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ON STABLE HOMEOMORPHISMS AND IMBEDDINGS OF THE PSEUDO ARC

BY

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1. Introduction

Basically, this paper addresses the following two problems, due to Bing [3]:

(1) STABLE HOMEOMORPHISM PROBLEM. If $h: P \twoheadrightarrow P$ is a homeomorphism of the pseudo arc P onto itself such that h is the identity on some open set, must h be the identity?

(2) COMPOSANT PROBLEM. If $h: P \twoheadrightarrow P$ is a homeomorphism of the pseudo arc P onto itself such that h carries each composant onto itself, must h be the identity?

We obtain only partial results for the above problems—for homeomorphisms extendable to S^2—and we use prime ends heavily in our work. The necessary preliminary ideas and theorems are developed in Sections 3–5. Along the way, we obtain some interesting by-products. In another paper [8], we use many of these ideas and theorems to study the extendable, periodic homeomorphisms of the pseudo arc.

In Section 2, we describe prime end theory briefly, stating some of the important theorems, and giving examples, so that the reader can understand the rest of the paper.

In Section 3, we provide a complete proof of a theorem of Iliadis [13], which is stated, but not proved, in the literature. This is a theorem about prime ends and is important in our work. It states that if M is an indecomposable continuum in S^2 and $\phi: (S^2 - M) \twoheadrightarrow \text{Int } B$ is a C-map (conformal map) to the interior of the unit disk, then ϕ induces a decomposition of $S^1 = \text{Bd } B$ into a collection of at most countably many intervals, and the points of the complement. The intervals are in 1-1 correspondence with those composants of M which contain more than one accessible point. An "interval," however, is either an open interval or S^1 minus a point.

In Section 4, we show that there are inequivalent imbeddings of the pseudo arc in S^2, by constructing imbeddings such that their accessible points are contained in exactly one and exactly two composants, respectively. This result shows (see the introduction to Section 4) that a certain statement made by Mason in [15], without proof and unnecessary to his paper, is false. The state-

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ment said that if P is a pseudo arc in S^2 and E is a prime end of $S^2 - P$, then the impression of E is P.

In addition, we construct an imbedding of the pseudo arc which we *conjecture* satisfies Mason's statement. We also raise some interesting questions.

In Section 5, we define the notion of *essentially extendable* homeomorphism of a continuum $M \subseteq S^2$. We say that $h: M \twoheadrightarrow M$ is *essentially extendable to* S^2, iff there is some imbedding $\phi: M \to S^2$ such that $\phi h \phi^{-1}: \phi(M) \twoheadrightarrow \phi(M)$ is extendable to a homeomorphism of S^2 onto itself. We give examples of homeomorphisms of the pseudo arc onto itself which are not extendable, but which are conjugate to extendable homeomorphisms. We also give an example of a chainable continuum and homeomorphism of it which is not essentially extendable to S^2. Other examples are given and some interesting questions are raised. (We note that in another paper [7] we will show that every homeomorphism of a chainable continuum onto itself, is essentially extendable *to* S^3.)

In Section 6, we deal with the stable homeomorphism and composant problems, obtaining some partial results. One of the main theorems of Section 6 is Theorem 6.5, and it answers the stable homeomorphism problem in the affirmative, for homeomorphisms of P which are extendable. Thus we show that if h is a homeomorphism of the standard (see Section 4) pseudo arc P onto itself which is the identity on some open set, *and* h is extendable, then h is the identity. The other important theorems of this section are Theorems 6.4, 6.9, and 6.17.

The results of this section were motivated by the theorem of Iliadis [13]—proved in Section 3—together with Mason's statement [15], discussed in Section 4. An easy argument (Corollary 6.4) is given to show that there are no homeomorphisms other than the identity keeping each composant fixed, in case $h: Q \twoheadrightarrow Q$ is extendable, where Q is a pseudo arc satisfying Mason's statement.

Finally, in Section 7, we list and discuss some important and interesting questions raised by the results of this paper.

The author wishes to thank the referee for his helpful comments.

The important continua studied in this paper are indecomposable continua. See [12] for definition and basic properties. We note, however, that the picture 3-22 of [12] of the pseudo arc, is incorrect, as is the picture 3-21 of the "3-point continuum."

DEFINITIONS AND NOTATION. Let X be a space, $h: X \twoheadrightarrow X$ a homeomorphism and $A \subseteq X$. We say that A is *invariant* (under h) iff $h(A) \subseteq A$, and that A is *fully invariant* (under h) iff $h(A) = A$. We use a double arrow ($\twoheadrightarrow$) to denote an *onto* function. $O(x)$ denotes the orbit of x under the iterates of h.

2. Prime ends

In this section we give some basic definitions and examples, and state some well-known theorems about prime ends. We will use prime end theory through-out the paper, since it is essential for our main results about extendable stable homeomorphisms. See [9], [10], [14], [20] for detailed discussions of prime ends.

2.1. DEFINITIONS. A *domain* is a connected open set. If U is a domain in S^2 with nondegenerate boundary, then a *crosscut* of U is an open arc in U, whose closure is an arc which intersects Bd U in its two endpoints exactly. An *endcut* of U is a half open arc in U whose closure is an arc which intersects Bd U in one point only—its other endpoint.

Let U be a simply connected domain in S^2 with a nondegenerate boundary. A *C-transformation* or *C-map* from U onto the interior of the unit disk B is a homeomorphism $\phi\colon U \twoheadrightarrow \operatorname{Int} B$ such that (1) the image of any crosscut of U is a crosscut of B and (2) the endpoints of images of crosscuts of U, are dense in Bd B, the unit circle. C-transformations have the topological properties of conformal maps, and by [20] this is all that is necessary for prime end theory. Prime end theory is originally due to Caratheodory, and in most discussions, conformal maps are used.

A sequence of crosscuts $\{Q_i\}_{i=1}^{\infty}$ of the simply connected domain U, is a *chain* iff

(1) the arcs $\bar{Q}_1, \bar{Q}_2, \ldots,$ are pairwise disjoint,
(2) Q_n separates Q_{n-1} from Q_{n+1} in U, and
(3) diam $Q_i \to 0$. We further require that lim Q_i be a point.

Corresponding to each Q_n, there is a domain U_n of $U - Q_n$, containing Q_{n+1}. Then $U_1 \supseteq U_2 \supseteq \cdots$.

If $\{Q_i\}_{i=1}^{\infty}$ and $\{R_i\}_{i=1}^{\infty}$ are chains of crosscuts and $\{U_i\}_{i=1}^{\infty}$ and $\{H_i\}_{i=1}^{\infty}$ are their respective corresponding domains, then $\{Q_i\}$ and $\{R_i\}$ are *equivalent* chains iff for each n, there is an m such that $H_m \subseteq U_n$ and $U_m \subseteq H_n$. A *prime end* is an equivalence class of chains of crosscuts of U.

The map ϕ determines a 1-1 correspondence between the prime ends of U and the points of the unit circle, Bd B.

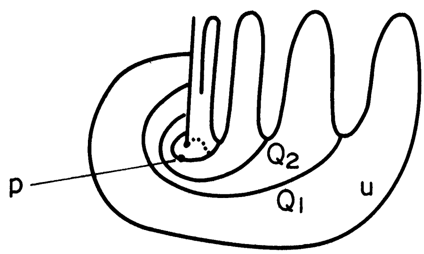
If $\{Q_n\}_{n=1}^{\infty}$ is a chain of crosscuts defining the prime end E, and $\{U_i\}_{i=1}^{\infty}$ is the sequence of corresponding domains of the chain, then the *impression of* E, $I(E)$, is the set $I(E) = \bigcap_{i=1}^{\infty} \bar{U}_i$. It can be shown that $I(E)$ is independent of which defining sequence of chains is used, and thus is well defined. Also $I(E) \subseteq$ Bd U, and is a continuum. Further, different prime ends may have the same impressions, and the impression of one prime end may be a proper subset of the impression of another. Examples will be given to illustrate this.

There are two equivalent ways to define the set of *principal points* of a prime end, and we give them both below. Let U be a simply connected domain, $\phi\colon U \twoheadrightarrow \operatorname{Int} B$ a C-map, E a prime end of U which corresponds to the point e in Bd B determined by ϕ. A half open arc A in U *defines the prime end* E iff $\phi(A)$ is an endcut of B, whose closure has endpoint e. (Note that $\bar{A}$ is not necessarily an arc.) Among the half open arcs defining E, there is one A such that $\bar{A} - A$ is minimal; that is $\bar{A} - A \subseteq \bar{A}' - A'$ for every half open arc A' defining E. This minimal set, which will be a subset of $I(E)$, is the set of *principal points* of E.

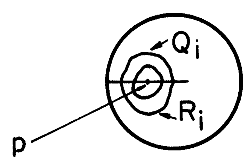
An alternate definition is the following: If E is a prime end of U, $\{Q_i\}$ is a chain defining E, and $p = \lim Q_i$, $p \in \operatorname{Bd} B$, then p is a *principal point* of E. We note that different equivalent chains of crosscuts may converge to different points of $I(E)$. An example will be given (2.2.4).

2.2. *Examples.*

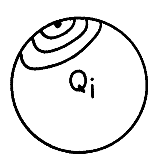
(1) E is determined by $\{Q_i\}$. $\{p\}$ is the only principal point of E. $I(E) =$ limit segment.



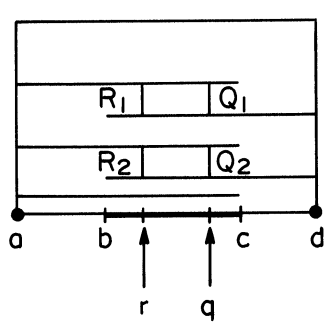
(2) $\{Q_i\}$ determines E. $\{R_i\}$ determines F. $E \neq F$. But $I(E) = I(F) = \{p\}$. Thus different prime ends may have the same impression.



(3)



(4) $\{R_i\} \equiv \{Q_i\}$ determines prime end E. $\lim R_i = r$. $\lim Q_i = q$. Interval $[b, c] =$ set of principal points of E. Interval $[a, d] = I(E)$.



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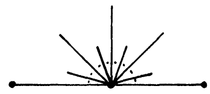


The continuum is a dendrite with p as the only branch point, and p has infinite order. At each successive stage, put in arcs emanating from p, cutting each previous angle, as well as length, in half. Then there are uncountably many different prime ends E such that $I(E) = \{p\}$. See page 53 of [14].

2.3. WELL-KNOWN RESULTS AND A DEFINITION.

2.3.1. THEOREM (See page 29 of [20].) *Let U be any simply connected domain in S^2 with nondegenerate boundary. Then there exists a C-map ϕ: $U \twoheadrightarrow$ Int B, where B is the unit disk in S^2.*

2.3.2. THEOREM [14, Theorem 2.20]. *The map ϕ of Theorem* 2.3.1 *establishes a* 1-1 *correspondence between the prime ends of U and the points of* Bd B, *which (correspondence) is continuous in both directions. That is, if E is a prime end determined by a chain of crosscuts $\{Q_i\}$ with corresponding domains $\{U_i\}$ and $e \in$* Bd B *is the point corresponding to E, then a sequence of points $\{x_i\}$ has the property that $x_i \in U_i$ for each i iff $\phi(x_i) \to e$ on* Bd B.

2.3.3. THEOREM (Ursell and Young [20], 4.10, page 6, and A17, page 27). *Let h: $\bar{U} \twoheadrightarrow \bar{U}$ be a homeomorphism with $h(U) = U$, and let ϕ: $U \twoheadrightarrow$* Int B *be a C-transformation. Then $\phi h \phi^{-1}$:* Int $B \twoheadrightarrow$ Int B *is a homeomorphism which can be extended to a homeomorphism $\overline{\phi h \phi^{-1}}$: $B \twoheadrightarrow B$.*

2.3.4. THEOREM [Rutt, 19]. (1) *If M is an indecomposable continuum, then $S^2 - M$ has a prime end whose impression is M.*

(2) *If M is a nondegenerate continuum not separating S^2, and some prime end of $S^2 - M$ has M as its impression, then M is indecomposable or the union of two indecomposable continua.*

(3) *If $S^2 - M$ has a prime end whose set of principal points is M, then M is indecomposable.*

2.3.5. DEFINITION. Let E be a prime end of the simply connected domain U, and $e \in$ Bd B, the point that corresponds to E under ϕ (a C-map). Then the prime end E is *fixed* under h iff $\overline{\phi h \phi^{-1}}(e) = e$, where h: $\bar{U} \twoheadrightarrow \bar{U}$ is a homeomorphism. Since, by Theorems 2.3.2 and 2.3.3 above, we may identify the prime ends of U with the points of Bd B, this definition makes sense.

