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PLANES FOR WHICH THE LINES ARE THE SHORTEST PATHS BETWEEN POINTS

BY

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Dedicated to Merrill E. Shanks

1. Introduction

This paper treats the planar case of Hilbert's fourth problem [14] by a direct geometric argument which is entirely free from the notion of differentiability. The key idea (Lemma 1) is a combinatorial version of Crofton's arclength formula which can be established strictly on the basis of Hilbert's simple axioms of plane incidence and order. In principle this allows a purely axiomatic treatment of the problem, although we will use the more convenient language of convexity, general topology and measure theory.

Even though the removal of cumbersome variational techniques is a major benefit, there are other virtues of the present approach. Since there is no need to assume a Euclidean incidence structure, the theorem of Desargues may be ignored. This greatly increases the method's scope. Also, the three basic lemmas on the foundations of integral geometry require absolutely no continuity assumptions. This ultimately allows the treatment of discontinuous path-length functions and a clear description of the possible sets of discontinuities.

Briefly stated, the fourth problem requests an investigation of the various geometries obtained by replacing the usual triangle congruence axiom with the requirement that the triangle inequality hold for the sides of any triangle. The axiom of parallels is dropped, but the remainder of the Hilbert axiom scheme is retained. The heart of the problem is to identify all metrics on such a plane which add continuously along the lines. They give rise to the continuous path-length functions for which "the lines are the shortest connection between points".

In order to avoid an axiomatic description we make the following four assumptions about the planes to be investigated: (1) the points of the plane carry a topology making the plane homeomorphic to the Euclidean plane; (2) the lines are certain pointsets which are homeomorphic to Euclidean lines in the relative topology; (3) two distinct points lie on precisely one line; (4) each line separates the plane into two distinct nonempty open convex sets, or equivalently, the axiom of Pasch is valid. The method does extend to some other general line-systems, certainly the projective plane, but we will use the above conditions which allow a simple exposition.

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There is a natural topology on the lines of a plane which may be described in a variety of ways. A description which is particularly apt for this paper is to let a lineset be a basic open set if it is the set of lines, possibly empty, which strictly separate a pair of nonempty finite pointsets. With this topology the set of lines which cut the interior of a triangle is topologically an open Möbius band.

As usual, a *pseudometric* is a real-valued function d on pairs of points such that

(i) $d(p, q) = d(q, p)$,
(ii) $d(p, p) = 0$, and
(iii) $d(p, q) + d(q, r) \geq d(p, r)$.

A plane pseudometric is said to *add along lines* if $d(p, q) + d(q, r) = d(p, r)$ whenever p, q, r are collinear in the given order. A pseudometric d is said to be *continuous at p* if $\text{limit}_i\, d(p, p_i) = 0$ whenever $\{p_i\}$ is a sequence of points tending to p in the plane topology. It is now possible to state the main theorem.

THEOREM 1. *Let d be a plane pseudometric which adds along lines.*

(1) *The points at which d is discontinuous form a set $A \cup B$ where A is a countable union of lines and B is countable.*

(2) *There is a unique Borel measure on the lines of the plane such that for any pair of points p, q at which d is continuous, $d(p, q)$ is equal to half the measure of the lines which cut the segment $\overline{pq}$.*

In Section 3 it is shown that d is continuous at p if and only if the pencil of lines at p has measure zero. This explains the nice structure of the set of discontinuities of d.

Furthermore, there is a simple necessary and sufficient condition that the line segments be uniquely the shortest paths connecting their endpoints, namely, that all nonempty open linesets be assigned positive measure. If p, q, r is a noncollinear triple of points, the set of lines which separate $\{p, r\}$ and $\{q\}$ is open and nonempty, and it follows from Section 3 that $d(p, q) + d(q, r) - d(p, r)$ is at least as large as the measure of this lineset. The sufficiency of the condition follows at once. The necessity is established by similar observations.

Since no single article can deal with all aspects of even the planar case of Hilbert's problem, we refer the reader to H. Busemann's recent survey [8] for a more complete list of references and a brief history of the problem beginning with the classical paper of G. Hamel [13] and proceeding to the recent work of A. V. Pogorelov [18].

Using variational methods and an integral averaging process, Pogorelov proves Theorem 1 for the case in which the metric is continuous and the plane is Desarguesian. Much earlier Blaschke [3] used variational methods to establish related theorems for certain line systems, not necessarily Desarguesian, with differentiable metrics. However, we have been unable to extend the methods of Blaschke and Pogorelov to obtain a second proof of Theorem 1. Busemann has given numerous synthetic constructions based on the ideas of integral geometry, and his book [7] contains a number of examples related to our present work in content and spirit.

