] = R[R, U][U, P]$. The last two factors are contained in U. Thus, $[RU, P] = RU$ is equivalent to $[R, U][U, P] = U$. In additive form, this may be written as $U = [U, R] + [U, P]$. Notice, $[U, R]$ is the radical of U when viewed either as an $F[R]$-module or as an $F[PR]$-module, and the equation is equivalent to the statement that U does not have the principal $F[PR]$-module as a homomorphic image, i.e., $\hom_{F[PR]}(U, F) = \{0\}$.

LEMMA 3.2. *Let V be an irreducible $F[G]$-module and W an irreducible $F[H]$-module where H is a subgroup of G. Assume F is a finite field which is a splitting field for all subgroups of G. Assume also that the Brauer characters of V and W may be lifted to ordinary irreducible characters, say χ and λ respectively, and that $(\chi_H, \lambda)_H \neq 0$. Then W is a homomorphic image of V_H.*

Proof. Let the characteristic of F be q. Then, F is obtained from the prime subfield by adjoining a primitive mth root of unity, where $q \nmid m$. Moreover, since F is a splitting field for all the cyclic subgroups of G, it follows that the exponent of G divides $q^a m$ for some a. Let K denote the algebraic number field obtained by adjoining a primitive $q^a m$th root of 1 to the rationals. Hence, K is a splitting field (in characteristic 0) for all subgroups of G. Let R be the localization of the algebraic integers of K relative to some prime ideal containing q, and let $\mathscr{P}$ denote the unique prime ideal of R. Then $R/\mathscr{P} \cong F$, and we may regard $R/\mathscr{P} = F$. As R is a P.I.D., χ is realizable in R, and we choose an R-free $R[G]$-module, say X_0 which affords χ. Let $X = X_0 \otimes_R K$, and regard $X_0 \subseteq X$. Thus, X is an irreducible $K[G]$-module affording χ. Since χ is a lift of the Brauer character affording V, the $F[G]$-module $X_0/\mathscr{P}X_0$ is isomorphic to V.

By hypothesis, $(\chi_H, \lambda)_H \neq 0$, so that λ is a constituent of χ_H. Since X_H is completely reducible, it follows that X_H contains a maximal $K[H]$-module, say M, such that X/M affords λ. Let $M_0 = M \cap X_0$. Then M_0 is an R-pure $R[H]$-submodule of X_0, and the quotient X_0/M_0 is a free R-module. (This construction of M_0 in X_0 is the same idea appearing in Theorem 1 of [10]). Now, X_0/M_0 affords λ, and it follows that the $F[H]$-module $(X_0/M_0)/\mathscr{P}(X_0/M_0)$ is isomorphic to W, as λ is a lift of the Brauer character for W. Thus, $X_0/\mathscr{P}X_0$ maps onto W, and since $V \cong X_0/\mathscr{P}X_0$, we have $\hom_H(V_H, W) \neq \{0\}$.

The next technical lemma is the first indication that, in the situation of Hypothesis 1.1, exceptional sets of primes will have to be considered in case p is 3 or 5.

LEMMA 3.3. *Let G be a group of the form $G = PSQ$, where P is a cyclic group of prime order p, and p is a Fermat prime. Assume $Q = [Q, P]$ is a normal q-subgroup of G and that Q is an extra special q-group of order q^p and exponent q. Also, assume that $S = [S, P]$ is an extra special 2-group of order $2(p-1)^2$ and that PS is faithful and irreducible on $Q/Z(Q)$. Let λ be a nonprincipal irreducible character of G with kernel SQ, and let χ be a faithful irreducible character of PSQ whose restriction to Q is irreducible, and which is canonical for χ_Q. Then $(\chi_{PS}, \lambda^k_{PS})_{PS} \neq 0$ for $0 \leq k \leq p-1$ unless $p = 3$ and $q \in \{5, 7, 11, 13, 23\}$ or unless $p = 5$ and $q \in \{3, 7, 11\}$.*

Proof. Since χ_Q is irreducible, it follows that $Z(Q) = Z(G)$. Moreover, if $Z(S) = \langle s \rangle$, then s inverts $Q/Z(Q)$ and centralizes $Z(Q)$. It follows that $I = \{y \in Q \mid y^s = y^{-1}\}$ is a set of coset representations for $Z(Q)$ in Q, and obviously, PS permutes I.

Let X be a $K[PSQ]$-module affording χ where K is a splitting field of characteristic zero for χ. Now, if $\rho\colon G \to GL(X)$ is the corresponding representation (i.e., $v\rho(g) = vg$ for $v \in X$, $g \in G$) then G acts on $\operatorname{End}_K(X)$ as follows: for $f \in \operatorname{End}_K(X)$ and $g \in G$, $f^g = \rho(g)^{-1}f\rho(g)$. The space $\operatorname{End}_K(X)$ may then be viewed as a $K[PSQ]$-module in the natural way, and the character of this module is $\chi\bar{\chi}$.

Since ρ_Q is irreducible, we know from the representation theory of Q that $\rho(I)$ is a basis for $\operatorname{End}_K(X)$. Thus, $\operatorname{End}_K(X)$ is a permutation module for PS. Moreover, since $1 \in I$ is the only element fixed by P, P acts fixed point freely on $I - \{1\}$ and hence on $\rho(I) - \{\rho(1)\}$. Thus, except for 1, all point stabilizers are contained in S. Thus, we have

$$(\chi\bar{\chi})_{PS} = 1_{PS} + \sum_i (1_{S_i})^{PS}$$

where the S_i are subgroups of S (possibly with repetition).