2.4. THEOREM. *Let U be a simply connected domain in S^2 with nondegenerate boundary, and ϕ: $U \twoheadrightarrow$ Int B a C-map. Let E be a prime end of U such that $I(E) = \{x\}$. Then if A is any half open arc defining E, then $\bar{A} \cap$* Bd $U = \{x\}$, *so that $\bar{A}$ is an arc.*

Proof. Since $\{x\}$ is a principal point of $I(E)$ there exists a chain of concentric circular arcs with radii $\to 0$, and common center x, such that this chain is *crossed* by every half open arc defining E. (See 4.9, page 5, of [20].) Earlier in [20], following 4.6, the authors define *crossing* a chain of crosscuts to mean that for all n sufficiently large, a beginning arc of the half-open path, is separated from the remainder of the path by the nth crosscut of the chain. But if $I(E) = \{x\}$, we see that the diameters of the domains corresponding to the crosscuts, $\to 0$, so that $\bar{A}$ is arc and $\bar{A} \cap \text{Bd } U = \{x\}$.

3. Accessibility and Iliadis' theorem

In this section we study the relationship between accessible points of an indecomposable continuum, and its prime ends. The purpose of this section is to provide a complete proof of a theorem of Iliadis, Theorem 4 of [13], which is stated but not proved in that paper. In addition, we make his result a little more precise. These results will be used later in the paper.

3.1. THEOREM. *Let M be a nonseparating indecomposable continuum in S^2, C a composant of M, and a, b two points of C which are accessible from $S^2 - M$. Let A be a crosscut in $S^2 - M$ with endpoints a and b, and let U and V be the complementary domains of A in $S^2 - M$. Then there exists a unique proper subcontinuum K of M, $K \subseteq C$, such that $A \cup K$ is the boundary of either U or V.*

Proof. Since M is a nonseparating indecomposable continuum in S^2, no subcontinuum of M separates. Thus, by Theorem 11.5 on page 117 of Newman's *Elements of the topology of plane sets of points*, Cambridge Univ. Press, 1964, M is hereditarily unicoherent.

Now, since a and b lie in the same composant of M, there exists a proper subcontinuum containing both a and b. By the above paragraph, the intersection of any two such continua is another such continuum. Let K be the intersection of a maximal tower of such subcontinua. Then K is a minimal proper subcontinuum of M containing a and b. By the hereditary unicoherence of M, K is unique.

We show that $A \cup K$ is the boundary of either U or V. Note that $A \cup K$ separates S^2 into two components U' and V'. But since M is indecomposable, $A \cup K$ cannot separate M, or K would separate M. Thus $A \cup K$ contains the boundary of one of U' and V'. Let $A \cup K$ contain the boundary of U'. Then $U' = U$ or V, and without loss of generality, $U' = U$. Now suppose Bd U is proper in $A \cup K$, say Bd $U = A \cup K_1$ where K_1 is proper in K. Then $A \cup K_1$ fails to separate S^2, since K was the *minimal* subcontinuum of M containing both a and b. Thus Bd $U = A \cup K$.

3.2. THEOREM. *Let $M \subseteq S^2$ be an indecomposable continuum, and let A_1, A_2 be two endcuts of $S^2 - M$, each of whose endpoints on M is the same point x. Then A_1 and A_2 define the same prime end. Thus, each accessible point of an indecomposable plane continuum corresponds uniquely to a prime end.*

Proof. Suppose A_1 and A_2 define different prime ends E_1 and E_2. Then there exists a neighborhood U of x such that any endcut in U from a point of A_1 to a point of A_2, must meet M. (See [14, pp. 52–54] for discussion.) Thus M separates A_1 from A_2 in U. Let A_3 be an arc in $S^2 - M$ so that $A_1 \cup A_2 \cup A_3$ is a "crosscut" of $S^2 - M$, with both endpoints being x. Then $A_1 \cup A_2 \cup A_3$ is a simple closed curve whose intersection with M is x, and which separates M. Thus x is a cut point of M, and therefore M is not indecomposable. This is a contradiction, and the theorem follows.

3.3. THEOREM. *Let M be a nonseparating indecomposable continuum in S^2, C a composant of M, and a, b two points of C which are accessible from $S^2 - M$. Let a' and b' be the points of* Bd B *which correspond to a and b respectively, by the C-map, ϕ. Then one of the two open intervals I and J which are the components of* $\text{Bd } B - \{a', b'\}$, *say I, has the property that if $e \in I$, and E is the prime end associated with e, then $I(E) \subseteq C$.*

Proof. Let A be a crosscut in $S^2 - M$ from a to b, K be the continuum of Theorem 3.1, and U the complementary domain bounded by $A \cup K$. Let $I = \text{Int } (\overline{\phi(U)} \cap \text{Bd } B)$. Let E be a prime end of $S^2 - M$ determined by a chain of crosscuts $\{C_i\}_{i=1}^{\infty}$ each of which lies in U. Then $\{\phi(C_i)\}_{i=1}^{\infty}$ is a chain of crosscuts converging to a point $e \in I$. Further, for each $e \in I$, by Theorem 2.15 of [14], there exists a prime end of $S^2 - M$ with image e. Thus if $\{\phi(C_i)\}$ is a chain of crosscuts of $\phi(U)$ converging to e, then $\{C_i\}$ is a chain of crosscuts of U representing the prime end E corresponding to e. Clearly each $C_i \subseteq U$. Thus $I(E) \subseteq K \subseteq C$.

3.4. THEOREM. *For each composant C of the nonseparating indecomposable continuum $M \subseteq S^2$, where C has more than one accessible point, there exists a maximal "open interval" I (which may be all of* Bd B $(= S^1)$ *minus one point) such that $I(E) \subseteq C$ for each $e \in I$, where E is the prime end corresponding to e.*

Proof. We note first that there is at least one prime end E such that $I(E) = M$, by [19]. Thus the interval of the theorem must be a subset of (S^1 less one point). We also note that I may, in fact, be all of S^1 minus one point, as in the case if M is the chainable "U-continuum" (see Section 4) with its endpoint in an accessible composant.

Now let a and b be two points of C which are accessible from $S^2 - M$. By Theorem 3.3, there exists an open interval I_{ab} on Bd B, such that $I(E) \subseteq C$, if E is the prime end corresponding to $e \in I_{ab}$. Thus for each accessible point $c \neq a$ in C, there exists an open interval I_{ac}. Let $I' = \bigcup I_{ac}$. If I' is connected and $I' = S^1 - a'$, then by the above paragraph, $I = I'$. If I' is connected and has proper closure in S^1, then $I = I'$. If I' is not connected, $I' = (xa') \cup (a'y)$ where a' is the point of $S^1 = \text{Bd } B$ corresponding to a. (Note that x and y may be equal.)

We show that if F is the prime end corresponding to a', then $I(F) \subseteq C$. Let p', q' be points of I' on either "side" of a', which correspond to accessible points p, q of M in C. Let A be a crosscut of $S^2 - M$ from p to q. Then by Theorem 3.1, there exists a unique proper subcontinuum K of M, $K \subseteq C$, and complementary domain U of $(S^2 - M) - A$ such that $K \cup A = \text{Bd } U$. By Theorem 3.3, if $\phi(U) = U'$, then Int $[(\text{Bd } U') \cap S^1] = I_{pq}$ is such that $I(E) \subseteq C$ for any prime end E corresponding to $e \in I_{pq}$. But I_{pq} must contain a' since I' misses at least one other point.* Thus if F is the prime end corresponding to a', then $I(F) \subseteq C$. Further, if Z is a prime end corresponding to a point z' not in $I' \cup \{a'\}$, then $I(Z) \nsubseteq C$.** It follows that $I = I' \cup \{a'\}$ is the maximal interval of the theorem.

Proof of *. For if I_{pq} does not contain a', then $\phi(U) = U'$ is a domain whose closure contains that complementary interval of $\{p', q'\}$ on Bd B which misses a'. Thus *if* $z' \notin I'$ and $z' \neq a'$, then $z' \in \text{Int } (\overline{U'} \cap \text{Bd } B)$, and if the prime end Z corresponds to z', then $I(Z) \subseteq C$.

Let w' be a point of Bd B corresponding to both the accessible point w and the prime end W, such that w' lies between p' and x'. Then $w' \in I'$ and therefore if Q' is a crosscut of Int B from a' to w', $\phi^{-1}(Q') = Q$ is a crosscut of $S^2 - M$ from a to w, by Theorem 2.4. Now Q separates $S^2 - M$ into two domains O_1 and O_2, one of which, say O_1, is bounded by Q and a proper subcontinuum L of C. Let $\phi(O_1) = O_1'$ and suppose without loss of generality, that O_1' is that domain which contains p' in its closure. There exists an arc T' from a' to q' such that $T' \subseteq U' + O_1'$ and therefore $\phi^{-1}(T') = T$ is a crosscut (by 2.4) in $U + O_1$ from a to q. We may also require that $T' \cap A'$ is a single point. Thus $T + K + L$ contains the boundary of a complementary domain of $(S^2 - M) - T$, and $K + L$ is proper in C. It follows that $z' \in I'$. This is a contradiction. Thus if I_{pq} does not contain a', then I' is connected. This is again a contradiction. So I_{pq} contains a'.

Proof of **. Suppose $I(Z)$ is proper in C. Let $\{Q_i\}$ be a chain of crosscuts defining Z, with endpoints $\{(a_i, b_i)\}$. Then for n sufficiently large, $a_n, b_n \in C$. Thus $a_n' = \phi(a_n)$ and $b_n' = \phi(b_n)$ are sequences converging to z' on either "side" in Bd B, and correspond to accessible points in C. Since a is accessible in C we may argue that

$$(z', a') \cup (a', z') \subseteq I'$$

and thus $I(E)$ is proper in C, and therefore in K, for *all* prime ends E of $S^2 - M$. But by [19], there is at least one prime end whose impression is all of M. This is a contradiction.

3.5. Main Theorem (Iliadis [13]). *Let M be a nonseparating indecomposable continuum in S^2 and let ϕ: $(S^2 - M) \twoheadrightarrow \text{Int } B$ be a C-map. Then ϕ induces a "decomposition" on $S^1 = \text{Bd } B$ into a (possibly $\emptyset$) collection of pairwise disjoint open intervals and the points of the complement of this collection. (Recall from Theorem 3.4 that an "open interval" may be S^1 minus one point.) These open intervals are in 1-1 correspondence with those composants containing more than one accessible point. If e is a point of such an interval and E is the corresponding prime end, then $I(E)$ is proper in M, while if e is a point of the complement of the union of these intervals, and E is a prime end corresponding to e, then $I(E) = M$.*

Proof. Again, as observed in the first paragraph of Theorem 3.4, the decomposition may be a point and the complementary "open interval." By Theorem 3.4, each composant C containing more than one accessible point corresponds to a maximal open interval I_{ab} on $S^1 = \text{Bd } B$, such that the impression of each prime end corresponding to a point of that interval is a subset of C. Further, from that proof it follows that no two disjoint open intervals can correspond to the same composant. Thus there is a 1-1 correspondence between nonempty open intervals and composants containing more than one accessible point.

We know there exists at least one point $f \in \text{Bd } B$ such that $I(F) = M$, where F is the prime end corresponding to f. We show that if $e \neq f$, $e \in \text{Bd } B$, e is not in any of these open intervals, and E is the prime end corresponding to e, then $I(E) = M$, also. Let $\{C_i\}_{i=1}^{\infty}$ be a chain of crosscuts of $S^2 - M$, defining E, with endpoints $\{(a_i, b_i)\}$. Then $\{\phi(C_i)\}$ is a chain of crosscuts in Int B, with endpoints $\{(a'_i, b'_i)\}$ on Bd B, and $a'_i \to e$ and $b'_i \to e$. Since e is not an element of our defined set of open intervals, there exists N such that for $n > N$, not both a'_n and b'_n are in same one of these open intervals. Thus for $n > N$, a_n and b_n belong to different composants of M, and the smallest subcontinuum of M containing a_n and b_n is M. It follows that $I(F) = M$.