The proof of the main theorem is essentially self-contained with the exception of certain claims concerning convexity and separation in a plane. An example is the assertion that two disjoint convex polygons share two unique internal joint tangent lines. Let us briefly say that with a slight modification the arguments of H. Brunn [5] apply just as well in the planes presently under consideration to justify these claims. Also, the angle concept will not be discussed. The early paper of Busemann [6] shows that if need be, angles may be introduced in the presence of a metric which adds continuously along the lines of a plane.

2. Lemmas on the foundations of integral geometry on a plane

A pseudometric is a very special example of a mapping d from the line segments of a plane to a commutative group G. The most important construction of this paper uses only this fact together with the incidence and order properties of the points and lines of the geometry.

If $H = \overline{p_0 p_1 \cdots p_k}$ is a polygonal line, $d(H)$ will denote $\sum_{i=1}^{k} d(p_{i-1}, p_i)$, and it is natural and convenient to define $d(p, p)$ to be the identity of G. For the present we do not assume that d adds along lines.

Let $Q = \{p_1, \ldots, p_n\}$ be a set of distinct points in a plane, *no three collinear*, and let S denote the collection of $C(n, 2)$ distinct line segments $\overline{p_i p_j}$. A line l is said to *separate* the points Q if (i) none of the points lie on l, and (ii) each of the open halfplanes determined by l contains at least one of the points. Two separating lines will be called *equivalent* if each separates Q into the same pair of sets. An inductive argument shows that there are precisely $C(n, 2)$ equivalence classes of separating lines. Form the lineset L by choosing exactly one representative from each equivalence class.

Suppose K and K^* are disjoint convex polygons whose vertices are members of Q. There will be two distinguished members of S, $\overline{pp^*}$ and $\overline{qq^*}$, which lie on the internal joint tangent lines of K and K^*. (See Figure 1.) Let T and T^* be the portions of the boundaries of K and K^*, respectively, which are contained in the convex set, Conv $\{p, p^*, q, q^*\}$. We define $\sigma(K, K^*)$ in G by

$$\sigma(K, K^*) = d(p, p^*) + d(q, q^*) - d(T) - d(T^*). \tag{1}$$

The segments $\overline{pp^*}$, $\overline{qq^*}$ together with the various segments which comprise T and T^* will be called *extreme segments* for the pair (K, K^*). For each line l in L, K_l and K_l^* will denote the convex hulls of the two sets which are separated by l, and $d'(l)$ will denote $\sigma(K_l, K_l^*)$.

In the following lemma K_l or K_l^* is allowed to be a single point or a line segment. If $n \geq 3$, not both sets can be points. The formula (1) for $\sigma(K, K^*)$ is still defined even when T or T^* consists of a single point by setting $d(T)$ to be the identity of G.

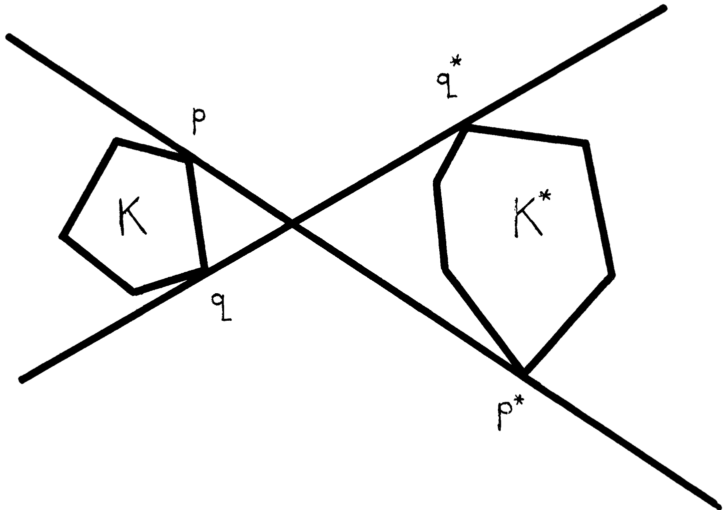

FIGURE 1

LEMMA 1. *Let Q, S, and L be as described above. Then for any i, j there is a combinatorial Crofton formula given by*

$$2d(p_i, p_j) = \sum \{d'(l): l \cap \overline{p_i p_j} \neq \varnothing\}. \tag{2}$$

Proof. We will show that each segment $\overline{qr}$ in S occurs as an extreme segment in such a manner that $+d(q, r)$ appears exactly the same number of times as $-d(q, r)$ on the right side of equation (2), unless $\overline{qr} = \overline{p_i p_j}$, in which case $+d(q, r)$ appears twice. There are four distinct cases to consider.