Now, with λ as in the statement of the lemma, we have, for any i,

$$(1_{PS} - \lambda_{PS}, 1_{S_i}^{PS})_{PS} = (1_{S_i} - \lambda_{S_i}, 1_{S_i})_{S_i} = 0$$

as ker $\lambda \supseteq S \supseteq S_i$. Clearly $(1_{PS} - \lambda_{PS}, 1_{PS})_{PS} = 1$, and it follows that

$$(1_{PS} - \lambda_{PS}, \chi\bar{\chi})_{PS} = 1.$$

But $(1_{PS} - \lambda, \chi\bar{\chi})_{PS} = ((1_{PS} - \lambda)\chi, \chi)_{PS}$. Write

$$\chi_{PS} = \sum_{j=0}^{p-1} a_j\lambda^j + \sum_{j=0}^{p-1} b_j\lambda^j\psi + \eta$$

where $\psi \in \operatorname{Irr}(PS)$ is the unique faithful, rational character, and η is a sum of characters of the form μ^{PS}, where μ is a linear character of S. Then $(1 - \lambda)\eta = 0$ so that

$$((1_{PS} - \lambda_{PS})\chi_{PS}, \chi_{PS})_{PS} = \sum_{j=0}^{p-1} a_j(a_j - a_{j-1}) + \sum_{j=0}^{p-1} b_j(b_j - b_{j-1}) = 1$$

where all subscripts are read mod p. Hence,

$$\sum_{j=0}^{p-1} (a_j - a_{j-1})^2 + \sum_{j=0}^{p-1} (b_j - b_{j-1})^2 = 2\left\{\sum_{j=0}^{p-1} a_j(a_j - a_{j-1}) + \sum_{j=0}^{p-1} b_j(b_j - b_{j-1})\right\}$$
$$= 2.$$

We already know χ_{PS} is rational valued, so $a_1 = a_2 = \cdots = a_p$ and $b_1 = b_2 = \cdots = b_p$. Thus, the above equality yields either $a_0 = a_1$ (and $|b_0 - b_1| = 1$) or $|a_0 - a_1| = 1$ (and $b_0 = b_1$). The lemma asserts that $a_0 a_1 \neq 0$ except for $p = 3$, $q \in \{5, 7, 11, 13, 23\}$ and $p = 5$, $q \in \{3, 7, 11\}$.

To compute these inner products, we need to compute some values of χ_{PS}. We remark that I. M. Isaacs, in a different, more general context, has described an algorithm for computing values of the canonical extension χ of χ_Q. See [6].

Let $x \in S$ be a noncentral involution (x exists only for $p > 3$). Now $\psi(x) = 0$ where $\psi \in \operatorname{Irr}(PS)$ is the unique faithful, rational character. Thus, exactly half of the "eigenvalues for $\psi(x)$" are negative ones, and the other half are ones. As ψ is the Brauer character for $Q/Z(Q)$, it follows that $|\mathbf{C}_{Q/Z(Q)}(x)| = q^{(p-1)/2}$. Since $(\chi\bar{\chi})_{PS}$ is the permutation character of PS on $Q/Z(Q)$, it follows that $\chi(x)^2 = q^{(p-1)/2}$ and so $\chi(x) = \delta q^{(p-1)/4}$, where δ is a sign. We use the fact that det $\chi(x) = 1$ to compute δ. Suppose $\rho(x)$ has u eigenvalues equal to 1, and v eigenvalues equal to -1. Then

$$\chi(x) = u - v = \delta q^{(p-1)/4} \quad \text{and} \quad \chi(1) = u + v = q^{(p-1)/2}.$$

As det $\chi(x) = 1$ it follows that v is even. Solving for v yields

$$v = \frac{q^{(p-1)/2} - \delta q^{(p-1)/4}}{2}$$

Therefore, the numerator must be congruent to 0 mod 4. Now if $p > 5$, then $(p-1)/2$ and $(p-1)/4$ are powers of 2, both ≥ 2, and

$$q^{(p-1)/2} \equiv q^{(p-1)/4} \equiv 1 \mod 4.$$

This proves $\delta = 1$. If $p = 5$, then the numerator is congruent to $1 - \delta q$ mod 4 so $q \equiv \delta \pmod 4$. Hence δ is determined.

If $x \neq 1$ is any element of PS which is not a noncentral involution, then $\langle x \rangle$ acts Frobeniusly on $Q/Z(Q)$, which implies that $\chi(x)^2 = |\mathbf{C}_{Q/Z(Q)}(x)| = 1$. Thus $\chi(x) = \pm 1$.

The case $p \geq 17$. In this case, all character values for χ_{PS} are ≥ -1 and we have

$$\begin{aligned} a_0 &= (\chi_{PS}, 1_{PS})_{PS} \\ &= \frac{1}{|PS|} \sum_{x \in PS} \chi(x) \\ &\geq \frac{1}{|PS|}(\chi(1) - (|PS| - 1)) \\ &> \frac{\chi(1)}{|PS|} - 1. \end{aligned}$$

Now $\chi(1) = q^{(p-1)/2}$ and $|PS| = (1 + 2^s) \cdot 2^{2s+1}$ where $p = 1 + 2^s$, $s \geq 4$. Thus a_0 will be greater than one if

$$q^{2^{s-1}} > 2 \cdot (1 + 2^s) \cdot 2^{2s+1}.$$

This last inequality is implied by the inequality

$$q^{2^{s-1}} > 2 \cdot 2^{s+1} \cdot 2^{2s+1} = 2^{3s+3}$$

which is equivalent to $q > 2^{(3s+3)/2^{s-1}}$. Since $p = 1 + 2^s$ is a Fermat prime, s is a power of 2, and in our case $s \geq 4$. If $s \geq 8$, the fractional exponent in the last inequality is less than 1 so that all odd primes q satisfy it. For $s = 4$ the exponent is $15/8 < 2$, so all odd primes $q > 2^2 = 4$ satisfy the inequality. It suffices now to compute a_0 for $p = 17$ and $q = 3$. In this case, S cannot be a central product of 4 dihedral groups of order 8, as, in this group there are $2^8 - 2^4 = 240$ noncentral involutions (which could not be fixed point freely permuted by an automorphism of order 17, as 17 does not divide 240). Hence, S is the central product of three dihedral groups with one quaternion group. For this group, the number of noncentral involutions is $17 \cdot 14$. Hence

$$a_0 \geq \frac{1}{17 \cdot 512}(3^8 + 17 \cdot 14 \cdot 3^4 - (17 \cdot 512 - 17 \cdot 14 - 1)) > 1.$$

We have now shown that for $p \geq 17$, $a_0 > 1$ and hence $a_1 \geq 1$. The lemma is now proved for $p \geq 17$.