3.6. COROLLARY (Iliadis [13] and Mazurkiewicz [16].) *There are at most countably many composants of an indecomposable continuum M in S^2 with more than one accessible point.*

Proof. Clear, since, in each complementary domain, there are at most countably many with more than one accessible point.

3.7. *Remark.* If M is a nonseparating indecomposable continuum such that all the accessible points of M lie in exactly one composant of M, then the "interval" corresponding to this composant is S^1 minus one point. For example, the "U-continuum" (see Section 4) with its standard construction has this property.

4. Inequivalent imbeddings

In this section we ask whether every two pseudo arcs in S^2 are equivalently imbedded, and show that the answer is "no." We do this by constructing imbeddings which are inequivalent, because their accessible points are contained in different numbers of composants.

In Section 3 of [15], Mason states: "The reader unfamiliar with prime ends might attempt to show as an *exercise* that if K is a pseudo arc and E is a prime end of $S^2 - K$, then $I(E) = K$"; that is, the impression of each prime end of $S^2 - K$ is all of K. The statement fails to take into account different possible imbeddings, and presumably meant only the "standard" one. It is an easy consequence of this statement, if it were true, and the results of Section 3, that each composant can contain at most one accessible point. See Section 6 for proof and applications of this result.

The results of this section show that Mason's statement is *false—even for the standard pseudo arc*. Mason's otherwise excellent paper was the motivation for our interest in prime ends and this statement, which we initially believed, was the motivation for the portion of this paper dealing with stable homeomorphisms.

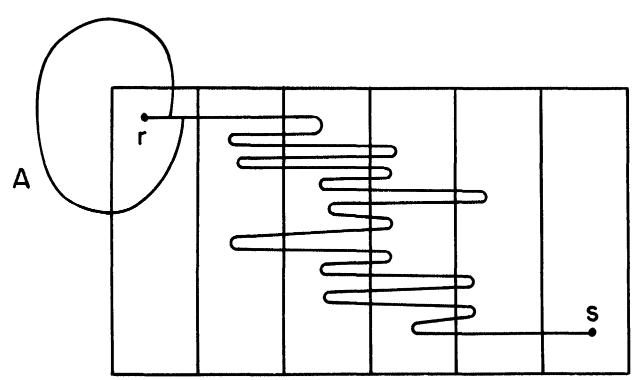
Our special imbeddings will be imbeddings with the following properties:

(a) *The standard pseudo arc P in S^2*. This continuum has exactly 2 composants with accessible points, and the two "endpoints" r and s lie in different composants.

(b) *The U-continuum pseudo arc P_u*. This continuum has only one composant with accessible points and all its accessible points lie in the same composant as the "endpoint" of P_u.

(c) *The special pseudo arc P_s*. We *conjecture* that this continuum is imbedded in S^2 in such a way that no composant contains more than one accessible point. This continuum would then have the property that the impression of each prime end of $S^2 - P_s$ is all of the continuum. Thus it would show that, in some cases, Mason's statement does hold.

4.1. *Example*. The standard pseudo arc P. See [17]. The construction is outlined—essentially the refinements are made by "descending chains"—that is, if the big chain is straightened out, keeping r to the left and s to the right, the refinement is descending. The construction should be carried out in a symmetric manner, so that at each stage, $\mathscr{C}_n$ is carried onto itself in reverse order, by a 180° rotation of the plane. We will show below that P contains exactly 2 composants with accessible points, one containing r, and the other containing s. Further, if E_1 is the prime end corresponding to r (i.e., defined by a chain of crosscuts going around r, as does A in diagram) then $I(E_1) = P$. Similarly if E_2 is crosscut corresponding to s, then $I(E_2) = P$.



4.1.1. DEFINITION. We think of P as a subset of E^2. Let L and M be two vertical lines containing r and s respectively. Then P separates the strip between L and M into an upper open set and a lower open set. The accessible points, other than r and s, which are accessible from the upper open set are called the *top accessible points*, and the others are the *bottom accessible points.*

4.1.2. THEOREM. *All the top accessible points are in the same composant as r.*

Proof. Let L and M be vertical lines through r and s respectively. Let K' be a crosscut joining two top accessible points, a and b. Then from the geometry, K' can be homotoped to a crosscut K which lies between L and M and above P, in such a way that (1) the homotopy occurs outside a neighborhood of P, and (2) for any initially given fixed integer n, K meets only those links of $\mathscr{C}_n$ containing a and b.

Now $K \cup P$ separates S^2 into two connected open sets U and V, one of which is bounded, say U. Let $K \cup P'$ be the boundary of U, and by Theorem 3.1, P' is a (sub)continuum of P.

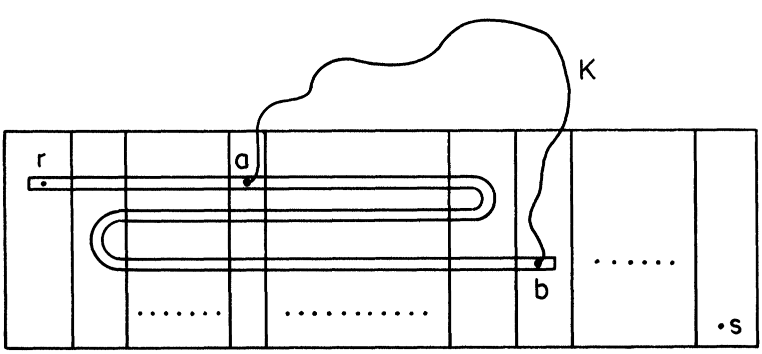
We show that P' contains r but not s. Let n be so large that a and b are separated by at least five links of $\mathscr{C}_n$, as are b and s, and also let a, b be named so that the links containing r, a, and b, are in the order r-a-b reading from left to right. Let $\mathscr{C}_{n+1}(1, m)$ be the smallest subchain of $\mathscr{C}_{n+1}$ from the first link of $\mathscr{C}_{n+1}$ to a link containing b. Then

$$\mathscr{C}_{n+1}(1, m) = \mathscr{C}_{n+1}(1, x) + \mathscr{C}_{n+1}(x, y) + \mathscr{C}_{n+1}(y, z)$$

where the xth link is in the link adjacent to that link of $\mathscr{C}_n$ containing b, and yth link is in the 2nd link of $\mathscr{C}_n$.

Let $h: S^2 \twoheadrightarrow S^2$ be a straightening homeomorphism for the chain $\mathscr{C}_n$. We assume that K was chosen so that the "n" of property (2) was for the chain $\mathscr{C}_n$. Thus $h(K) \cup h(P')$ bounds $h(U)$, and we see from the geometry (see diagram below) that $h(U)$ contains a point of the third link of $h(\mathscr{C}_n)$. Then U contains a point of the third link of $\mathscr{C}_n$. But as $n \to \infty$, $\mu(\mathscr{C}_n) \to 0$, so that r is a limit point of U, and therefore $r \in \bar{U}$, and therefore $r \in P'$.

We observe that $s \notin P'$, since there exists a subcontinuum from r to b contained in $\mathscr{C}_n(1, l)$, where lth link is the first link of $\mathscr{C}_n$ containing b. This is true because $\mathscr{C}_{n+1}$ is decreasing in $\mathscr{C}_n$. It follows that there is a proper subcontinuum containing a and b, so that $a, b \in C_r$, the composant containing r.



4.1.3. THEOREM. *All the bottom accessible points are in the same composant as s.*

Proof. The argument is symmetric to that of 4.1.2, since the construction is symmetric.

4.1.4. COROLLARY. *There are exactly two composants containing the accessible points—one containing r and one containing s.*

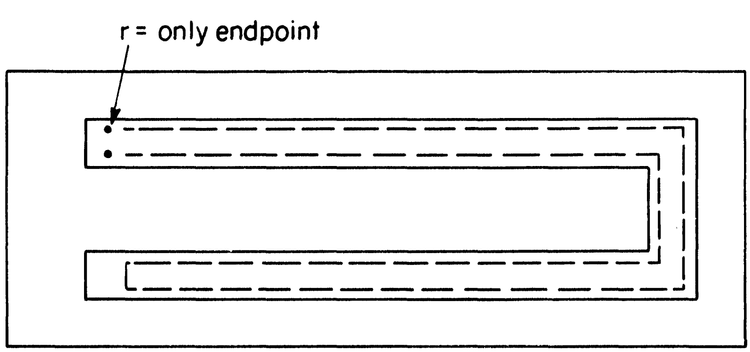
Proof. Clearly r and s are in different composants since P is irreducible between r and s.

4.1.5. THEOREM. *Let E and F be the unique prime ends of P which correspond to r and s, respectively (Theorem 3.2). Let $\{K_i\}$ be a chain of crosscuts defining E (or F). Then $I(E) = P(I(F) = P)$.*

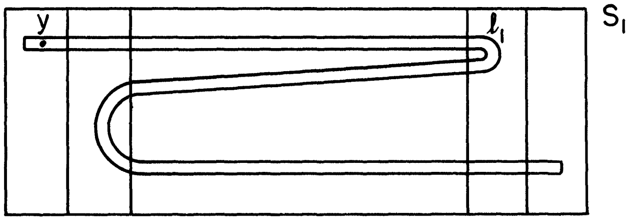
Proof. From the proof of 4.1.2, it follows that each K_i must have one endpoint in the composant of r and one in the composant of s. Therefore the bounded domain cut off by $K_i \cup P$ is bounded by all of $K_i \cup P$, since no proper subcontinuum of P can contain both a_i and b_i. Thus $I(E) = P$.

4.2. *Example.* The standard "U-continuum" K_U. This continuum, with its standard construction, is a chainable continuum in S^2 with exactly one endpoint, r, such that the composant of r, C_r, contains all the accessible points of K_U. The construction is carried out, as indicated in the diagram, with 2-1 refinements at each stage, the end links at each stage always being in the first link of the preceding stage, and the first link always lying above the last link. The endpoint r is the intersection of the first links.

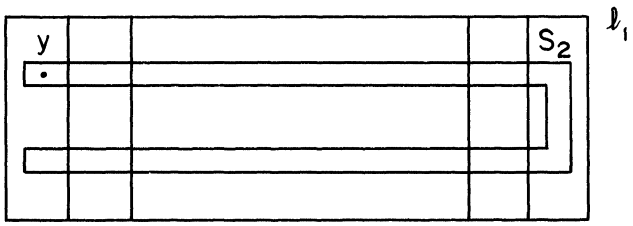
If E is the prime end corresponding to r, then $I(E) = K_U$. Otherwise $I(E)$ is a point. Note that if ϕ: $(S^2 - K_U) \twoheadrightarrow \operatorname{Int} B$ is a C-map, then the "interval" of Bd B that corresponds to C_r, by Section 3, is Bd B less one point.



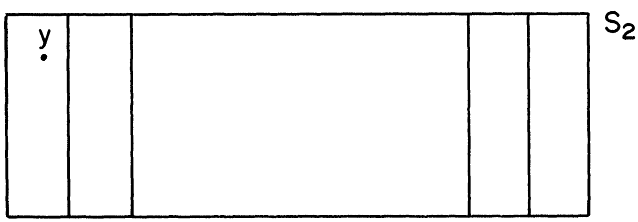
4.3. *Example.* The U-continuum pseudo arc $P_u \subseteq S^2$. The idea is to alternate straight chains $\{\mathcal{S}_i\}$ and crooked chains $\{\mathcal{C}_i\}$, in the 2-1 manner indicated for the U-continuum. We illustrate:



To construct $\mathscr{S}_2$, straighten $\mathscr{C}_1$ to get



and refine with $\mathscr{S}_2$, where $\mathscr{S}_2$ is 2-1 in $\mathscr{C}_1$. To construct $\mathscr{C}_2$, straighten $\mathscr{S}_2$



and put in a descending crooked chain $\mathscr{C}_2$ from y to last link of $\mathscr{S}_2$. Etc. Clearly P_u is chainable and hereditarily indecomposable, and thus it is a pseudo arc [4]. We will show that all the accessible points of P_u lie in C_y, the composant containing the "endpoint" y of P_u. We will also show that if F is the prime end corresponding to y, then $I(F) = P_u$, while if E is any other prime end, then $I(E)$ is proper in P_u.