Case i. Suppose p_i, p_j, q, r are distinct points and that the line containing $\overline{qr}$ does not cut $\overline{p_i p_j}$. Here we see that $\pm d(q, r)$ cannot appear on the right side of (2). For if l in L is such that p_i is in K_l and p_j is in K_l^*, then any line which contains an extreme segment must cut $\overline{p_i p_j}$.

Case ii. Again, suppose p_i, p_j, q, r are distinct, and that the line containing $\overline{qr}$ does cut $\overline{p_i p_j}$. There are precisely four lines in L which separate p_i and p_j in such a manner that qr is an extreme segment. We will identify the lines by giving the pairs (K_l, K_l^*).

Let the line containing $\overline{qr}$ separate $Q - \{q, r\}$ into the necessarily nonempty sets A, B. We now list the four possible choices for the pairs (K_l, K_l^*):

1. (Conv $(A \cup \{r\})$, Conv $(B \cup \{q\})$),
2. (Conv $(A \cup \{q\})$, Conv $(B \cup \{r\})$),
3. (Conv (A), Conv $(B \cup \{q, r\})$),
4. (Conv $(A \cup \{q, r\})$, Conv (B)).

We observe that the first two pairs will contribute $+d(q, r)$ while the last two pairs will contribute $-d(q, r)$.

Case iii. Suppose $\overline{p_i p_j} \neq \overline{qr}$, but $r = p_i$. There are precisely two lines in L which separate p_i and p_j in such a manner that $\overline{qr}$ is an extreme segment. We use the notation of Case ii, except that we assume that p_j is in B and we allow A to be empty. The possible pairs are:

1. (Conv $(A \cup \{r\})$, Conv $(B \cup \{q\})$),
2. (Conv $(A \cup \{q, r\})$, Conv (B)).

The first pair leads to a contribution of $+d(q, r)$ in equation (1) while the second gives $-d(q, r)$.

Case iv. Suppose $\overline{qr} = \overline{p_i p_j}$. There are exactly two lines in L which separate p_i and p_j in such a way that $\overline{qr}$ is an extreme segment. Here we allow either A or B to be empty. The possible pairs are:

1. (Conv $(A \cup \{q\})$, Conv $(B \cup \{r\})$),
2. (Conv $(A \cup \{r\})$, Conv $(B \cup \{q\})$).

Each pair leads to a contribution of $+d(q, r) = +d(p_i, p_j)$.

The four cases taken together show that equation (2) is indeed valid. If K is a convex polygon whose vertices $q_1, q_2, \ldots, q_k$ are in Q, and G possesses no elements of order 2, then equation (2) and a simple counting argument gives a combinatorial formula

$$P(K) = \sum \{d'(l) \colon l \cap K \neq \varnothing\} \tag{3}$$

where $P(K)$ is the generalized perimeter $d(\overline{q_1 \cdots q_k q_1})$.

Richard L. Bishop kindly points out that the requirement that G contain no elements of order 2 assures the uniqueness of the function d' in equation (2). Also, it turns out that σ as a function on pairs of convex sets has occurred in the work of Crofton [10] and Sylvester [19]. Here d was the Euclidean metric and $\sigma(K, K^*)$ was interpreted as the probability that a line would separate K and K^*.

LEMMA 2. *Let d be a plane pseudometric which adds along lines. If K_1 and K_2 are convex polygons such that $K_1 \subset K_2$, then*

$$P(K_1) \leq P(K_2). \tag{4}$$

Proof. Suppose the successive vertices of K_1 are $p_0, p_1, \ldots, p_{k-1}$. Let l be the line containing $\overline{p_0 p_1}$, and let q_0 and q_1 be the points on l where l leaves K_2. (See Figure 2.)

The line l breaks the boundary of K_2 into two polygonal lines having q_1 and q_0 as endpoints. Since d is linearly additive, $P(K_2)$ is the sum of the lengths of

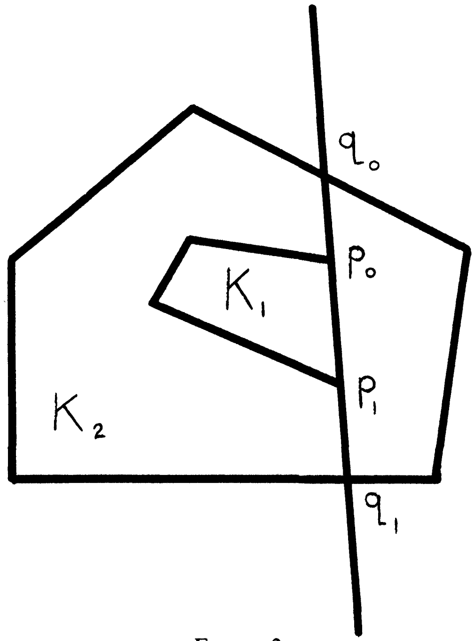

FIGURE 2

these two polygonal lines. Each of these lengths is at least $d(q_0, q_1)$ by the extended triangle inequality. Therefore the segment $\overline{q_0 q_1}$ together with the polygonal line on the same side of l as K_1 bounds a convex polygon $K_2^{(1)}$ where $K_1 \subset K_2^{(1)} \subset K_2$ and $P(K_2^{(1)}) \leq P(K_2)$.