The cases $p = 5$ and $p = 3$. The explicit values of χ_{PS} are needed in order to handle the cases $p = 5$ and $p = 3$.

For simplicity, write $\chi(x) = \chi(k)$ if the order of x is k and $k \neq 2$. Let $\chi(2)$ denote the value $\chi(s)$ where s is the unique central involution of S, and let $\chi(2')$ denote $\chi(x)$ where x is a noncentral involution of S (which exists for $p = 5$, but not when $p = 3$). We already know $\chi(2') = \delta q$ where δ is the unique sign satisfying $q \equiv \delta \pmod 4$. The values of χ_{PS} at all other nonidentity elements are signs. Let s be the central involution of S. If s has 1 as an eigenvalue with multiplicity u on X, and -1 with multiplicity v, then

$$u - v = \chi(2), \qquad u + v = \chi(1) = q^{(p-1)/2}.$$

Thus $2v = q^{(p-1)/2} - \chi(2)$. Now, $\det \chi_{PS} = 1_{PS}$, so v must be even, and $q^{(p-1)/2} \equiv \chi(2) \pmod 4$. This determines $\chi(2)$ as $\chi(2)$ is a sign. In fact, for $p = 5$, $\chi(2) \equiv q^2 \equiv 1 \pmod 4$, so $\chi(2) = 1$.

Now let x be an element of order 4. As x^2 is a central involution, and $\chi(x)$ is a sign, we have

$$\chi(4) = \chi(x) = u_1 - u_2 + (v/2)i + (v/2)(-i) = u_1 - u_2$$

where $u_1 + u_2 = u = (q^{(p-1)/2} + \chi(2))/2$. Now, u_2 must be even, as $\det \chi(x) = 1$. But

$$u_2 = \frac{q^{(p-1)/2} + \chi(2) - 2\chi(4)}{4},$$

and thus $q^{(p-1)/2} + \chi(2) - 2\chi(4) \equiv 0 \pmod 8$. This determines $\chi(4)$ uniquely. Notice that for $p = 5$,

$$q^{(p-1)/2} = q^2 \equiv 1 \pmod 8,$$

so $\chi(2) = 1$ and hence $\chi(4) = 1$. The only elements remaining are those of orders $2p$ and p. Now, since χ_{PS} is rational valued, we have $\chi(g^p) \equiv \chi(g) \pmod p$ for any $g \in PS$. Thus

$$\chi(2p) \equiv \chi(2) \pmod p \quad \text{and} \quad \chi(p) \equiv \chi(1) \pmod p.$$

Thus $\chi(2p) = \chi(2)$ and $\chi(p)$ is the unique sign satisfying $q^{(p-1)/2} \equiv \chi(p) \pmod p$.

We now tabulate these values below. Let δ, δ_4 and δ_p denote the unique signs satisfying the congruences

$$\delta \equiv q \pmod 4, \quad \delta_4 \equiv q^{(p-1)/2} \pmod 4, \quad \delta_p \equiv q^{(p-1)/2} \pmod p.$$

Also, let ε be the sign satisfying $q + \delta_4 - 2\varepsilon \equiv 0 \pmod 8$.

Element of PS	Value when $p = 5$	Value when $p = 3$
1	q^2	q
2 (central)	1	δ_4
2′ (noncentral)	δ_q	(no such elements when $p = 3$)
4	1	ε
p	δ_p	δ_p
$2p$	1	δ_4

We note that for $p = 5$, S must be the central product of a dihedral group with a quaternion group. Otherwise, S would have 12 elements of order 4, which could not be permuted fixed point freely by an automorphism of order 5. Hence, when $p = 5$, S has 10 noncentral involutions, and 20 elements of order 4. Also, there are $\frac{1}{2}(p-1)|S| = 64$ elements of order p, and the same number of order $2p$. Thus

$$\begin{aligned} a_0 &= (\chi_{PS}, 1_{PS})_{PS} \\ &= \tfrac{1}{160}(q^2 + 1 + 10\delta q + 20 + 64(1 + \delta_p)) \\ &= \tfrac{1}{160}((q + 7\delta)(q + 3\delta) + 64(1 + \delta_p)). \end{aligned}$$

It is easy to check that $a_1 = (\chi_{PS}, \lambda)_{PS}$ satisfies $a_1 = a_0$ when $\delta_p = -1$ and $a_1 = a_0 - 1$ when $\delta_p = 1$. Now

$$a_0 \geq \tfrac{1}{160}((q-7)(q-3)) > 1 \quad \text{when } q \geq 19,$$

and hence it suffices to consider odd primes q less than 19 (and $\neq 5$). If q is 17 or 13 then $\delta = 1$ so

$$a_0 \geq \tfrac{1}{160}((13+7)(13+3)) > 1.$$

For the remaining primes ($q = 3, 7, 11$) we tabulate the following.

q	δ	δ_p	a_0	a_1
3	−1	−1	0	0
7	−1	−1	0	0
11	−1	1	1	0

This proves the lemma when $p = 5$.