4.3.1. THEOREM. *Let y be the intersection of the first links of each chain, as illustrated in the diagram, and let C_y be the composant of y. Then all the accessible points of P_u are in C_y.*

Proof. The argument is similar to that of Theorem 4.1.2. Here we choose a crosscut A from y to the accessible point x, in such a way that $h_n(A)$ lies in the upper half "plane" determined by $h_n(P_u)$ for some n, where h_n is a straightening homeomorphism for $\mathscr{C}_n$. Then show that $y = h_n(y)$ and $h_n(x)$ lie in a proper subcontinuum of P_u.

4.3.2. THEOREM. *Let E be any prime end not corresponding to the accessible point y of P_u. Then $I(E)$ is proper in P_u, and contains y.*

Proof. Let $\{Q_i\}_{i=1}^{\infty}$ be a chain of crosscuts defining E. Then $Q_1 \cup P_u$ separates S^2 and the bounded complementary domain contains Q_i for all $i \geq 2$. Thus there exists n such that if h_n is the straightening homeomorphism for the nth chain $\mathscr{S}_n$, then Q_1 is homotopic to a crosscut Q which lies above $h_n(\mathscr{S}_n)$ in such a way that (1) the homotopy is the identity outside some neighborhood of $h_n(P_u)$ and (2) Q meets only those links of $h_n(P_u)$ which contain the endpoints of Q. Thus there exists an integer J such that for $i \geq J$, $h_n(Q_i)$ is a subset of the bounded complementary domain of $h_n(P_u) \cup Q$. Let a and b be the endpoints of Q.

Now choose a towered sequence of subchains of $\{\mathscr{C}_n\}$, beginning with the first link of $\mathscr{C}_n$ for each n, in such a way that the last links are towered, and so that a and b are always in an interior link. This sequence defines a proper subcontinuum K of P_u, with the two endpoints y and z, say. Then, as in the proof of Theorem 4.3.1, z is not in the composant of y *in* K, but a and b are. And each crosscut of the chain $\{h_n(Q_i)\}_{i=J}^{\infty}$ determines a proper subcontinuum *of* K containing a, b, and y. Thus $h_n^{-1}(K)$ is proper in P_u, and contains both y and $I(E)$. It follows that $I(E)$ contains y.

4.3.3. THEOREM. *Let F be the prime end corresponding to y. Then $I(F) = P_u$.*

Proof. Since P_u is indecomposable, there is at least one prime end E such that $I(E) = P_u$ [19]. But if E is any prime end, $E \neq F$, then $I(E)$ is proper by 4.3.2. Thus $I(F) = P_u$.

4.4. *Example.* The special pseudo arc P_s. We obtain P_s as the intersection of a tower of chains from fixed endpoint r to fixed endpoint s, with the crooked refinements *alternating* with respect to "ascending" and "descending" patterns. Here, "ascending" and "descending" means with respect to a straightening out process, as indicated in Example 4.3.

We conjecture that if E is a prime end of P_s, then $I(E) = P_s$. It would then follow from Theorem 3.1, that no composant of P_s contains more than one accessible point.

4.5. *Remark.* From the above, we have at least two inequivalently imbedded pseudo arcs in S^2. Probably one can get countably many, by following a pattern for chainable continua in S^2 with exactly n endpoints in S^2. (We are thinking of generalizing the example of the "3-point continuum" constructed as a chainable continuum, irreducible between each two of three given points. See pp. 141–142 of [12] for (incorrect) picture and (correct) precise description of construction.) That is, construct a pseudo arc, using these chainable continua as a guide, in a manner similar to our example P_u.

QUESTION 1. Are there uncountably many inequivalently imbedded pseudo arcs in S^2?

QUESTION 2. Does the special pseudo arc P_s have the property that no composant contains more than one accessible point?

5. Essentially extendable homeomorphisms

We call a homeomorphism h of a planar continuum M onto itself, *essentially extendable* iff there exists an imbedding $\phi: M \to E^2$ such that

$$\phi h \phi^{-1}: \phi(M) \twoheadrightarrow \phi(M)$$

can be extended to a homeomorphism $\overline{\phi h \phi^{-1}}: E^2 \twoheadrightarrow E^2$. ($S^2$ may replace E^2 in the definition.)

In this section we construct several examples to show the following:

(1) The standard pseudo arc P admits homeomorphisms which are not extendable, but are conjugate to extendable homeomorphisms. That is, these homeomorphisms are essentially extendable by an imbedding $\phi: P \twoheadrightarrow P \subseteq E^2$.

(2) The U-continuum pseudo arc P_u admits homeomorphisms which are essentially extendable, but are not conjugate to extendable homeomorphisms. That is, M and $\phi(M)$ are inequivalently imbedded.

(3) There exists a chainable continuum $M \subseteq E^2$ and a homeomorphism $h: M \twoheadrightarrow M$ such that h is not essentially extendable.

We obtain the above results using prime end theory and, in the process, prove some theorems which we will use in a forthcoming paper [8] on periodic homeomorphisms. The idea of essential extendability will be very important if the results of Section 6 can be improved so that (1) the Stable Homeomorphism Problem has an affirmative solution for *all* essentially extendable homeomorphisms, and (2) every homeomorphism of the pseudo arc onto itself, is essentially extendable. In another forthcoming paper [7], we show that every homeomorphism of the pseudo arc onto itself, is essentially extendable *to* E^3, by an imbedding $\phi: P \to E^2 \subseteq E^3$. Does this imbedding, or a modification of it, work for E^2?

The results of this section raise many interesting questions, and we list these in Section 8 of this paper. We also mention the most basic ones at the end of this section.

5.1. THEOREM. *Let M be a nonseparating indecomposable continuum in S^2, $h: S^2 \twoheadrightarrow S^2$ a homeomorphism such that (1) $h(M) = M$ and (2) for some accessible point $e \in M$, $h(e) = e$. Let E be the prime end corresponding to e. Then E is a fixed prime end.*

Proof. Let $\phi: (S^2 - M) \twoheadrightarrow \mathrm{Bd}\, B$ be a C-map, and let e' be that point of $\mathrm{Bd}\, B$ which corresponds to e. Then $\overline{\phi h \phi^{-1}}: B \twoheadrightarrow B$ is a homeomorphism, and we show that $\overline{\phi h \phi^{-1}}(e') = e'$.

Let A be an endcut in Int B with endpoint e'. From Theorem 2.4, it follows that $\phi^{-1}(A)$ is an endcut in $S^2 - M$ with endpoint e. Thus $h(\phi^{-1}(A))$ is also an endcut in $S^2 - M$ with endpoint e. By Theorem 3.2, the accessible point e corresponds to exactly one prime end, so $h(\phi^{-1}(A))$ must also correspond to E. That is, A and $\phi h \phi^{-1}(A)$ are endcuts in Int B each with endpoint e'. Thus $\overline{\phi h \phi^{-1}}(e') = e'$.

5.2. THEOREM. *Let Q be a pseudo arc in S^2 and h: $S^2 \twoheadrightarrow S^2$ a homeomorphism such that* (1) $h(Q) = Q$ *and* (2) $h \mid Q$ *is of period* 2. *Let*

$$\phi\colon (S^2 - Q) \twoheadrightarrow \text{Int } B$$

be a C-map. Then $\overline{\phi h \phi^{-1}} \mid$ Bd B is also of period 2.

Proof. If x is any nonfixed accessible point of Q, then $O(x) = \{x, h(x)\}$, and the two points x and $h(x)$ correspond to distinct points (Theorem 3.2) x' and $h(x)'$ on Bd B, under $\overline{\phi h \phi^{-1}}$.

Since $\overline{\phi h \phi^{-1}}$ is a homeomorphism, it interchanges x' and $h(x)'$. But the images of accessible points of Q are dense in Bd B. Thus $\overline{\phi h \phi^{-1}} \mid$ Bd B is of period ≤ 2 on a dense subset of Bd B and therefore must be of period ≤ 2. But at least one point has order 2. Thus $\overline{\phi h \phi^{-1}} \mid$ Bd B is of period 2.

Remark. A more general version of this theorem will be proved in another paper [8].

5.3. THEOREM. *Let P_u be the U-continuum pseudo arc in S^2 with endpoint y. Let A_1 and A_2 be two crosscuts from y to a_1 and a_2 respectively. Let $A_1 \cup K_1$ bound a complementary domain W_1 of $S^2 - (P_u \cup A_1)$, and let $A_2 \cup K_2$ bound a complementary domain W_2 of $S^2 - (P_u \cup A_2)$, where K_1 and K_2 are proper subcontinua of P_u. Let ϕ: $(S^2 - P_u) \twoheadrightarrow$ Int B be a C-map, and let $A'_1 = \phi(A_1)$ and $A'_2 = \phi(A_2)$. Let D_1 be the arc of* Bd B *such that $A'_1 \cup D_1$ bounds $\phi(W_1)$ and let D_2 be the arc of* Bd B *such that $A'_2 \cup D_2$ bounds $\phi(W_2)$. Then if $K_1 \subseteq K_2$ then $D_1 \subseteq D_2$.*

Proof. Let $y' \in$ Bd B be the point corresponding to y. This represents the only prime end E for which $I(E) = P_u$. Since each accessible point lies in the composant of y, the circle can be thought of as a "half open interval" $[y', y') \cong [0, 1)$ (see Section 3), where the accessible points each correspond to exactly one prime end, and their images are dense in Bd B.

Now by the geometry and arguments of Section 4, we may see that A_1 and A_2 are homotopic to crosscuts X_1 and X_2 with the same endpoints as A_1 and A_2 respectively, and such that $\overline{O_1} \subseteq \overline{O_2}$, where O_i is the domain bounded by $X_i \cup K_i$ for $i = 1, 2$. If $D_i^* = \overline{\phi(O_i)} \cap$ Bd B, we see that $D_i^* = D_i$ for $i = 1, 2$. But $\overline{O_1} \subseteq \overline{O_2}$ implies that $\overline{\phi(O_1)} \subseteq \overline{\phi(O_2)}$ which implies that $D_1^* \subseteq D_2^*$. Thus $D_1 \subseteq D_2$.

5.4. THEOREM. *Let P be the standard pseudo arc in S^2, chained from r to s,* *and let h: $S^2 \twoheadrightarrow S^2$ be a homeomorphism such that $h(P) = P$. Then the endpoints r and s are either fixed or interchanged.*

Proof. In Section 4, we showed that if E_1 is the prime end corresponding to r, and E_2 is the prime end corresponding to s, then $I(E_1) = I(E_2) = P$. Further, if E is any other prime end of $S^2 - P$, then $I(E)$ is proper in P. Thus these prime ends must be carried to themselves under h and it follows that either $h(r) = r$ and $h(s) = s$, or $h(r) = s$ and $h(s) = r$.

5.5. COROLLARY. *If h: $P \twoheadrightarrow P$ is conjugate to an extendable homeomorphism, then either h has at least 2 fixed points in different composants, or there are a pair of points of different composants which are interchanged by h.*

5.6. THEOREM. *Let P be the standard pseudo arc in S^2, chained from r to s. Then there is a homeomorphism h: $P \twoheadrightarrow P$ such that h is of period 2, and r is the only fixed point.*

Proof. In [6] we showed that there exists a period 2 homeomorphism g: $P \twoheadrightarrow P$ such that g is a restriction of a period 2 rotation of E^2, g interchanges the endpoints r and s, and g has exactly one fixed point x.

By the homogeneity of P [2, 18], there is a homeomorphism f: $P \twoheadrightarrow P$ such that $f(x) = r$. Then $h = fgf^{-1}$ is of period 2, and has as its only fixed point, the endpoint r.