The above argument may be repeated with the successive segments $\overline{p_1 p_2}, \ldots, \overline{p_{k-1} p_0}$. At the final stage $K_2^{(k)} = K_1$, and since $P(K_2^{(i+1)}) \leq P(K_2^{(i)})$ for each i, the lemma is true.

LEMMA 3. *Let d be a plane pseudometric which adds along lines. Then $d'(l) \geq 0$ for each l in L in equation* (2).

Proof. (See Figure 1.) Let us assume that not both K and K^* are singletons. Let r be the point of intersection of the segments $\overline{pp^*}$ and $\overline{qq^*}$. If K is not a singleton, Lemma 2 allows the conclusion that $d(p, r) + d(r, q) \geq d(T)$, after subtraction of $d(p, q)$ from both sides. This conclusion is trivial if K is a singleton. Similarly, $d(p^*, r) + d(r, q^*) \geq d(T^*)$. Adding these two inequalities and using the linear additivity of d gives $\sigma(K, K^*) \geq 0$.

The three lemmas show that in a plane the fundamental reasons for the existence of a Crofton-type formula for arc length are independent of the continuity (or differentiability) properties of the metric d as well as the collineation group of the plane. Also, the theorem of Desargues plays no role.

The classical Crofton measure on the lines of the Euclidean plane may be obtained in a simple manner. If L is a Borel lineset, let $L(\theta)$ denote those members of L which are orthogonal to a line $l(\theta)$, making a positive angle θ with respect to some fixed line $l(0)$ which acts as an axis. Let $s(\theta)$ denote the linear Lebesgue measure of the pointset $L(\theta) \cap l(\theta)$. Define $\eta[L]$ to be $\int_0^\pi s(\theta)\, d\theta$. It is easily checked that $\eta[L] = 2$ if L consists of the lines which cut a unit segment, and that $\eta[L] = 2\pi$ if L consists of the lines which cut a unit circle. This construction for Crofton's measure utilizes many special properties of the Euclidean plane. A much more general construction for line measures, based on the three lemmas, is given in the next section.

3. Construction of the line measure η

If A is a planar pointset, $L(A)$ will denote the set of lines having nonempty intersection with A. Note that $L(\bigcup A_i) = \bigcup_i L(A_i)$, and that if $\{A_i\}$ is a decreasing sequence of convex pointsets whose intersection is a nonempty compact set, then $L(\bigcap A_i) = \bigcap_i L(A_i)$. The symbol A^0 will denote the interior of A.

Let $\bar{K}$ be a convex polygon with $\bar{K}^0$ being nonempty. Choose $p_1, p_2, \ldots$ to be a countable dense subset of $\bar{K}$ which contains the vertices of $\bar{K}$, and which has the property that no three p_i are collinear. Let $Q_n = \{p_1, p_2, \ldots, p_n\}$ be an initial segment of the sequence which contains the vertices of $\bar{K}$.

If d is a pseudometric which adds along lines, we define an atomic measure η_n on $\bar{L} = L(\bar{K})$, a compact lineset, by putting $\eta_n(l) = \sigma(K_l, K_l^*)$ for each line l in a set of $C(n, 2)$ separating lines chosen for Q_n. From equation (3) it follows that $\eta_n[\bar{L}] = P(\bar{K})$, a finite number which does not depend on n.

By the Helly compactness theorem there will be a subsequence of the measures $\{\eta_n\}$ which converge weakly with respect to the line topology on $\bar{L}$ to a Borel measure η. By reindexing we may assume that $\{\eta_n\}$ does converge weakly to η. Our strategy will be to prove Theorem 1 for d restricted to $\bar{K}^0$. The complete theorem will then follow easily.