When $p = 3$, S is the quaternion group of order 8, in which there are no noncentral involutions, and 6 elements of order 4. In PS there are 8 elements of order 3 and 8 of order 6. We have

$$a_0 = (\chi_{PS}, 1_{PS})_{PS} = \tfrac{1}{24}(q + \delta_4 + 6\varepsilon + 8\delta_p + 8\delta_4).$$

Thus, $a_0 \geq (1/24)(q - 23) > 1$, when $q > 47$, and hence a_0, $a_1 \neq 0$ when $q > 47$. We remark that when δ_p and δ_4 are of opposite signs, then $a_0 = a_1$ and for $\delta_p = \delta_4 = 1$ we have $a_1 = a_0 - 1$, while for $\delta_p = \delta_4 = -1$ we have $a_1 = a_0 + 1$. The following table is easily worked out:

q	5	7	11	13	17	19	23	29	31	37	41	43	47
δ_4	1	-1	-1	1	1	-1	-1	1	-1	1	1	-1	-1
ε	-1	-1	1	-1	1	1	-1	-1	-1	-1	1	1	-1
δ_p	-1	1	-1	1	-1	1	-1	-1	1	1	-1	1	-1
a_0	0	0	0	1	1	1	0	1	1	2	2	2	1
a_1	0	0	1	0	1	1	1	1	1	1	2	2	2

From the table, we have $a_1 a_0 = 0$ exactly when $q \in \{5, 7, 11, 13, 23\}$, and this completes the entire proof of Lemma 3.3.

COROLLARY 3.4. *Let G have a normal series $Q \lhd SQ \lhd PSQ$ where P is a cyclic group of prime order p, $S = [S, P]$ is an extra special 2-group of order $2(p-1)^2$ and $Q = [Q, P]$ is an extra special q group of order q^p and exponent q (where $2 \neq q \neq p$). Assume PS acts faithfully and irreducibly on $Q/Z(Q)$. Let U be a faithful irreducible $F[G]$-module where F is a finite field of characteristic 2 which is a splitting field for all subgroups of G. Assume U_Q is irreducible. Then*

$$\hom_{F[PS]}(U_{PS}, F) \neq \{0\}$$

unless $p = 3$ and $q \in \{5, 7, 11, 13, 23\}$ or unless $p = 5$ and $q \in \{3, 7, 11\}$.

Proof. Let L be a faithful, irreducible $F[PSQ/SQ]$-module, regarded as an $F[PSQ]$-module. Write

$$L^k = L \otimes_F L \otimes_F \cdots \otimes_F L \ (k \text{ times}).$$

Then, $U, U \otimes L, U \otimes L^2, \ldots, U \otimes L^{p-1}$ is a complete list of irreducible $F[PSQ]$-modules whose restriction to Q is U_Q. Notice

$$\begin{aligned}\hom_{F[PS]}((U \otimes L^k)_{PS}, F) &\cong \hom_{F[PS]}(U_{PS}, ((L^k)^\wedge \otimes F)_{PS}) \\ &\cong \hom_{F[PS]}(U_{PS}, L_{PS}^{p-k}).\end{aligned}$$

It follows that $(U \otimes L^k)_{PS}$ maps onto F for all k if and only if U_{PS} maps onto $(F \oplus L \oplus L^2 \oplus \cdots \oplus L^{p-1})_{PS}$. Notice that this last module is really just the regular $F[PSQ/SQ]$-module. It therefore suffices to prove that U_{PS} maps onto $F \oplus L \oplus L^2 \oplus \cdots \oplus L^{p-1}$ unless $p = 3$ and $q \in \{5, 7, 11, 13, 23\}$ or unless $p = 5$ and $q \in \{3, 7, 11\}$. Clearly, we may replace U by any of the modules $U \otimes_F L^k$, so that we may assume U is the "canonical" extension of U_Q. (This is the unique extension of U_Q to U satisfying $\det x_U = 1$ for every $x \in PS$).

From now on, U is an irreducible $F[G]$-module with U_Q irreducible and U canonical for U_Q. Let ϕ be the Brauer character of U. By the Fong-Swan Theorem, there exists an ordinary irreducible character χ of G such that χ agrees with ϕ on elements of odd order. (For a proof of the Fong-Swan Theorem, see Theorem 72.1 on p. 473 of [2]. There is a more conceptual, character theoretic proof of this theorem, given in [7]. In fact, because of the specific nature of the group $G = PSQ$, a separate argument may be given to prove the existence of χ, without appealing to the Fong-Swan Theorem.)

Clearly, the module L has a Brauer character which may be lifted to an ordinary character of PSQ/SQ, say λ. Since χ is the canonical extension of χ_Q, the previous lemma implies $(\chi_{PS}, \lambda^k_{PS})_{PS} \neq 0$ for $0 \leq k \leq p-1$ except when $p = 3$ and $q \in \{5, 7, 11, 13, 23\}$ or when $p = 5$ and $q \in \{3, 7, 11\}$. Thus, by Lemma 3.2, U_{PS} maps onto $(F \oplus L \oplus \cdots \oplus L^{p-1})_{PS}$ unless $p = 3$ and $q \in \{5, 7, 11, 13, 23\}$ or $p = 5$ and $q \in \{3, 7, 11\}$, and this completes the proof of Corollary 3.4.

4. Main results

The first theorem of this section is a generalization of Corollary 3.4.

THEOREM 4.1. *Let G be a group of the form $G = PSQ$ where $|P| = p > 2$ is a prime. Assume $Q = [Q, P] \lhd G$ is a q-group, and $S = [S, P]$ is a 2-group, where $2 \neq q \neq p$. Assume also that $\mathbf{C}_{PS}(Q) = 1$. Let U be a faithful $F[G]$-module where F is a finite field of characteristic 2, and suppose $U = [U, S] + [U, P]$. Then:*

(a) *$p = 3$ and $q \in \{5, 7, 11, 13, 23\}$ or $p = 5$ and $q \in \{3, 7, 11\}$.*
(b) *Q is a nonabelian group of exponent q and class 2.*

Proof. The hypothesis $U = [U, S] + [U, P]$, which by Lemma 3.1 is equivalent to $\hom_{F[PS]}(U_{PS}, F) = \{0\}$, is unchanged if we replace F by any finite extension field, say E, and U by $U \otimes_F E$. We may therefore assume that F is a splitting field for all subgroups of G. We now prove (a) and (b) together by induction on $\dim_F U + |G|$.