5.7. THEOREM. *Let P be the standard pseudo arc in S^2 chained from r to s, and let h: $P \twoheadrightarrow P$ be the period 2 homeomorphism given in the proof of Theorem 5.6. Then h is not extendable to a homeomorphism of S^2 onto itself, but h is conjugate to one which is extendable.*

Proof. Suppose h is extendable. Then by Theorem 5.4, $h(r + s) = r + s$. By construction, $h(r) = r$. Thus $h(s) = s$. But h has exactly one fixed point. This is a contradiction. Thus h is not extendable.

We show that h is conjugate to an extendable homeomorphism. Let f and g be as in the proof of Theorem 5.6. Then $f^{-1}hf = f^{-1}(fgf^{-1})f = g$. But g is a restriction of a rotation, so g is extendable. Thus $f^{-1}hf = g$ is a conjugate of h, and is extendable.

5.8. THEOREM. *Let P_u be the U-continuum pseudo arc in S^2. Then there exists a homeomorphism α: $P_u \twoheadrightarrow P_u$ such that α is essentially extendable, but is not conjugate to any extendable homeomorphism of P_u onto itself.*

Proof. Let P be the standard pseudo arc in S^2 chained from r to s. Let g: $P \twoheadrightarrow P$ be a period 2 homeomorphism, obtained from a rotation of S^2 onto itself, and let x be the fixed point of P under g. Let h: $P \twoheadrightarrow P_u$ be a homeomorphism such that $h(x) = y$, the endpoint of P_u. Then hgh^{-1} is a homeomorphism of P_u onto itself, of period 2, with y as its only fixed point.

We show that hgh^{-1} is not conjugate to any extendable homeomorphism of P_u onto itself; that is, there does not exist a homeomorphism $\psi\colon P_u \twoheadrightarrow P_u$ such that

$$\psi(hgh^{-1})\psi^{-1}\colon P_u \twoheadrightarrow P_u$$

is extendable. Suppose such a ψ exists. Let $f = (hgh^{-1})\psi^{-1}$. We assume that ψ and f are defined on all of S^2. Let $\phi\colon (S^2 - P_u) \twoheadrightarrow \operatorname{Int} B$ be a C-map, and let $y' \in \operatorname{Bd} B$ be the point of Bd B corresponding to the prime end whose impression is P_u. In Section 4, we showed that there is exactly one such prime end E, and it corresponds to the accessible point y of P_u. Thus $f(y) = y$ and $\overline{\phi f \phi^{-1}}\colon B \twoheadrightarrow B$ takes y' to y', by Theorem 5.1.

Now $f\colon P_u \twoheadrightarrow P_u$ is of period 2, and its only fixed point is y. Thus by Theorem 5.2, $\overline{\phi f \phi^{-1}}\,|\,\operatorname{Bd} B$ is of period 2, and, since y' is fixed, it must be (conjugate to) a reflection on Bd B. Let A be a crosscut in $S^2 - P_u$ joining the endpoint y of P_u to another accessible point z in P_u, and let $A' = \phi(A)$. $A \cup P_u$ separates S^2 and one of the two components of $S^2 - (A \cup P_u)$, say W, is bounded by a proper subcontinuum K of P_u, together with A. Then $\phi(W)$ is bounded by $A' \cup D$, where D is an arc in Bd B, joining y' to z', the point of Bd B corresponding to z. Thus $\overline{\phi f \phi^{-1}}(\phi(\bar{W}))$ meets Bd B in an arc $\overline{\phi f \phi^{-1}}(D)$ from y' to some (other) point of Bd B, in such a way that D neither contains nor is contained in $\overline{\phi f \phi^{-1}}(D)$ since $\overline{\phi f \phi^{-1}}$ is of period 2 on Bd B.

However, since P_u has only one composant containing accessible points, and both K and $h(K)$ are proper subcontinua containing the endpoint y of P_u, and the pseudo arc is hereditarily indecomposable [2 and 17], it follows that $K \subseteq h(K)$ or $h(K) \subseteq K$. Suppose, without loss of generality, that $K \subseteq h(K)$. Then it follows from Theorem 5.3 that $D \subseteq \overline{\phi f \phi^{-1}}(D)$. (Similarly, if $h(K) \subseteq K$, then $\overline{\phi f \phi^{-1}}(D) \subseteq D$.) This is a contradiction to the fact that $\overline{\phi f \phi^{-1}}\,|\,\operatorname{Bd} B$ is a conjugate of a reflection.

It follows that $f = \psi(hgh^{-1})\psi^{-1}$ could not be extendable, and so hgh^{-1} is not conjugate to an extendable homeomorphism of P_u onto itself. Thus hgh^{-1} is the α of our theorem.

5.9. THEOREM. *There exists a chainable continuum $M \subseteq S^2$ and a homeomorphism $h\colon M \twoheadrightarrow M$ such that h is not essentially extendable.*

Proof. Let P be the standard pseudo arc in S^2 with right endpoint s on the y-axis. That is, P lies to the left of the y-axis. Let $f\colon S^2 \twoheadrightarrow S^2$ be a reflection through the y-axis, and let $M = P \cup f(P)$. Let $g\colon P \twoheadrightarrow P$ be a period 2 homeomorphism, keeping the right endpoint s fixed, as in Theorem 5.6. Let $h\colon M \twoheadrightarrow M$ be defined by

$$h(x) = \begin{cases} g(x) & \text{if } x \in P, \\ x & \text{if } x \in f(P). \end{cases}$$

We will show that h is not essentially extendable. We note that there are exactly two prime ends E and F corresponding to s.*

We first show that if $\alpha\colon M \twoheadrightarrow M$ is any extendable homeomorphism, then $\alpha(E) = E$ and $\alpha(F) = F$. Let $\{A_i\}_{i=1}^{\infty}$ be a chain of crosscuts defining E. Then for sufficiently large i, A_i lies either "above" or "below" M. We assume that A_i lies above M for $i \geq N$. Then each of the complementary domains of $A_i \cup M$ is bounded by all of $A_i \cup M$. (The arguments are similar to those in Section 4.) Now if $\{B_i\}_{i=1}^{\infty}$ is a chain of crosscuts representing F, then for $i \geq L$, B_i lies "below" M. However, again by arguments like those of Section 4, we see that for each $i \geq L$, one of the complementary domains of $A_i \cup M$ is bounded by all of $A_i \cup M$, and the other is bounded by $A_i \cup K_i$, where K_i is a continuum in M such that $K_i \cap P$ is proper in P and $K_i \cap f(P)$ is proper in $f(P)$. Thus no homeomorphism of S^2 onto itself carrying M onto M, can take a chain of crosscuts like $\{A_i\}$ to a chain of crosscuts like $\{B_i\}$. It follows that $\alpha(E) = E$ and $\alpha(F) = F$.

We next show that h is not extendable. Suppose h is extendable to the homeomorphism $H\colon S^2 \twoheadrightarrow S^2$. Let $\phi\colon (S^2 - M) \twoheadrightarrow \operatorname{Int} B$ be a C-map. Let $\underline{t = f(r)} \in f(P)$, and let t' be the corresponding point on Bd B. Then $\overline{\phi H \phi^{-1}}(t') = t'$, by the arguments of Theorems 3.2 and 5.1. From the last paragraph, the cut point $\underline{s \text{ of } M}$ corresponds to two points s' and s'' on Bd B, and $\overline{\phi H \phi^{-1}}(s') = s'$ and $\overline{\phi H \phi^{-1}}(s'') = s''$. Thus the induced homeomorphism on Bd B has at least three fixed points. But by Theorem 5.2, $\overline{\phi H \phi^{-1}} \,|\, \text{Bd } B$ must be of period 2. However, this is impossible, and we have a contradiction. It follows that h is not extendable.

Finally we show that h is not essentially extendable. Suppose $\psi\colon M \to S^2$ is an imbedding such that

$$\psi h \psi^{-1}\colon \psi(M) \twoheadrightarrow \psi(M)$$

is extendable to a homeomorphism

$$\beta\colon S^2 \twoheadrightarrow S^2.$$

Then $\overline{\phi_0 \beta \phi_0^{-1}} \,|\, \text{Bd } B$ is the identity on the open interval $(\psi(s)', \psi(s)'')$ which contains $\psi(t)'$, where $\phi_0\colon (S^2 - \psi(M)) \twoheadrightarrow \operatorname{Int} B$, and $\psi(s)'$, $\psi(s)''$, and $\psi(t)'$ correspond to $\psi(s)$, $\psi(s)$, and $\psi(t)$, respectively. That is, there will again be two prime ends corresponding to $\psi(s)$. Also $\overline{\phi_0 \beta \phi_0^{-1}} \,|\, \text{Bd } B$ is of period 2 on the complementary open interval. But this is impossible, and we see that h is not essentially extendable.

Proof of *. Suppose there are at least three prime ends of $S^2 - M$ corresponding to s. Then there exist endcuts A_1 and A_2 of $S^2 - M$, each of whose endpoints on M is the point s, and such that A_1 and A_2 are endcuts from (say) the "top" of M, with A_1 and A_2 representing different prime ends. Then, as in the proof of Theorem 3.2, there exists a neighborhood U of s such that M separates A_1 from A_2 in U.

Now suppose there exist arbitrarily small neighborhoods containing s, such that in each of these neighborhoods, P separates A_1 from A_2. Then s would not correspond uniquely to a prime end of $S^2 - P$. But this contradicts Theorem 3.2. A similar statement holds if P is replaced by $f(P)$.

It follows that there is exactly one prime end of $S^2 - M$ corresponding to s from above, and similarly, exactly one prime end of $S^2 - M$ corresponding to s from below.

5.10. QUESTIONS. The basic question raised by the results of this section is the following: Is every homeomorphism of the pseudo arc onto itself essentially extendable to E^2? In another paper [7], we show that every such homeomorphism *is* essentially extendable *to* E^3.

Another related question is: Is every homeomorphism of P onto itself conjugate to an extendable homeomorphism of P onto itself?

6. Stable homeomorphisms

The *Stable Homeomorphism Problem* for the pseudo arc P, is the following: If $h: P \twoheadrightarrow P$ is a homeomorphism which is the identity on some open set, must h be the identity? The *Composant Problem* asks: If $h: P \twoheadrightarrow P$ is a homeomorphism carrying each composant onto itself, must h be the identity? Clearly, a positive answer to the Composant Problem implies a positive answer to the Stable Homeomorphism Problem.

In this section we first give an easy argument to show that if Q is a pseudo arc in S^2 such that each prime end of $S^2 - Q$ has Q as its impression, and if $h: Q \twoheadrightarrow Q$ is a homeomorphism *extendable* to S^2 and carrying each composant to itself, then h must be the identity. (Recall that in Section 4, we showed that the prime end hypothesis of this statement need not hold.) As a corollary (6.4) we have our *First Main Theorem* of this section: If every homeomorphism of Q onto itself is essentially extendable by means of an imbedding ϕ such that $\phi(Q)$ satisfies the above prime end hypothesis, then both the composant and the stable homeomorphism problems have positive solutions.

We next obtain the *Second Main Theorem* (6.5) of this section: If $h: P \twoheadrightarrow P$ is an extendable (to S^2) homeomorphism which is the identity on an open set, then h is the identity. *Question:* Does this result hold if h is only essentially extendable? The remainder of this section is devoted to investigating this question. We show that, under certain hypotheses, the answer is "yes," but we have not been able to resolve the question in general. Thus we obtain our *Fourth Main Theorem* (6.17) of this section (whose proof includes the Third Main Theorem (6.9) as a corollary): Let Q be a pseudo arc in S^2, $h: Q \twoheadrightarrow Q$ a homeomorphism such that h is the identity on some open subset U of Q *and* h is extendable to a homeomorphism of S^2 onto S^2. Suppose also that for each $\varepsilon > 0$, there are no more than a finite number of *inequivalent* accessible points which *span* subcontinua of diameter $> \varepsilon$. (See Definitions 6.10 and 6.11.) Then h is the identity.

Finally, we summarize the important related questions.

6.1. THEOREM. *Let M be a nonseparating indecomposable continuum in S^2 such that, for each prime end E of $S^2 - M$, $I(E) = M$. Then no composant contains more than one accessible point.*

Proof. Let C be a composant of M, and suppose that C contains two accessible points, a and b. Then by Lemma 3.1, if A is a crosscut from a to b, one of the complementary domains, say U, of A in $S^2 - M$ is bounded by $A \cup K$, where K is a proper subcontinuum of M. Thus, if $\{A_i\}_{i=1}^{\infty}$ is any chain of crosscuts in U, representing some prime end E, then $I(E) \subseteq K \subsetneq M$. This is a contradiction. Thus no composant contains more than one accessible point.