LEMMA 4. *Let $K \subset \bar{K}^0$ be a convex polygon. Then*

$$\eta[L(K^0)] \le P(K) \le \eta[L(K)]. \tag{5}$$

Proof. Suppose K^0 is nonempty, and that the vertices of K belong to the previously described dense sequence $\{p_i\}$. Let $K' \subset K^0$ be a convex polygon with L' denoting $L(K')$. Let L'' denote the compact lineset $\bar{L} \sim L(K^0)$, the lines which cut $\bar{K}$ but not K^0. There is a continuous function $f\colon \bar{L} \to [0, 1]$ such that $f \mid L'' = 0$ and $f \mid L' = 1$. If we choose n large enough that $\eta_n[L(K)] = P(K)$, as guaranteed by equation (3) as soon as the vertices of K belong to Q_n, we have $\int f\, d\eta_n \le \eta_n[L(K)] = P(K)$. Therefore, since $\eta[L'] \le \int f\, d\eta$, the weak convergence implies that $\eta[L'] \le P(K)$. The fact that $L(K^0)$ can be expressed as a monotone union of such linesets L' gives the left inequality in (5) for the special choice of K.

Again for the special K, except that we allow K^0 to be empty and thus allow K to be a point or segment, let $K \subset H^0$ where $H \subset \bar{K}$ is a convex polygon. Let

L' denote $\bar{L} \sim L(H^0)$, and let f: $\bar{L} \to [0, 1]$ be chosen so that $f \mid L' = 0$ and $f \mid L(K) = 1$. If n is large enough that $\eta_n[L(K)] = P(K)$, then $P(K) \leq \int f \, d\eta_n$. Since $\int f \, d\eta \leq \eta[L(H^0)]$, $P(K) \leq \eta[L(H^0)]$ because of the weak convergence of $\{\eta_n\}$. The fact that $L(K)$ is a countable intersection of sets $L(H^0)$ gives the right inequality in (5) for the special K.

For the general K chose $K_1 \subset K \subset K_2$ where the K_i are special. Our previous work, including Lemma 2, gives

$$\eta[L(K_1^0)] \leq P(K_1) \leq P(K) \leq P(K_2) \leq \eta[L(K_2)].$$

Taking monotone limits gives the complete result so that the inequalities (5) are valid.

LEMMA 5. *Let p, q be points in $\bar{K}^0$. Then d is continuous at q if and only if $\eta[L(q)] = 0$. Also, if d is continuous at both p and q, then $\eta[L(\overline{pq})] = 2d(p, q)$.*

Proof. Suppose $\eta[L(q)] = 0$, and let $q_1, q_2, \ldots$ be a sequence of points which converge to q in the plane topology. Let $K_1, K_2, \ldots$ be a monotone decreasing sequence of triangles with $\bigcap K_i = \{q\}$ and q_i in K_i^0 for each i. By Lemma 4, $P(K_i) \leq \eta[L(K_i)]_j$ and since $\text{limit}_i\, \eta[L(K_i)] = 0$ because of measure continuity from above, $\text{limit}_i\, P(K_i) = 0$. It follows from Lemma 2 that $d(q, q_i) \leq \frac{1}{2}P(K_i)$ and therefore that $\text{limit}_i\, d(q, q_i) = 0$.

Next suppose that $\eta[L(q)] = \delta > 0$, and let the triangles $\{K_i\}$ be as above. If r_1, r_2, r_3 are the vertices of K_i, we have

$$\delta \leq \eta[L(K_i^0)] \leq P(K_i) \leq 2[d(q, r) + d(q, r_2) + d(q, r_3)].$$

Thus $d(q, r_j) \geq \delta/6$ for some choice of j, and it is clear that we can find a sequence $\{q_i\}$ tending to q such that $\liminf_i d(q, q_i) \geq \delta/6$.

Finally, suppose d is continuous at both p and q. Let

$$\{K_i\} = \{\text{Conv}\,(r_1^{(i)}, r_2^{(i)}, r_3^{(i)}, r_4^{(i)})\}$$

be a monotone decreasing sequence of quadralaterals chosen so that (i) $\overline{pq} \subset K_i^0$ for each i, (ii) $\bigcap K_i = \overline{pq} \cup \{p, q\}$, and (iii) $\{r_1^{(i)}\}$, $\{r_2^{(i)}\}$ each converge to p and $\{r_3^i\}$, $\{r_4^i\}$ each converge to q. Since

$$2d(p, q) \leq \eta[L(\overline{pq})] \leq \eta[L(K_i^0)] \leq P(K_i),$$

the triangle inequality together with continuity at p and q insures that $\text{limit}_i\, P(K_i) = 2d(p, q)$. Thus $2d(p, q) = \eta[L(\overline{pq})]$.

An important consequence of Lemma 5 is that if d is continuous at each vertex of a convex polygon $K \subset \bar{K}^0$, then the right inequality of Lemma 4 becomes equality, as does the left if K^0 is nonempty.

We are now easily able to prove part (1) of Theorem 1 for d restricted to $\bar{K}^0$. The lines which represent atoms of η will be called *comets*. A pencil of lines which, with comets dileted, possesses positive η-measures will be called a *star*. It is clear that the finite measure η induces only countable many stars and comets. Lemma 5 implies that d is discontinuous at p if and only if p either lies on a comet or is the vertex of a star. This proves the restricted case.