Since $O_2(G) = 1$, G acts faithfully on $U/J(U)$ where $J(U)$ is the radical of U. If $J(U) \neq \{0\}$, we are done by induction, so assume $J(U) = \{0\}$. Hence, U is completely reducible, and we may write $U = U_1 \dotplus \cdots \dotplus U_l$ where the U_i are simple $F[G]$-modules. If $Q \subseteq \mathbf{C}_G(U_i)$ for some i, then G is faithful on U/U_i and induction applies again. Thus, we may assume $Q \nsubseteq \mathbf{C}_G(U_i)$ for all i. Suppose $S \subseteq \mathbf{C}_G(U_i)$ for some i and let $\bar{G} = G/\mathbf{C}_G(U_i)$. Then $\bar{G} = \bar{P}\bar{Q}$ acts faithfully on U_i with $[\bar{Q}, \bar{P}] = \bar{Q}$ and $[U_i, \bar{P}] = U_i$. The hypotheses of Lemma 2.4 are satisfied with $N = \bar{Q}U_i$ and $r = 2$. By that lemma, $\bar{Q}$ centralizes U_i, which is a contradiction. Thus, $S \nsubseteq \mathbf{C}_G(U_i)$ for all i. Since $O_2(G/\mathbf{C}_G(U_i)) = \bar{1}$ for all i, it follows that $PS\mathbf{C}_G(U_i)/\mathbf{C}_G(U_i)$ acts faithfully on $Q\mathbf{C}_G(U_i)/\mathbf{C}_G(U_i)$ for all i. If $l > 1$, then induction applies and (a) follows. From (b), $Q\mathbf{C}_G(U_i)/\mathbf{C}_G(U_i)$ is a nonabelian q-group of exponent q and class 2, for each i. Since Q is a subdirect product of these groups, the same is true of Q itself.

Thus, we may assume $l = 1$, i.e., U is an irreducible $F[G]$-module. If U_{SQ} reduces, then $U \cong Y^G$ for some irreducible $F[SQ]$-module Y. Then $U_{PS} \cong Y^G|_{PS} \cong (Y_S)^{PS}$, and

$$\hom_{F[PS]}(U_{PS}, F) \cong \hom_{F[PS]}((Y_S)^{PS}, F) \cong \hom_{F[S]}(Y_S, F) \neq \{0\},$$

a contradiction. Hence, U_{SQ} is irreducible.

Suppose U_Q is not homogeneous. Then, by standard arguments, U is induced from a proper subgroup of the form PS_0Q where S_0 is a P-invariant subgroup of S. We may assume that PS_0Q is a maximal subgroup of PSQ so that $S_0 \lhd S$, and P acts irreducibly on S/S_0. Write $U \cong V^{PSQ}$ where V is an $F[PS_0Q]$-module.

Set $S_1 = [S_0, P]$, and assume $[V, PS_1] < V$. Now, $S_0 = S_1\mathbf{C}_{S_0}(P)$, and $\mathbf{C}_{S_0}(P)$ normalizes PS_1. Hence, S_0 normalizes PS_1 and also $[V, PS_1]$. Thus $[V, PS_1]$ is an $F[S_0]$-submodule of V. Clearly, the $F[S_0]$-submodules of V which contain $[V, PS_1]$ are stabilized by P and hence are $F[PS_0]$-submodules. Let W be a maximal $F[S_0]$-submodule of V containing $[Y, PS_1]$. As char $F = 2$, it follows that V/W is the principal $F[PS_0]$-module, and $\hom_{F[PS_0]}(V, F) \neq \{0\}$. However, $U \cong V^G$, so $U_{PS} \cong (V_{PS_0})^{PS}$ and

$$\begin{aligned}\{0\} &= \hom_{F[PS]}(U_{PS}, F)\\ &\cong \hom_{F[PS]}((V_{PS_0})^{PS}, F)\\ &\cong \hom_{F[PS_0]}(V_{PS_0}, F)\\ &\neq \{0\}.\end{aligned}$$

This contradiction proves that $[V, PS_1] = V$.

Let $\overline{PS_1Q} = PS_1Q/\mathbf{C}_{PS_1Q}(V)$. Since $PS_1Q \lhd PS_0Q$ and V is an irreducible $F[PS_0Q]$-module, it follows that V_{PS_1Q} is completely reducible, and hence $O_2(\overline{PS_1Q}) = 1$. From this, it follows that $\overline{PS_1}$ acts faithfully on $\bar{Q}$. If $\bar{S}_1 = 1$, then the hypotheses of Lemma 2.4 are satisfied with $N = \bar{Q}V$ and $r = 2$, so $\bar{Q}$ centralizes V. But then $\bar{Q} = 1$, so Q centralizes $V^G \cong U$, as $Q \lhd G$. This contradiction proves $\bar{S}_1 \neq 1$ so that induction applies in the group $\overline{PS_1Q}$ (with U replaced by V_{PS_1Q}). Thus, (a) is satisfied, and $Q\mathbf{C}_{PS_1Q}(V)/\mathbf{C}_{PS_1Q}(V)$ is a q-group of exponent q and class 2. Since the core of $\mathbf{C}_{PS_1Q}(V)$ is trivial, Q itself has exponent q and class 2.

We are now led to the case in which U_Q is homogeneous. Since SQ/Q is a 2-group, this implies U_Q is irreducible in fact.

Suppose U_{Q_0} is not homogeneous for some normal subgroup Q_0 of G contained in Q. Choose Q_0 with $|Q_0|$ as large as possible with this property. Choose a homogeneous component of U_{Q_0} in such a way that PS is contained in its inertia group. If PSQ_1 is the inertia group of this module, then $U \cong Y^G$ for some $F[PSQ_1]$-module. Let PSQ_2 be a maximal subgroup of G containing PSQ_1, and let $Y_2 = Y^{PSQ_2}$. Hence $Y_2^G \cong U$ and $Q_2 \lhd G$. Now

$$U_Q \cong (Y_2)^G|_Q = ((Y_2)_{Q_2})^Q$$

so $U_{Q_0} \cong ((Y_2)_{Q_2})^Q|_{Q_0}$. Hence U_{Q_0} reduces into $|Q:Q_0|$ distinct conjugates. By maximality of Q_0, and $Q_0 \subseteq Q_1 \subseteq Q_2$ it follows that $Q_0 = Q_2$ (and $Y_2 = Y$).