6.2. THEOREM. *Let M be a nonseparating indecomposable continuum in S^2 such that no composant contains more than one accessible point. Let $h\colon M \twoheadrightarrow M$ be a homeomorphism carrying each composant of M onto itself, such that h is extendable to S^2. Then h is the identity.*

Proof. Let $H\colon S^2 \twoheadrightarrow S^2$ be an extension of h. Then H must take accessible points to accessible points. Since no composant contains more than one accessible point, and each composant maps to itself, each accessible point goes to itself. But the accessible points are dense in M. It follows that h is the identity.

6.3. THEOREM. *Let Q be a pseudo arc in S^2 such that $I(E) = Q$ for every prime end E of $S^2 - Q$. Let $h\colon Q \twoheadrightarrow Q$ be a homeomorphism such that h carries each composant onto itself and h is extendable to S^2. Then h is the identity.*

Proof. This is just a corollary of Theorems 6.1 and 6.2 above.

6.4. COROLLARY (FIRST MAIN THEOREM). *If every homeomorphism of the pseudo arc P onto itself, is essentially extendable by means of an imbedding $\phi\colon P \to E^2$, where $\phi(P)$ has the property that for each prime end E of $S^2 - \phi(P)$, $I(E) = \phi(P)$, then if h is any homeomorphism of P onto itself such that h carries each composant of P onto itself, then h is the identity. In particular, if h is the identity on some open set, then h is the identity.*

Proof. Clear.

6.5. SECOND MAIN THEOREM. *Let $h\colon P \twoheadrightarrow P$ be an extendable (to S^2) homeomorphism which is the identity on some open set $U \cap P$. Then h is the identity.*

Proof. We note that, since h is the identity on an open set, each composant maps to itself, so that $h(r) = r$ and $h(s) = s$.

We first assume that U contains a neighborhood of s. Now some accessible point x of C_s must move off itself, since these points are dense in P. Let $h(x) = y$, and let K be the smallest proper subcontinuum of P which contains both x and y. From Section 4, we see that K must contain s, and also if K_1 is the component of $P - U$ containing x, and K_2 is the component of $P - U$ containing y, then K_1 and K_2 are disjoint continua, each meeting U. Thus $h(x) \neq y$. This is a contradiction. It follows that h is the identity.

Clearly the above argument also works, in case U contains a neighborhood of r. Thus we must show that there is no loss of generality by assuming U contains a neighborhood of s. Let H be an extension of h to S^2, and suppose U does not contain a neighborhood of either r or s. Since U is open, $U \cap P \neq \emptyset$, and $h | U \cap P$ is the identity, it follows that there exist points $a, b \in U \cap P$ such that $a \in C_r$, $b \in C_s$, a and b are accessible from $S^2 - P$, and a and b are so close together that if B is the straight line segment joining a and b, then $[B \cup H(B) \cup H^{-1}(B)] \subseteq U$. Note that B meets P in many points. Let A be an arc from a to b such that

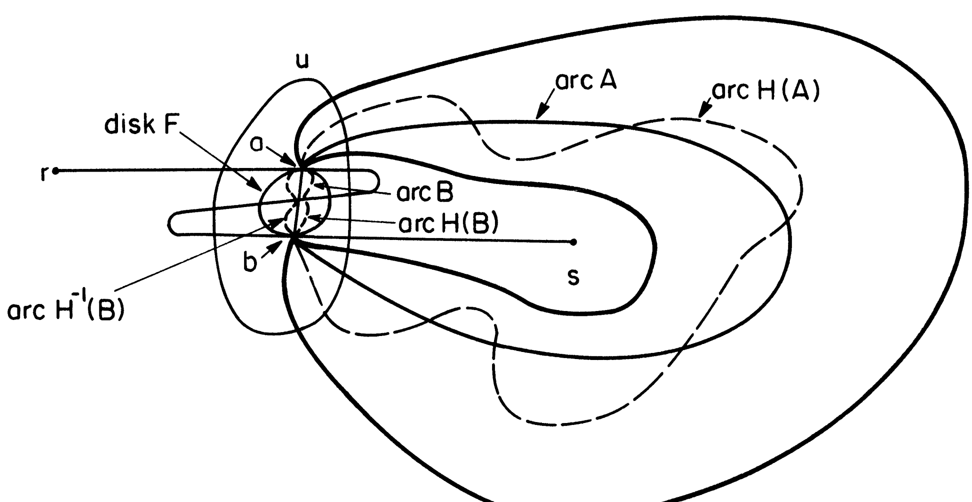
$$A \cap [B \cup H(B) \cup H^{-1}(B)] = \{a, b\},$$

and $A - \{a, b\}$ lies in $S^2 - P$. Then $H(A) \cap (B \cup H(B)) = \{h(a), h(b)\} = \{a, b\}$, and $A \cup B$ is a simple closed curve separating r from s.

We will modify H on a "neighborhood" of $\overline{\text{Int}\,(A \cup B)}$, to obtain a homeomorphism $g\colon S^2 \twoheadrightarrow S^2$ such that

(1) $g | \overline{\text{Int}\,(A \cup B)}$ is the identity, and
(2) $g(x) = h(x)$ for $x \in P \cap C(\overline{U \cup \text{Int}\,(A \cup B)})$.

The following diagram illustrates the modification described below.



We show there exists a disk D such that $[A \cup H(A) - \{a, b\}] \subseteq \text{Int}\, D$, a,

$b \in \text{Bd } D$, $D \cap P = \{a, b\}$, and $D \cap (B \cup H(B)) = \{a, b\}$. Also there exists a disk F such that $[B \cup H(B) - \{a, b\}] \subseteq \text{Int } F$, $a, b \in \text{Bd } F$, and $F \cap D = \{a, b\}$.

Case (i). If $A \cup H(A)$ fails to separate S^2, then $A \cup H(A)$ is an arc lying in $(S^2 - P + \{a, b\})$, and there exists a disk "neighborhood" D of $A \cup H(A)$ such that $(A \cup H(A) - \{a, b\})$ lies in Int D, a, $b \in \text{Bd } D$, $D \cap P = \{a, b\}$, and $D \cap (B \cup H(B)) = \{a, b\}$.

Case (ii). If $A \cup H(A)$ separates S^2, then exactly one complementary domain of $A \cup H(A)$ in S^2 contains

$$(P - \{a, b\}) \cup (B \cup H(B) - \{a, b\}).$$

Thus there exists a disk E such that E is a "neighborhood" of P in the sense that $P - \{a, b\} \subseteq \text{Int } E$, $a, b \in \text{Bd } E$, and $E \cap (A \cup H(A)) = \{a, b\}$. We also require that

$$(B \cup H(B) - \{a, b\}) \subseteq \text{Int } E.$$

Then $D = C(E)$ is a disk "neighborhood" of $A \cup H(A)$ with the desired properties.

Now, let ϕ_1 be a homeomorphism supported on D such that ϕ_1 takes $H(x)$ to x for each $x \in A$. Then $\phi_1 H$ is a homeomorphism of S^2 onto S^2 such that $\phi_1 H \mid A$ is the identity. Let F be a disk neighborhood of $B \cup H(B)$ such that

$$(B \cup H(B) - \{a, b\}) \subseteq \text{Int } F, \quad a, b \in \text{Bd } F,$$

and $F \cap D = \{a, b\}$. Clearly F exists. Let ϕ_2 be a homeomorphism supported on F such that ϕ_2 takes $H(x)$ to x for each $x \in B$. Then $\phi_2 \phi_1 H$ is a homeomorphism of S^2 onto S^2 which is the identity on $A \cup B$. Let $g\colon S^2 \twoheadrightarrow S^2$ be defined by

$$g(x) = \begin{cases} \phi_2 \phi_1 H(x) & \text{for } x \in C[\text{Int } F \cup \text{Int } D], \\ x & \text{for } x \in \overline{\text{Int } (A \cup B)}. \end{cases}$$

Then g is the desired homeomorphism.

6.6. THEOREM. *Let M be an indecomposable chainable continuum in S^2 such that each composant C_i with more than one accessible point has the property that all of its accessible points are contained in a single proper subcontinuum K_i of M. Suppose also that* $\lim_{i \to \infty} \text{diam } K_i = 0$. *Then there is a monotone map $f\colon S^2 \twoheadrightarrow S^2$ such that the nondegenerate inverses are precisely the collection of (pairwise disjoint) subcontinua $\{K_i\}$.*

Proof. Recall that by Corollary 3.6, there are only countably many composants with more than one accessible point. We will construct a sequence of homeomorphisms $\{f_i\}$, $f_i\colon S^2 \twoheadrightarrow S^2$ such that each f_i shrinks the $\{K_i\}$ a little in such a way that $g_n \to f$, where $g_n = f_n \circ \cdots \circ f_1$ and where f is a continuous, monotone map from S^2 onto S^2, and the nondegenerate inverses of f are precisely the collection $\{K_i\}$.

Let $\mathscr{C}_i$: $C_{i_1}, C_{i_2}, \ldots, C_{i_{n_i}}$ be a finite open chain cover of M, of mesh $< 1/2^i$, with each element of $\mathscr{C}_i$ being a cell in S^2, and $\mathscr{C}_i^*$ being a cell. Let $\mathscr{C}_{j_1} = \mathscr{C}_1$ and let $f_1\colon S^2 \twoheadrightarrow S^2$ be a homeomorphism such that (1) f_1 is supported on $D_1 = \mathscr{C}_{j_1}^*$ and (2) $\operatorname{diam} f_1(K_1) < 1/2$.

Let j_2 be so large that (1) $f_1(K_1)$ and $f_1(K_2)$ are separated by at least three links in $f_1(\mathscr{C}_{j_2})$, and (2) $\operatorname{diam} f_1^{-1}$ (each link of $f_1(\mathscr{C}_{j_2})) < 1/2$. This latter condition can be obtained since f_1^{-1} is a uniformly continuous homeomorphism. (Also since $\mu(\mathscr{C}_i) \to 0$.) Now let $f_2\colon S^2 \twoheadrightarrow S^2$ be a homeomorphism such that

(1) $\operatorname{diam} f_2(f_1(K_1)) < 1/4$,
(2) $\operatorname{diam} f_2(f_1(K_2)) < 1/4$,
(3) f_2 is supported on $D_{2,1} \cup D_{2,2}$ where $D_{2,i}$ is the union of the elements of $f_1(\mathscr{C}_{j_2})$ which meet $f_1(K_i)$ for $i = 1, 2$.

Let j_3 be so large that (1) $f_2 f_1(K_1)$ and $f_2 f_1(K_2)$ and $f_2 f_1(K_3)$ are pairwise separated by at least three links of $f_2 f_1(\mathscr{C}_{j_2})$, and (2) diam $(f_2 f_1)^{-1}$ (each link of $f_2 f_1(\mathscr{C}_{j_2})) < 1/4$. This latter condition can be obtained since $(f_2 f_1)^{-1}$ is a uniformly continuous homeomorphism. Now let $f_3\colon S^2 \twoheadrightarrow S^2$ be a homeomorphism such that

(1) $\operatorname{diam} f_3(f_2 f_1(K_1)) < 1/8$,
(2) $\operatorname{diam} f_3(f_2 f_1(K_2)) < 1/8$,
(3) $\operatorname{diam} f_3(f_2 f_1(K_3)) < 1/8$,
(4) f_3 is supported on $D_{3,1} \cup D_{3,2} \cup D_{3,3}$ where $D_{3,i}$ is the union of the links of $f_2 f_1(\mathscr{C}_{j_3})$ meeting $f_2 f_1(K_i)$ for $i = 1, 2, 3$.

We continue inductively. Let $g_n = f_n \circ \cdots \circ f_2 \circ f_1$ and let $f = \lim_{n\to\infty} g_n$. We show that f is continuous and monotone, shrinks each K_i to a point, and the collection $\{K_i\}$ is precisely the collection of nondegenerate inverses of f.