A line which is not a comet contains only countably many points in $\bar{K}^0$ which are points of discontinuity for d. Because of this, one can choose a dense sequence of points of continuity $\{p_i\}$ in $\bar{K}^0$ so that (1) no three are collinear, and (2) any point of intersection between two segments $\overline{p_i p_j}$ is also a point of continuity. We omit the simple inductive argument for this construction.

LEMMA 6. *Let η' be a Borel measure on $L(\bar{K}^0)$ such that*

$$\eta'[L(\overline{pq})] = \eta[L(\overline{pq})] = 2d(p, q)$$

for all pairs p, q in $\bar{K}^0$ where d is continuous. Then $\eta' = \eta$.

Proof. Choose a dense sequence $\{p_i\}$ of points of continuity for d in $\bar{K}^0$ which possesses properties (1) and (2) described above. Let Q and Q^* be finite subsets of the points in this sequence, and suppose that Q and Q^* are separated by a nonempty lineset L. Then Lemma 5, together with several applications of equation (3), shows that it must be true that $\eta'[L] = \eta[L] = \sigma(\operatorname{Conv}(Q), \operatorname{Conv}(Q^*))$. The fact that d is continuous at the point of intersection of the internal tangents facilitates this computation.

Next observe that if L and L' are two such separating linesets, then $L \cap L' = L'' \cup L'''$ where L'' and L''' are again separating linesets, possibly empty, which are disjoint. This means that η and η' must agree on $L \cap L'$. By very standard arguments used in the theory of measures (see [12, chapter 2]), η and η' must agree on the sigma-ring of sets generated by these separating linesets L. These linesets form a base for the topology of $L(\bar{K}^0)$ because the sequence $\{p_i\}$ is dense in $\bar{K}^0$. Therefore η and η' agree on all the Borel linesets in $L(\bar{K}^0)$. This completes the proof of Theorem 1 for d restricted to $\bar{K}^0$.

Now let $\{\bar{K}_i\}$ be a monotone increasing sequence of convex polygons whose union is the whole plane. Let η_i be the unique Borel measure on $L(\bar{K}_i^0)$ constructed in the manner described above. For each Borel lineset L define $\eta[L] = \operatorname{limit}_i \eta_i[L \cap L(\bar{K}_i^0]$. The fact that $\eta_i[L \cap L(\bar{K}_i^0)]$ is nondecreasing in i makes it a simple matter to show that η is indeed a Borel measure on the lines of the plane.

It is clear that $\eta[L(\overline{pq})] = 2d(p, q)$ if p, q are points at which d is continuous, and that $\eta[L(p)] = 0$ if and only if d is continuous at p. The uniqueness of η follows at once from the uniqueness of each η_i.

With the exception that η is sigma-finite, the previous discussion of comets and stars carries over so that the set of discontinuities of d satisfies the requirements. This completes the proof of Theorem 1.

As a final remark on the proof of Theorem 1 we wish to point out that the original dense sequence $\{p_i\}$ in $\bar{K}^0$ used in defining η on $L(\bar{K}^0)$ could have had the unfortunate property that $2d(p_i, p_j) \neq \eta[L(\overline{p_i p_j})]$ for all i, j. Thus the discrete version of the theorem given by the first three lemmas is in a sense a stronger statement about the linearly additive pseudometric d.

4. The converse of Theorem 1

It is clear from Theorem 1 and its proof that there is a unique one-one correspondence between the pseudometrics which add continuously along the lines of a plane and those Borel measures on the lines of the plane for which (1) $\eta[L(p)] = 0$ for each point p, and (2) $\eta[L(\overline{pq})]$ is finite for each segment $\overline{pq}$. The correspondence is given by setting $d(p, q) = \frac{1}{2}\eta[L(\overline{pq})]$.

If η is required to satisfy condition (2) alone, then there may exist a nonempty countable family of stars and comets. In this situation there is a rather complicated class of pseudometrics, each of which gives rise to the line measure η. While we do not intend to give a complete characterization of this convex set of discontinuous pseudometrics, we wish to show that it is certainly nonempty.

Suppose η is concentrated on a comet l with $\eta[l] = 2\alpha$. If p and q lie in opposite open halfplanes for l, set $d(p, q) = \alpha$, and let $d(p, q) = 0$ if the points lie in the same open halfplane. There is a large number of ways in which d can be extended to a linearly additive pseudometric on the plane. The most natural way seems to be as follows: define $d(p, q) = 0$ if both p and q lie on l, and define $d(p, q) = \alpha/2$ if p lies on l and q does not.