Let $\mathcal{O}$ be a nontrivial orbit of PS on Q/Q_0, and choose $xQ_0 = \bar{x} \in \mathcal{O}$. Since P acts fixed point freely on Q/Q_0, the stabilizer of $\bar{x}$ in PS is some subgroup of S, say S_0. Then, S_0 acts on xQ_0 by conjugation, and since S_0 is a 2-group, S_0 must centralize some element of xQ_0. We may assume that x is centralized by S_0. Clearly $Y \otimes x$ is an $F[S_0]$-submodule of Y^G, and $(Y \otimes x)^{PS}$ is a direct summand of $Y^G|_{PS}$ from Mackey's theorem. Since $Y^G|_{PS} \cong U_{PS}$, this implies

$$\hom_{F[PS]}((Y \otimes x)^{PS}, F) = \{0\}.$$

But

$$\hom_{F[PS]}((Y \otimes x)^{PS}, F) \cong \hom_{F[S_0]}((Y \otimes x)_{S_0}, F) \neq \{0\}.$$

This contradiction proves that U_{Q_0} is homogeneous for all normal subgroups Q_0 of G contained in Q.

In particular, every characteristic abelian subgroup of Q is contained in $Z(Q)$. Also, $U_{Z(Q)}$ is homogeneous, so $Z(Q)$ is cyclic and is contained in $Z(G)$. By Lemma 1 (with $r = q$), Q is extra special of exponent q, and (b) follows. It remains to prove (a).

Choose $Q_1 \subseteq Q$ with $Z(Q) \subsetneq Q_1 \lhd G$ such that PS acts irreducibly on $Q_1/Z(Q)$. Now, Q_1 is nonabelian as every normal abelian subgroup of G contained in Q is necessarily contained in $Z(Q)$. Thus $Q_1' = Q' = Z(Q)$. Now P acts fixed point freely on Q_1/Q_1', so $[Q_1, P] = Q_1$. If $[S, Q_1] = 1$, then Q_1 normalizes SU and Lemma 2.4 applies with $N = Q_1SU$ and $r = 2$. But then Q_1 centralizes $SU \supseteq U$ which contradicts that Q_1 is faithful on U. Hence $\mathbf{C}_{PS}(Q_1) < S$.

Let $G_1 = PSQ_1$. Notice that $O_2(G_1) = \mathbf{C}_{PS}(Q_1)$, and if $J(U)$ denotes the radical of U when viewed as an $F[G_1]$-module, then $\mathbf{C}_{G_1}(U/J(U)) = O_2(G_1) = \mathbf{C}_{PS}(Q_1)$. Thus, $G_1/O_2(G_1)$ acts faithfully on $U/J(U)$ and the hypotheses of the lemma are satisfied with G replaced by G_1 and U replaced by $U/J(U)$. If $Q_1 < Q$ then induction applies, and (a) follows.

Therefore, we may assume $Q_1 = Q$, which means that PS acts faithfully and irreducibly on $Q/Z(Q)$. By Lemma 2.2, S is extra special of order $2(p-1)^2$, and Corollary 3.4 now applies to this minimal situation.

THEOREM 4.2. *Let P and N be groups satisfying hypothesis* 1.1. *Then:*

(a) *If p is not a Fermat prime, then N is nilpotent.*

(b) *If $|N|$ is odd, then N is nilpotent.*

(c) *If p is a Fermat prime, then N has a nilpotent normal π_p-complement where*

$$\pi_3 = \{2, 5, 7, 11, 13, 23\}, \qquad \pi_5 = \{2, 3, 7, 11\}$$

and

$$\pi_p = \{2\} \quad \textit{for every Fermat prime } p \geq 17.$$

Proof. First assume that the hypothesis of (a) or (b) is satisfied. If every Hall $\{q, r\}$-subgroup of N is nilpotent, then so is N itself. We may replace N by a P-invariant Hall $\{q, r\}$-subgroup so as to assume that N itself is a $\{q, r\}$-group. Clearly, we may assume that N is neither a q-group nor an r-group. Let U be a minimal normal subgroup of NP contained in N. Without loss, we may assume that U is an r-group. By induction, N/U is nilpotent, so that $1 = O^{q,r}(N) < O^q(N) < N$, and the hypotheses of Lemma 2.4 are satisfied. By that lemma then, $O^q(N) \cap O^r(N) = 1$ and N is nilpotent.

It remains now to prove part (c). Assume that p is a Fermat prime and that N is a minimal counterexample to part (c). Let H be a P-invariant Hall 2-complement in N. By part (b), H is nilpotent. If N has a normal π_p-complement, say K, then $K \subseteq H$ and K is nilpotent. Thus, N does not have a normal π_p-complement.

If $O_{\pi_{p'}}(N) \neq 1$, then $N/O_{\pi_{p'}}(N)$ has a normal π_p-complement, and we're done. Let $\pi(N)$ denote the prime divisors of $|N|$. If $\pi(N) \subseteq \pi_p$ we are done, as the identity subgroup is then a normal π_p-complement. Let $q \in \pi(N)$, $q \notin \pi_p$ and let N_0 be a P-invariant Hall $\{2, q\}$-subgroup of N. We may assume that $Q = N_0 \cap H$ is the Sylow q-subgroup of H. Hypothesis 1.1 holds for the action of P on N_0, so if $N_0 < N$ then Q is normal in N_0. Also $Q \lhd H$ so $Q \lhd HN_0 = N$. But then $Q \subseteq O_{\pi_{p'}}(N) = 1$, a contradiction. Hence $N_0 = N$ and N is a $\{2, q\}$-group with $O_{\pi_{p'}}(N) = O_q(N) = 1$.