We see that

$$\operatorname{diam} f(K_i) = \operatorname{diam} \lim_{n\to\infty} f_n \circ \cdots \circ f_2 \circ f_1(K_i),$$

but

$$\operatorname{diam} f_n \circ \cdots \circ f_2 \circ f_1(K_i) < 1/2^n \quad \text{for } i = 1, \ldots, n.$$

Thus the limit has diameter 0. Therefore $f(K_i)$ is a point y_i.

Now suppose $f^{-1}(y_i)$ contains some point $x \notin K_i$. Let $\varepsilon = d(x, K_i)$. Then there exists n such that $1/2^n < \varepsilon$, and

$$\operatorname{diam}\, (f_n \circ \cdots \circ f_2 \circ f_1)^{-1} \text{ (each link of } f_n \circ \cdots \circ f_1(\mathscr{C}_{j_n})) < 1/2^n < \varepsilon.$$

Thus $(f_n \circ \cdots \circ f_1)(x) \notin D_{n+1,i}$, so that $f(x) \notin D_{n+1,i}$, for $i = 1, 2, \ldots, n+1$. It follows that $f(x) \neq y_i$. Thus $f^{-1}f(K_i) = K_i$.

By Corollary 3.11 on page 174 of [21], it follows that f is monotone and continuous, since f is the uniform limit of a sequence of monotone maps on the disk $\overline{\mathscr{C}_1^*}$.

It only remains to show that if $f^{-1}(y)$ is a nondegenerate continuum K, then

$K = K_i$ for some i. So suppose not. Then there are points $x_1 \neq x_2$, and y such that $f(x_1) = f(x_2) = y$, and $y \neq f(K_i)$ for any i. We note that

$$y \in \bigcap_n (f_n \circ \cdots \circ f_1)^{-1} D_{n+1,i_y},$$

where D_{n+1,i_y} is that element of $\{D_{n+1,i}\}_{i=1}^{n+1}$, which contains

$$f_n \circ \cdots \circ f_1(x_1) \quad \text{and} \quad f_n \circ \cdots \circ f_1(x_2).$$

But these intersections are precisely the $f(K_i)$'s. This is a contradiction.

6.7. THEOREM. *Let $M \subseteq S^2$ be a hereditarily indecomposable continuum, $h\colon S^2 \twoheadrightarrow S^2$ a homeomorphism such that $h(M) = M$ and each composant maps to itself, and let C be a composant of M such that all the accessible points of M which lie in C, are contained in a proper subcontinuum of M (and therefore of C). Then there exists a fully invariant, proper subcontinuum K of M, such that K contains the set A of all accessible points of M which lie in C.*

Proof. Let

$$\mathscr{K} = \{K_\alpha \mid K_\alpha \text{ is a proper subcontinuum of } M \text{ and } K_\alpha \text{ contains } A\}.$$

By hypothesis, $\mathscr{K} \neq \emptyset$. Since M is hereditarily indecomposable, $\mathscr{K}$ is a tower of sets. Let the continuum K be the intersection of this tower of continua.

We show that K is fully invariant; that is, $h(K) = K$. Suppose not. Then either (1) $h(K) \subsetneq K$ or (2) $K \subsetneq h(K)$. Now $h(A) = A$. Thus case (1) cannot occur, since then K would not be the minimal subcontinuum of M containing A. In case (2), $h^{-1}(K) \subsetneq K$, and $A \subseteq h^{-1}(K)$. So again K is not minimal. Thus, in either case we get a contradiction. It follows that $h(K) = K$.

6.8. THEOREM. *Let Q be a pseudo arc in S^2, $h\colon S^2 \twoheadrightarrow S^2$ such that $h(Q) = Q$ and each composant maps to itself. We also assume that if C_i is a composant of Q with more than one accessible point, then the set of accessible points of M which belong to C_i, is contained in a minimal proper subcontinuum K_i of M. We further assume that $\lim_{i\to\infty} \operatorname{diam} K_i = 0$. Let $f\colon S^2 \twoheadrightarrow S^2$ be a monotone map such that each K_i is an inverse set and these are the only nondegenerate inverse sets. Then h induces a homeomorphism $g\colon S^2 \twoheadrightarrow S^2$ such that $g(f(Q)) = f(Q)$.*

Proof. By Corollary 3.6, there are at most countably many K_i's, and by Theorem 6.7, $h(K_i) = K_i$. Thus if $K_i \to K$, then K is a point and $h(K) = K$.

We define $g\colon S^2 \twoheadrightarrow S^2$ as follows:

$$g(x) = fhf^{-1}(x).$$

If $f^{-1}(x)$ is nondegenerate, then $h(f^{-1}(x)) = f^{-1}(x)$, so that $fhf^{-1}(x) = x$. If $x \in f(Q)$, $g(x) \in f(Q)$ and also $g^{-1}(x) = fh^{-1}f^{-1}(x) \in f(Q)$. Thus $g(f(Q)) = f(Q)$. It is easy to see that g is a homeomorphism of S^2 onto S^2. The theorem follows.

6.9. THIRD MAIN THEOREM. *Let Q be a pseudo arc in S^2 such that each composant C_i with more than one accessible point, has the property that all of its accessible points are contained in a minimal proper subcontinuum K_i of Q, and* lim diam $K_i = 0$. *Let $h: S^2 \twoheadrightarrow S^2$ be a homeomorphism such that $h(Q) = Q$ and h carries each composant of Q onto itself. Then h is the identity on Q.*

Proof. By Corollary 3.6, the collection $\{K_i\}$ is at most countable. By Theorem 6.7, each K_i is fully invariant under h. By Theorem 6.6, there is a monotone map $f: S^2 \twoheadrightarrow S^2$ such that the nondegenerate inverses are the K_i's. By Theorem 6.8, h induces a homeomorphism $g: S^2 \twoheadrightarrow S^2$ such that $g(f(Q)) = f(Q)$, where $g(x) = fhf^{-1}(x)$.

We note that $f(Q)$ is the monotone image of the pseudo arc, and therefore also a pseudo arc [4].

We now show that if $g \mid f(Q)$ is the identity, then $h \mid Q$ is the identity. Since there are at most countably many points of $f(Q)$ with nondegenerate inverse sets, and $f(Q)$ is a pseudo arc, there exists a composant C of $f(Q)$ such that $f \mid f^{-1}(C)$ is 1-1. Now $g(x) = fhf^{-1}(x) = x$ for all $x \in f(Q)$, and for $x \in C$, $f^{-1}(x)$ is a point. Thus $hf^{-1}(x) = f^{-1}(x)$, for $x \in C$; that is, h is the identity on $f^{-1}(C)$. But $f^{-1}(C)$ is dense in Q. It follows that h is the identity on Q.

Next, we observe that each composant of $f(Q)$ can contain at most one accessible point. For if T is an arc lying in $S^2 - Q$ except for one endpoint t, which is in Q, then $f(T)$ has the same properties. Conversely, if T is an arc lying in $S^2 - f(Q)$ except for one endpoint t which lies in $f(Q) - \bigcup f(K_i)$, then $f^{-1}(t)$ is accessible iff t is accessible. Thus each composant of $f(Q)$ can contain at most one accessible point.

Now, since $g: S^2 \twoheadrightarrow S^2$ takes $f(Q)$ onto itself, accessible points of $f(Q)$ must map to accessible points. But each composant maps to itself. Thus $g(a) = a$ for each accessible point of $f(Q)$. Since the accessible points of $f(Q)$ are dense in $f(Q)$, it follows that $g \mid f(Q)$ is the identity.

From the paragraph before the last one, we see that h is the identity on Q.

6.10. DEFINITION. Let M be a continuum and let $h: M \twoheadrightarrow M$ be a homeomorphism. Let $a \in M$ be a point such that no proper subcontinuum of M contains $\bigcup \{h^n(a)\}_{n=-\infty}^{\infty}$. Then we say that the point a *spans* the continuum M.

6.11. DEFINITION. Let M be a nonseparating indecomposable continuum in S^2, and let $h: S^2 \twoheadrightarrow S^2$ be an orientation preserving homeomorphism such that $h(M) = M$. Let

$$\phi: (S^2 - M) \twoheadrightarrow \text{Int } B$$

be a C-map, and let g: Bd $B \twoheadrightarrow$ Bd B be the homeomorphism induced on the boundary of the unit disk, as given in prime end theory. Then g is orientation preserving on Bd B.* Thus if g has at least one fixed point and is not the identity, then there are at most countably many pairwise disjoint intervals A_i in Bd B such that $g(A_i) = A_i$, and $g \mid \text{Int } A_i$ is a conjugate of a translation on the line. Also, if there is a fixed point, there is at least one such interval, where the "interval" might be all of Bd B, and its interior all of Bd B minus the fixed point.

Let A_i be such an interval, and let $a' \in \text{Int } A_i$ be a point of Int A_i corresponding to the accessible point a of M. Then the point a spans a (not necessarily proper) subcontinuum of M. We will call two points a and b of M *equivalent* iff (1) a and b are both accessible points of M and (2) a and b correspond to points a' and b' which lie in the interior of the same A_i.

Proof of *. We show that g is orientation preserving. Suppose not. Let $G = \phi h \phi^{-1}$, so that $g = G \mid \text{Bd } B$. Then RG, where R is a reflection on B, is a homeomorphism on B which is orientation preserving on Bd B. Thus there is an isotopy $\{F_\alpha\}_{0 \leq \alpha \leq 1}$ of B such that $F_0 \mid \text{Bd } B = RG \mid \text{Bd } B$ and $F_1 \mid \text{Bd } B$ is the identity. Then there is an isotopy (the Alexander isotopy [1]) $\{H_\alpha\}_{0 \leq \alpha \leq 1}$ of B such that $H_0 = F_1$ and H_1 is the identity. Thus $\{F_\alpha\}$ followed by $\{H_\alpha\}$ is an isotopy taking RG to the identity on B. Thus $R(RG) = G$ must be orientation reversing on B, and therefore on Int B. But h is orientation preserving on $S^2 - M$, so $\phi h \phi^{-1} = G \mid \text{Int } B$ must be orientation preserving on Int B. This is a contradiction, and it follows that g is orientation preserving.

6.12. THEOREM. *If M is a hereditarily indecomposable continuum,*

$$h: M \twoheadrightarrow M$$

a homeomorphism, and $a \in M$, then there is exactly one (sub)continuum K of M such that a spans K.

Proof. Suppose K_1 and K_2 are subcontinua of M, both of which are spanned by a. Then $a \in K_1 \cap K_2$, so $K_1 \cap K_2 \neq \emptyset$. Since M is hereditarily indecomposable, $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. But a spanning continuum is minimal. Thus $K_1 = K_2$.

6.13. THEOREM. *Let M be a hereditarily indecomposable, nonseparating continuum in S^2, and let $h: S^2 \twoheadrightarrow S^2$ be a homeomorphism such that $h(M) = M$, and such that h is the identity on an open set of M. Let a and b be equivalent points of M and let K_a and K_b be their respective (unique) spanning continua. Then $K_a = K_b$.*

Proof. Let $\phi: (S^2 - M) \twoheadrightarrow \text{Int } B$ be a C-map to the interior of the unit disk, and let $g = \phi h \phi^{-1}: B \twoheadrightarrow B$ be the induced homeomorphism on B, given by prime end theory. Since h is the identity on an open set, g fixes many points of Bd B. Thus $g \mid$ (fixed interval) is order preserving on that interval. Let A be the minimal invariant interval of Bd B containing the points a' and b' corresponding to a and b respectively. Let $a'_n = g^n(a')$ and $b'_n = g^n(b')$. Then a'_n and b'_n correspond to the accessible points $h^n(a)$ and $h^n(b)$, respectively, in M. The sequences $\{a'_n\}$ and $\{b'_n\}$ each span the interval A in Bd B.