Next suppose that η is concentrated on a star with vertex at r, and that $\eta[L(r)] = 2\alpha$. If neither p or q is r, define $d(p, q) = \frac{1}{2}\eta[L(\overline{p, q})]$. If $p \neq r$, set $d(p, r) = \alpha/2$. This gives only one of many pseudometrics associated with η.

The general measure η which satisfies requirement (2) may be written as $\eta = \eta_0 + \sum \eta' + \sum \eta''$ where η_0 possess no stars or comets, each η' is concentrated on a comet, and each η'' is concentrated on a star. The previous discussion shows that we may define a pseudometric d associated with η as $d = d_0 + \sum d' + \sum d''$.

As was remarked in the introduction, a necessary and sufficient condition that noncollinear triples always yield strict triangle inequalities is that η assign positive measure to nonempty open linesets. Even with this condition it is clear that there is a huge family of metrics on a plane for which the lines are the shortest paths between points.

5. A remark on the Hilbert geometries

In discussions of Hilbert's fourth problem it is traditional to indicate how the various ideas apply to Minkowski geometries and Hilbert geometries [15, appendix I]. G. D. Chakerian's paper [9] together with the work he references gives a very satisfactory description of the line measure η associated with a symmetric two dimensional Minkowski geometry.

A Hilbert geometry is defined in the interior of a bounded convex region D in Euclidean space. The lines are the open chords of D. If p, q are distinct points in the geometry, let p', q' be the endpoints of the chord containing p and q, the order being p', p, q, q'. The Hilbert distance $h(p, q)$ is defined to be

$$\ln \frac{e(p', q)}{e(p', p)} + \ln \frac{e(p, q')}{e(q, q')},$$

where e is the usual Euclidean metric. As shown by Hilbert, h is a metric which adds along lines.

The existence of an associated line measure η_h is guaranteed by Theorem 1 as well as by Pogorelov's theorem [18]. Nevertheless we wish to give a direct construction of the line measure η_h when D is a convex polygon which makes the nature of η_h transparent.

In the Euclidean plane let l and l' be rays which determine a nondegenerate angle at p_0. If p and q are points interior to this angle, let l_p and l_q be the rays from p_0 through the respective points. Define

$$d(p, q) = |\ln (ll', l_p l_q)|$$

where the ordered quadruple indicates the usual line cross ratio in the plane. It is easily checked that d is a linearly additive pseudometric on the interior of the angle. Also, $d(p, q) = 0$ only if p and q lie on a common ray through p_0. The line measure η associated with d is none other than the projective angle measure associated with the angle at p_0.

If D is a convex Euclidean polygon with vertices $p_1, \ldots, p_n$, let η_i be the projective angle measure on the angle at p_i. A computation shows that $2h = \sum_{i=1}^n d_i$ so that $\eta_h = \sum_{i=1}^n \eta_i$. The computation uses the successive vertices as centers of perspectivities, and is left as an exercise.

The above construction shows that h is a metric which adds along lines, and simultaneously identifies the line measure η_h.

6. Consequences

Because of the immense number of metrics on a plane for which the lines are the shortest paths between points, it would seem likely that few interesting properties are shared by these metric geometries. However, there certainly are some which should be recorded. In this section we will only consider continuous metrics.

THEOREM 2. *In a plane metric geometry for which the lines are the shortest paths between points, there is a nontrivial intrinsic area function which is invariant under any isometry.*

Proof. The area may be defined with the aid of a standard idea from integral geometry. If K is a convex polygon, define

$$\text{Area } (K) = \int s(l)\, d\eta(l) \tag{6}$$

where $s(l)$ is the length of the segment $l \cap K$ and η is the associated line measure. For more complicated sets K we may set $s(l) = \mu_l(K \cap l)$ where μ_l is the Stieltjes measure naturally induced on the line l by the metric d.

The fact that this area function is invariant under an isometry follows from a technical argument which is somewhat similar to the proof that η is unique in Theorem 1, and we omit this.

THEOREM 3. *Suppose d and d^* are metrics on a given plane such that for both, the lines are the shortest paths between points. If $d(p, q) = d^*(p, q)$ for all points p, q on the boundary of a convex polygon K, then d and d^* agree throughout K.*

Proof. Let Q be a finite set of points on the boundary of K, and let L be a set of separating lines chosen for Q. Observe that for any l in L either $d'(l) = d(p, s) + d(q, r) - d(p, r) - d(q, s)$ for some p, q, r, s in Q or $d'(l) = d(p, q) + d(q, r) - d(p, r)$ for some p, q, r in Q. The separating linesets for such finite sets Q form a base for the line topology on $L(K^0)$. It follows at once that η and η^* agree on all Borel subsets of $L(K^0)$ and hence $d = d^*$ throughout K. The proof extends to more general sets K whose boundary is a simple closed curve.