The Fitting subgroup $\mathbf{F}(NP)$ of NP must be a 2-group. If the Frattini subgroup $\phi(NP)$ is nontrivial, then $N/\phi(NP)$ has a normal 2-complement, which must be $Q\phi(NP)/\phi(NP)$. Hence $Q\phi(NP) \lhd N$. Thus

$$[Q, \mathbf{F}(NP)] \subseteq Q\phi(NP) \cap \mathbf{F}(NP) = \phi(NP).$$

As $\mathbf{C}(\mathbf{F}(NP)/\phi(NP)) \subseteq \mathbf{F}(NP)/\phi(NP)$, this proves that $Q \subseteq \mathbf{F}(NP)$, a contradiction. Thus, $\phi(NP) = 1$ so that $U = \mathbf{F}(NP)$ is an elementary abelian 2-group.

Since N/U is not a counterexample to part (c), N/U has a normal π_p-complement, which must be QU/U. Hence $QU \lhd NP$.

Let $G = N_{NP}(Q)$. By the standard Frattini argument, $NP = G \cdot U$. Let $C = G \cap U = \mathbf{C}_U(Q)$. Since U is abelian, $C \lhd U$ and hence $C \lhd NP$. If $C \neq 1$ then N/C is not a counterexample to part (c), so that N/C has a normal π_p-complement (which is QC/C). Thus $QC \lhd NP$, and so $[U, Q, Q] \subseteq [QC \cap U, Q] = [C, Q] = 1$. But this implies that $Q \subseteq \mathbf{C}(U) = U$, a contradiction. Thus, $C = 1$ and G is a complement for U in NP.

Notice that if S is a P-invariant Sylow 2-subgroup of G, then $G = PSQ$. Furthermore, U may be regarded as a $GF(2)[G]$-module. Since $[SU, P] = SU$ and $[S, P] = S$ it follows that $[U, S] + [U, P] = U$. The hypotheses of Theorem 4.1 are now satisfied, and this forces either $p = 3$ or 5 and $q \in \pi_p$. Either case is a contradiction, and the proof of Theorem 4.2 is now complete.

Theorem 4.1 is also useful in classifying all solvable groups N satisfying Hypothesis 1.1 and having small nilpotent (or Fitting) length. In fact, the structure of $N/\mathbf{F}(N)$ is completely determined. In order to state the result explicitly, we need some notation. Define $l(G)$ to be the nilpotent length for any solvable group G, and define $\mathbf{K}(G)$ to be the characteristic subgroup of G containing $\mathbf{F}(G)$ and satisfying $\mathbf{K}(G)/\mathbf{F}(G) = O_{2'}(G/\mathbf{F}(G))$. (Notice $\mathbf{K}(G) = O_{2,2'}(G)$.)

Finally, define Q_8 to be the quaternion group of order 8, D_8 the dihedral group of order 8, and $D_8 \gamma Q_8$ the central product of these groups.

THEOREM 4.3. *Let P and N satisfy Hypothesis* 1.1 *and assume* $l(N) \leq 3$.

(a) *If* $l(N) = 1$ *then N is nilpotent.*

(b) *If* $l(N) = 2$ *then p is a Fermat prime and* $N/\mathbf{F}(N)$ *is a subdirect product of isomorphic extra special groups, each having order* $2(p-1)^2$. *In particular, N has a normal 2-complement which is nilpotent, and the class of* $N/\mathbf{F}(N)$ *is* 2.

(c) *If* $l(N) = 3$ *then* $p = 3$ *or* 5, *and* $\mathbf{K}(N)/\mathbf{F}(N)$ *is the normal 2-complement for* $N/\mathbf{F}(N)$. *The group* $\mathbf{K}(N)/\mathbf{F}(N)$ *is the direct product of special q-groups for odd primes q in* π_p. *If* $p = 3$, $N/\mathbf{K}(N)$ *is a subdirect product of groups isomorphic to* Q_8 *while for* $p = 5$, $N/\mathbf{K}(N)$ *is a subdirect product of groups isomorphic to* $D_8 \gamma Q_8$. *In particular,* $N/\mathbf{K}(N)$ *and* $\mathbf{K}(N)/\mathbf{F}(N)$ *each have class* 2, *and* $N/\mathbf{F}(N)$ *is a* π_p*-group.*

Proof. Part (a) is a triviality.

Assume that $l(N) = 2$. Suppose $O_{2'}(N) = 1$. Then $\mathbf{F}(NP) = \mathbf{F}(N)$ is a 2-group which implies (since $N/\mathbf{F}(N)$ is nilpotent) $N/\mathbf{F}(N)$ has odd order. Let Q be a P-invariant Sylow q-subgroup of N for some odd q dividing $|N|$. Now, by Lemma 2.4 applied to the group $Q\mathbf{F}(N)$ with $r = 2$ we get the contradiction $Q \subset \mathbf{C}(\mathbf{F}(N)) \subseteq \mathbf{F}(N)$. Hence, $O_{2'}(N) \neq 1$, and by induction, $N/O_{2'}(N)$ has a normal 2-complement. Thus, N itself has a normal 2-complement which is nilpotent by Theorem 4.2(b). By part (a) of that same theorem, p must be a Fermat prime. As $\mathbf{F}(NP) = \mathbf{F}(N)$, the conclusion is unaffected if $\phi(NP)$ is factored out, so we may assume $\phi(NP) = 1$. Thus $\mathbf{F}(N) = A \times B$ where A and B are abelian groups, $|A|$ is odd, and B is a 2-group. The hypotheses of Corollary 2.3 are now satisfied in the action of $PN/\mathbf{F}(N)$ on A, and case (b) follows.