Let X be a crosscut in $S^2 - M$ from a to $h(a)$. Then $X \cup M$ separates S^2 into the connected open sets U and V. Let K be that subcontinuum of M such that $X \cup K$ bounds one of U and V, say $X \cup K$ bounds U. Now $\phi(X)$ is a crosscut in B from a_0' to a_1', and separates Int B into the two open sets $\phi(U)$ and $\phi(V)$. Note that each of $\phi(U)$ and $\phi(V)$ has some b_i' as an accessible point. Let b_n' be accessible from $\phi(U)$. Then there is an endcut T in $\phi(U)$ leading to b_n' such that $\phi^{-1}(T)$ is an endcut leading to $h^n(b)$. Thus $h^n(b) \in \bar{U}$ and therefore $h^n(b) \in K$ and therefore $h^n(b) \in K_a$. By a similar argument, there exists an integer m such that $h^m(a) \in K_b$. It follows that $K_b \cap K_a \neq \emptyset$ so that $K_b \subseteq K_a$ or $K_a \subseteq K_b$, since M is hereditarily indecomposable.

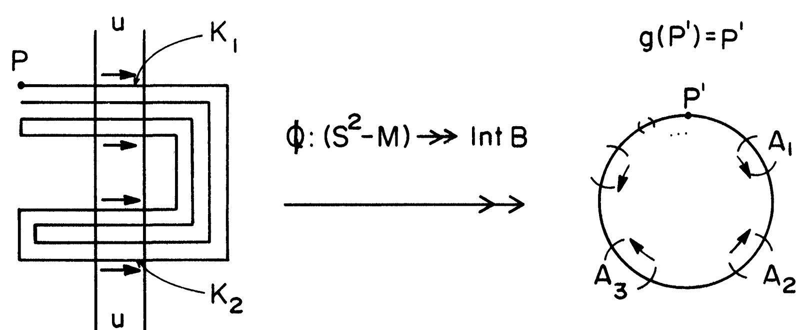
Without loss of generality, we assume that $K_b \subseteq K_a$. Suppose K_a properly contains K_b. Then there is an integer j such that $h^j(a) \in K_a - K_b$. Also from above, $h^n(b)$ and $h^m(a)$ are both in K_b. Then $h^{j-m}(a)$ takes $h^m(a)$ to $h^j(a)$. Thus $h^{j-m}(K_b)$ contains K_b properly, so that $h^{m-j}(K_b) = (h^{j-m})^{-1}(K_b)$ is proper in K_b, and contains $\bigcup_i h^i(b)$. But K_b was a minimal continuum with this property. This is a contradiction. It follows that $K_a = K_b$.

6.14. THEOREM. *Let M be a hereditarily indecomposable, nonseparating continuum in S^2, and let $h: S^2 \twoheadrightarrow S^2$ be a homeomorphism such that $h(M) = M$ and h is the identity on some open set. Let a be an accessible point in M, with infinite orbit, and let K be its spanning continuum. Then $h(K) = K$.*

Proof. $K \cap h(K) \neq \emptyset$, since $h(a) \in K \cap h(K)$. Thus $K \subseteq h(K)$ or $h(K) \subseteq K$. Without loss of generality, we assume $K \subseteq h(K)$; for otherwise replace h by h^{-1} in the argument below.

Now, suppose K is proper in $h(K)$. Then $h^{-1}(K)$ is proper in K. But $h^{-1}(K)$ contains $h(\bigcup_i h^i(a)) = \bigcup_i h^i(a)$. Thus $h^{-1}(K)$ is a smaller continuum (than K) containing all the iterates of a. This contradicts the fact that K was a spanning continuum for a. It follows that $h(K) = K$.

6.15. *Example.* Let M be the U-continuum and let $h: S^2 \twoheadrightarrow S^2$ be the homeomorphism described in the diagram. Note that $h(M) = M$. Let $\phi: (S^2 - M) \twoheadrightarrow \text{Int } B$ be a C-map, and let $g: B \twoheadrightarrow B$ be the induced homeomorphism, $\overline{\phi h \phi^{-1}}$. Then $g \mid \text{Bd } B$ is described in the diagram below.



Terminology. If A is an arc $[a, b]$ then a *translation* on A means a homeomorphism conjugate to $\sqrt{x}$ on $[0, 1]$.

(1) U is the open strip indicated in the diagram.
(2) h is identity on $S^2 - U$.
(3) h is (translation $\times$ identity) on U; as indicated.
(4) The A_i's are a countable sequence of intervals beginning on right side of p' and converging to p' on the other side of p'. $g \mid (\mathrm{Bd}\ B - \bigcup A_i)$ is the identity. $g(A_i) = A_i$ and $g \mid A_i$ is a translation in the direction indicated. Note that the directions alternate.

We see that the intervals A_i in Bd B determine unique spanning continua K_i in M, even though M is not *hereditarily* indecomposable.

6.16. THEOREM. *Let Q be a pseudo arc in S^2, h: $S^2 \twoheadrightarrow S^2$ a homeomorphism such that (1) $h(Q) = Q$ and (2) h is the identity on some open set U of Q. Then $h \mid Q$ is "determined" by an at-most-countable collection of proper spanning subcontinua $\{K_i\}_{i=1}^{\infty}$ of Q; that is, $h \mid (Q - \overline{\bigcup K_i})$ is the identity.*

Proof. Let ϕ: $(S^2 - Q) \twoheadrightarrow \mathrm{Int}\ B$ be a C-map and let g: $B \twoheadrightarrow B$ be the induced homeomorphism given by prime end theory. Since h is the identity on an open set of Q, g will fix many points of Bd B. Thus $g \mid \mathrm{Bd}\ B$ is order preserving and will carry each of an at-most-countable collection of pairwise disjoint open intervals $\{A_i\}$ onto itself. Further $h \mid A_i$ will be equivalent to the "translation" x^2 or $\sqrt{x}$ on $[0, 1]$. (Note that several of these intervals may correspond to one composant.) Let $a_i' \in A_i$ correspond to some accessible point $a_i \in Q$. Let K_i be the spanning subcontinuum for a_i. By Theorem 6.14, each K_i is fully invariant.

We first show that each K_i is proper in Q. Suppose, for some n, K_n is not proper in Q. Let L_j be the smallest proper subcontinuum of Q containing $h^i(a)$ for $-j \leq i \leq j$. Then $L_1 \subseteq L_2 \subseteq \cdots \subseteq L_n \subseteq \cdots$, and $\bigcup L_i$ is dense in Q. Let U' be open in U so that U' misses L_1. Then there is an integer m such that $L_m \cap U' = \emptyset$, but $L_{m+1} \cap U' \neq \emptyset$. This contradicts the fact that $h \mid U'$ is the identity. Thus each K_i is proper in Q. (Note that we do not claim that the K_i's are necessarily pairwise disjoint.)

Now, let $x \in Q - \overline{\bigcup K_i}$, and suppose $h(x) \neq x$. Then some neighborhood V of x moves off itself. Thus there is an accessible point b in V which moves off itself. But b corresponds to a point of Bd B which remains fixed. This is a contradiction. The theorem follows.

6.17. FOURTH MAIN THEOREM. *Let Q be a pseudo arc in S^2, h: $S^2 \twoheadrightarrow S^2$ a homeomorphism such that (1) $h(Q) = Q$ and (2) h is the identity on some open subset U of Q. Suppose also that for each $\varepsilon > 0$, there are no more than a finite number of inequivalent accessible points which span subcontinua of diameter $> \varepsilon$. Then h is the identity.*

Proof. Let $\{K_i\}_{i=1}^{\infty}$ be the collection of spanning subcontinua of Q which determine h, as given by Theorem 6.16. Then $h \mid (M - \overline{\bigcup K_i})$ is the identity. By Theorem 6.14, each of the K_i's is fully invariant. By an argument similar to that of Theorem 6.6, there exists a monotone map f: $S^2 \twoheadrightarrow S^2$ such that the only nondegenerate inverses are the K_i's. Then, as in the proofs of Theorems 6.8 and 6.9, h induces a homeomorphism g: $S^2 \twoheadrightarrow S^2$, $g(f(Q)) = f(Q)$, and $g \mid f(Q)$ is the identity. It follows, as in the proof of Theorem 6.9, that h is the identity on Q.

6.18. QUESTIONS. Let Q be a pseudo arc in S^2 and let h: $S^2 \twoheadrightarrow S^2$ be a homeomorphism such that $h(Q) = Q$.

1. STABLE HOMEOMORPHISM PROBLEM FOR ESSENTIALLY EXTENDABLE HOMEOMORPHISMS. If h is the identity on an open subset of Q and h is extendable to S^2, must h be the identity on Q?
2. COMPOSANT PROBLEM FOR ESSENTIALLY EXTENDABLE HOMEOMORPHISMS. If h carries each composant of Q onto itself and h is extendable to S^2, is h the identity?
3. Can h and Q be chosen so that h is the identity on an open set of Q, h is extendable, *and* for some $\varepsilon > 0$, there are infinitely many inequivalent accessible points of Q, whose spanning continua K_i all have diameter greater than ε? (A negative answer to this would answer #1 above, in the affirmative.)
4. Same as #3, except that h carries each composant onto itself (not necessarily the identity on an open set).
5. Does Theorem 6.17 hold if we only require invariant composants, rather than a pointwise fixed open set?

7. Questions and some possible ramifications

We recall some problems mentioned in previous sections, add some new ones, and explain the importance of the problems.

7.1. IMBEDDINGS.

7.1.1. PROBLEM. Are there uncountably many inequivalent imbeddings of P in S^2?

7.2. ESSENTIALLY EXTENDABLE HOMEOMORPHISMS.

7.2.1. PROBLEM. Is every homeomorphism of the pseudo arc P onto itself essentially extendable to E^2?

In another paper [7] we will show that such homeomorphisms are essentially extendable to E^3, by constructing an imbedding ϕ: $P \to E^2 \subseteq E^3$ such that

$$\phi h \phi^{-1}\colon \phi(P) \twoheadrightarrow \phi(P)$$

is extendable to E^3. This imbedding carries P to a plane and it may be that this imbedding works for E^2, though we do not yet know the answer.

In still another paper [8], we will show that extendable homeomorphisms of the standard pseudo arc, P, cannot be of period $n > 2$. If this result could be improved so that it holds for essentially extendable homeomorphisms, then a positive answer to this question (7.2.1) would tell us that there were no period $n > 2$ homeomorphisms of the pseudo arc onto itself.

7.2.2. PROBLEM. Is every homeomorphism of the standard pseudo arc P in E^2 conjugate to an extendable homeomorphism?

If so, then each such homeomorphism has either 2 fixed points in different composants, or a pair of points of different composants which interchange.

If so, then it would follow from our Second Main Theorem of Section 6 (6.5) that there are no nonidentity stable homeomorphisms of the pseudo arc onto itself.

7.2.3. PROBLEM. Characterize the continua X in E^2 for which each homeomorphism $h: X \twoheadrightarrow X$ is essentially extendable to E^2. Perhaps any non-separating continuum without cut points is a characterization?

Note that in Section 5 we gave an example of a chainable continuum and a homeomorphism on it, which is not essentially extendable to E^2. This continuum contains a cut point, which is where the trouble occurs.

7.2.4. PROBLEM. Does there exist a continuum M in E^2 such that not every homeomorphism of M is extendable, but every homeomorphism is *conjugate* to an extendable homeomorphism?

7.2.5. PROBLEM. Does there exist a continuum M in E^2 such that not every homeomorphism is extendable or conjugate to an extendable homeomorphism, but every homeomorphism is essentially extendable?

7.3. STABLE HOMEOMORPHISMS.

7.3.1. PROBLEM. Let $h: P \twoheadrightarrow P$ be a homeomorphism. If h is the identity on an open set *and* essentially extendable, is h the identity? If h carries each composant onto itself *and* is essentially extendable, is h the identity?

A positive answer to these questions, together with a positive answer to 7.2.1, would answer the stable homeomorphism and composant problems for the pseudo arc. That is, it would show that if h is a homeomorphism which is the identity on an open set (carries each composant to itself), then h is the identity.

Remark. The referee has informed the author that Lewis, a student of Bing at Texas, has answered both the Stable Homeomorphism and Composant Problems in the negative. This work was done quite recently, in fact, since the present paper was submitted for publication.

Added in proof. Questions 1 and 2 of Section 4 (and therefore also the conjecture of Section 4(c) and Problem 7.1.1) have recently been answered in the affirmative by Wayne Lewis, in a paper (preprint) entitled *Embeddings of the pseudo arc in E^2*. His solution of the Stable Homeomorphism and Composant Problems are contained in *Stable homeomorphisms of the pseudo arc*, to appear in the Canadian Journal of Mathematics.

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