J. Lindenstrauss [17], E. Bolker [4], and L. Dor [11] have shown that any two dimensional Minkowski space embeds isometrically in $\mathscr{L}_1(0, 1)$. The following theorem generalizes this result. We only offer a sketch of the proof since a more complete version will appear in a later article.

THEOREM 4. *Any plane metric space for which the metric adds continuously along lines embed isometrically in $\mathscr{L}_1(0, 1)$.*

Proof. Let $\{l_i\}$ be a dense sequence of lines in the line topology of the plane. In an arbitrary manner associate the open halfplanes determined by l_i with 0 and 1, respectively. Define a one-one mapping from the points of the plane into set of sequences $\{\alpha_1, \alpha_2, \alpha_3, \ldots : \alpha_i \in \{0, 1\}\}$ as follows: if p is on l_i, set $\alpha_i(p) = 1$, otherwise set $\alpha_i(p)$ to be 0 or 1 according to which open halfplane contains p. Define the lineset $H(p)$ to be the closure of $\{l_i : \alpha_i(p) = 1\}$. If η is the line measure associated with the metric d, then it may be shown that $d(p, q) = \eta[H(p) \Delta H(q)]$ where Δ indicates the symmetric difference. If η is a finite measure, $p \to X(p)$, where $X(p)$ is the characteristic function of $H(p)$, gives an isometric embedding of the plane into $\mathscr{L}_1(\eta) \cong {}_1(0, 1)$. A technical argument deals with the case of η being sigma-finite. Corollary 5.6 in the book [20] by J. Wells and L. Williams is also applicable.

The equation $d(p, q) = \eta[H(p) \Delta H(q)]$ shows that J. B. Kelly's theorem [16] holds, and therefore d is a hypermetric. In particular the hyperbolic plane is a hypermetric space, and this answers a question raised by Kelly. The articles by H. Witzenhausen [21] and L. Dor [11] give further results of interest concerning metric embedding and hypermetrics.

The articles [1], [2] by the author and K. Stolarsky show how various metric inequalities may be applied to extremal problems in geometry. Many of these ideas can now be applied in any plane metric geometry for which the lines are the shortest paths between points.

6. Nonsymmetric metrics

At present we do not have a complete theory of planar *nonsymmetric* metrics d which add along lines, but some interesting things do follow from the previous work. We say that d is *continuous at* p if $\text{limit}_i\, d(p, p_i) = \text{limit}_i\, d(p_i, p) = 0$ for any sequence $\{p_i\}$ tending to p in the plane topology. Applying Theorem 1 to the symmetric metric d^* defined by $d^*(p, q) = d(p, q) + d(q, p)$ shows that the set of points of discontinuity for d is a countable union of lines together with a countable set. Since d is nonnegative, the sets of discontinuities for d and d^* are identical.

We call d *weakly symmetric* if for any triple of points p, q, r,

$$d(p, q) + d(q, r) + d(r, p) = d(q, p) + d(p, r) + d(r, q).$$

This means that the two oriented perimeters of any polygon are equal. If we define

$$s(p, q) = d(p, q) - d(q, p)$$

so that $d(p, q) = d^*(p, q) + s(p, q)$, then it is clear that a weakly symmetric metric is obtained by adding a suitable conservative path function to a symmetric metric. Thus there is a reasonable global description for this type of metric.

Whether or not the global methods of this paper can be modified so as to characterize all planar nonsymmetric metrics which add along lines remains an interesting question. The treatise of E. Zaustinski [22] is the standard reference for nonsymmetric metrics on general geodesic spaces.

Addendum

Since the results of this paper were announced,[1] the following paper has appeared: R. V. Ambartzumian, *A note on pseudo-metrics on the plane*, Z. Wahrscheinlichkeitstheorie verw. Gebiete, vol. 37 (1976), pp. 145–155.

Here a theorem which is intermediate to Pogorelov's theorem and the main theorem of the present paper is proved. The arguments involve continuity, but not differentiability. In addition, Ambartzumian proves our Theorem 3 in the setting of his paper, and we wish to recognize his independent discovery of this result.

Also, we have been able to obtain a copy of Pogorelov's recent book *Hilbert's Fourth Problem*, in which he gives a beautiful extension of his results to higher dimensional spaces. At present the variational technique discovered by Pogorelov seems to be the only way of treating the higher dimensional problem in terms of integral geometry.

[1] Ralph Alexander, *A new approach to Hilbert's fourth problem*, Notices Amer. Math. Soc., vol. 26 (1976), pp. A593–A594.

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