Suppose now $l(N) = 3$. Then, N cannot have a nilpotent normal 2-complement, so by Theorem 4.2, p is 3 or 5. As $l(N/\mathbf{F}(N)) = 2$, it follows from case (b) that $N/\mathbf{F}(N)$ has a nilpotent normal 2-complement, which is therefore $\mathbf{K}(N)/\mathbf{F}(N)$. Now $\mathbf{K}(N)/O_2(N)$ is isomorphic to a P-invariant Hall 2-complement (say H) of N, and so is nilpotent. As $O_2(N/O_2(N))$ is trivial, $\mathbf{K}(N)/O_2(N)$ is the Fitting subgroup of $N/O_2(N)$. Clearly, $l(N/O_2(N)) = 2$, and so $N/\mathbf{K}(N)$ is a subdirect product of extra special groups of order $2(p-1)^2$. The only extra special group of this order which admits a nontrivial automorphism of order p is Q_8 when $p = 3$ and $D_8 \gamma Q_8$ when $p = 5$. It remains to determine the structure of $\mathbf{K}(N)/\mathbf{F}(N)$.

Define F by the equation $F/O_{2'}(N) = \mathbf{F}(N/O_{2'}(N))$. Since the Hall 2-complement H for N is nilpotent, $F/O_{2'}(N)$ must be a 2-group. Now $F \supseteq \mathbf{F}(N)$ and so $F \cap \mathbf{K}(N) = \mathbf{F}(N)$. Define F_2 by $F_2/F = \mathbf{F}(N/F)$. Clearly, F_2/F contains $\mathbf{K}(N)F/F$, and as F_2/F must have odd order, we have $F_2/F = \mathbf{K}(N)F/F$ so $F_2 = \mathbf{K}(N)F$. If $F_2 = N$ then $N/\mathbf{F}(N)$ is isomorphic to the direct product of

$\mathbf{K}(N)/\mathbf{F}(N)$ with $F/\mathbf{F}(N)$ and so is nilpotent, contradicting $l(N) = 3$. Thus $F_2 < N$, so that $l(N/O_{2'}(N)) = 3$. Also, $F_2 = \mathbf{K}(N)F$ so

$$F_2/F = \mathbf{K}(N)F/F \cong \mathbf{K}(N)/(\mathbf{K}(N) \cap F) = \mathbf{K}(N)/\mathbf{F}(N).$$

It follows that both the section $\mathbf{K}(N)/\mathbf{F}(N)$ and the length $l(N)$ are unaffected if $O_{2'}(N)$ is factored out. We may assume then that $O_{2'}(N) = 1$, and then $\mathbf{K}(N)/\mathbf{F}(N) = \mathbf{F}(N/\mathbf{F}(N))$. Clearly, $\phi(NP)$ may also be factored out, so that $\mathbf{F}(N) = \mathbf{F}(NP)$ is an elementary abelian 2-group. Also, $\mathbf{F}(NP)$ is complemented in NP by a group G which we may assume contains PH (recall that H is a P-invariant Hall 2-complement for N). Thus, $H \lhd G$ and $G = PSH$ where S is a P-invariant Sylow 2-subgroup of G.

Set $U = \mathbf{F}(NP) = \mathbf{F}(N)$. Then $\mathbf{C}(U) = U$, so U may be regarded as a faithful $GF(2)[G]$-module. Furthermore, $\mathbf{C}(U) = U$ also implies that $l(SQU) = 3$, where Q is the unique Sylow q-subgroup of H for any prime $q \mid |H|$. We may therefore assume $H = Q$ is a q-group. Since $[SU, P] = SU$, it follows that $U = [U, S] + [U, P]$ where U is denoted additively. All of the hypotheses of Theorem 4.1 are now satisfied, so $q \in \pi_p$ and Q has exponent q and class 2. Thus $Q' = \phi(Q) \subseteq Z(Q)$.

Suppose $Q' < Z(Q)$. Now $Z(Q)$ is elementary abelian, and since $[Q, P] = Q$, P is fixed point free on Q/Q'. By Maschke's theorem, there exists a PS-invariant subgroup Q_0 of $Z(Q)$ such that $Q_0 \cdot Q' = Z(Q)$ and $Q_0 \cap Q' = 1$. It follows that Q_0 admits P fixed point freely and that the hypotheses of Theorem 4.1 are satisfied in the action of PSQ_0 on U. By part (b) of that theorem, Q_0 must be nonabelian, and this contradiction proves that $Q' = Z(Q)$. Thus Q is special and Theorem 4.3 is now completely proved.

5. Concluding remarks

It is interesting to consider whether Hypothesis 1.1 implies that $l(N)$ is bounded. Because of Theorem 4.2, only the primes $p = 3$ and $p = 5$ need be considered. The fact that $N/\mathbf{F}(N)$ is completely determined when $l(N) = 3$ suggests that a bound is possible. The author suggests that $l(N) \leq 4$.

If $p = 3$ and $q \in \{5, 7, 11, 13, 23\}$, a $\{2, q\}$-group N may be constructed satisfying hypothesis 1.1 but $l(N) = 3$. This shows that Theorem 4.2 is no longer valid if any prime is removed from π_3. (Similarly, neither 2 nor 3 may be removed from π_5. It appears likely that the other two primes in π_5 can't be removed). The group PN has the form $PSQU$ where the hypotheses of Corollary 3.4 hold for the group $G = PSQ$ acting on the $F[G]$-module U. Using [1], the source for the module U may be computed. The action of P on $U/[U, S]$ is then determined, and replacing U by $U \otimes_F L$ if necessary (where L is a module for $PSQU/SQU$), the module U then satisfies $\hom_{F[PS]}(U_{PS}, F) = \{0\}$. Then $[SU, P] = SU$ by Lemma 3.1, and Theorem 4.2 is false if q is removed from π_3. I am indebted to Professor T. R. Berger for pointing out to me the relevance of Dade's important work in [1].

It is an open question whether Theorem 4.2 remains true if the solvability assumption is removed from hypothesis 1.1.

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