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SMOOTHNESS OF THE FREE BOUNDARY IN THE STEPHAN PROBLEM WITH SUPERCOOLED WATER

BY

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Introduction

In [3], van Moerbeke studied an optimal stopping problem and related it to a Stephan problem with supercooled water. Later, Friedman [1] generalized this result somewhat and simplified the proof.

In this paper we consider the same problem. As Friedman, we study the problem as a variational inequality: find $u = u(x, t)$ for $(x, t) \in \mathbf{R} \times (0, T)$ such that

$$
\begin{aligned}
&u \geq 0 \quad \text{a.e.},\\
(0.1) \qquad &(u_t - u_{xx})(v - u) \geq -(v - u) \quad \text{a.e. for any } v \geq 0,\\
&u(x, 0) = h(x).
\end{aligned}
$$

Under some general conditions this problem has a unique solution. By obtaining a new estimate on the Lipschitz smoothness of the free boundary we greatly simplify the conditions needed to prove that the free boundary of this problem is C^∞. In fact, we shall only require that $h'(x)$ changes sign once. In [1] and [3] the crucial condition is that h'' changes sign twice. Our proof will be based on an entirely new idea.

In Section 1 we state some results from [1] and prove some necessary facts for the application of the techniques of Section 2. Section 2 contains the essential "a priori" estimate. We study $-(u_t/u_x)(x, t)$ where u is the solution of (0.1). This can be interpreted as the derivative of the level curves of u when written as functions of t. We are able to bound this fraction uniformly on certain subsets of $Rx(0, T)$. This gives a Lipschitz bound on the free boundaries.

1. Preliminary results

We shall study the variational inequality: find $u = u(x, t)$, $(x, t) \in \mathbf{R} \times (0, \infty)$, satisfying

(1.1) u, u_x, u_{xx}, u_t are bounded functions,
(1.2) $u \geq 0$,
(1.3) $(u_t - u_{xx})(v - u) \geq -(v - u)$ a.e. for any $v \geq 0$,
(1.4) $u(x, 0) = h(x)$.

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We make the following assumptions.

(1.5) $h(x)$ is continuous for $x \in \mathbf{R}$, $h(x) = 0$ for $x \notin (x_1, x_2)$ $(-\infty < x_1 < x_2 < \infty)$.

(1.6) $h \in C^2([x_1, x_2])$.

(1.7) There exists a point $x^* \in (x_1, x_2)$ such that $h'(x) > 0$ if $x \in (x_1, x^*)$, $h'(x) < 0$ if $x \in (x^*, x_2)$,

$$\lim_{x \uparrow x_2} \frac{h''(x) - 1}{h'(x)} \quad \text{and} \quad \lim_{x \downarrow x_1} \frac{h''(x) - 1}{h'(x)} \tag{1.8}$$

both exist.

The next results are found in [1].

(1.9) [1, Theorem 1.1] There exists a unique solution u, of (1.1)–(1.4) and it has compact support.

(1.10) [1, Theorem 2.2] Let $\Omega \equiv \{(x, t) \mid u(x, t) > 0\}$. Then there are two functions $S^-(t) \le S^+(t)$, $t \in [0, T^+]$, such that S^- is upper semicontinuous and S^+ is lower semicontinuous and

$$\{(x, t) \mid 0 \le t < T^+, S^-(t) < x < S^+(t)\} = \Omega.$$

LEMMA 1.1. *Let η be a regular value of $u_x(x, t)$, $\eta \neq 0$. Then any connected component of $u_x(x, t) = \eta$ can be written as*

$$\begin{aligned} x &= x_\eta^-(t), \quad 0 \le t \le \tau_\eta \\ x &= x_\eta^+(t), \quad 0 \le t \le \tau_\eta \end{aligned} \tag{1.11}$$

where $x_\eta^\pm \in C^\infty((0, \tau_\eta)) \cap C([0, \tau_\eta])$, $x_\eta^-(t) < x_\eta^+(t)$ if $t < \tau_\eta$ and $x_\eta^-(\tau_\eta) = x_\eta^+(\tau_\eta)$.

Proof. Let $(x(\rho), t(\rho))$ for $\rho \in [a, b]$ be a smooth curve with $(x'(\rho), t'(\rho)) \neq 0$ and such that

$$u_x(x(\rho), t(\rho)) = \eta. \tag{1.12}$$

We shall show that $t'(\rho)$ vanishes exactly once (at a maximum of $t(\rho)$); this will prove the lemma. Suppose $t'(\rho_0) = 0$, then by differentiating (1.12),

$$u_{xx}(x(\rho_0), t(\rho_0))x'(\rho_0) = 0. \tag{1.13}$$

Without loss of generality we may parameterize the curve, $(x(\rho), t(\rho))$, so that

$$x = \rho \quad \text{for } \rho \text{ near } \rho_0. \tag{1.14}$$

So for x near $x_0 = \rho_0$ we have, from (1.12) and (1.13),

$$u_x(x, t(x)) = \eta, \quad \text{and} \quad u_{xx}(x_0, t(x_0)) = 0.$$

Differentiating the first equation above twice and evaluating at $x = x_0$ gives

$$u_{xxx}(x_0, t(x_0)) + u_{xt}(x_0, t(x_0))t''(x_0) = 0. \tag{1.15}$$

We have, in Ω, $u_{xt} - u_{xxx} = 0$ (since, by (1.3), $u_t - u_{xx} = -1$ in Ω). Furthermore, since η is a regular value of u_x, $\nabla u_x(x_0, t(x_0)) \neq 0$ but $u_{xx}(x_0, t(x_0)) = 0$. Therefore $u_{xt}(x_0, t(x_0)) \neq 0$. By this and (1.15) we see $1 + t''(x_0) = 0$ or $t''(x_0) = -1$. We conclude that $t(x_0)$ is a local maximum whenever $t'(x_0) = 0$. It follows easily that $t(\rho)$ is a smooth curve with at most one local maximum and no local minimums and $t'(\rho)$ vanishes only at the local maximum. Finally, there must be one local maximum of $t(\rho)$. Indeed, if not we could parameterize $t(\rho)$ so that it is monotone increasing and $t(0) = 0$. Then, there is a largest number ρ^* below which $t(\rho)$ is defined. We have $(x(\rho), t(\rho))$ approaching $\partial\Omega \backslash \{(x, t) \mid t = 0\}$ as $\rho \nearrow \rho^*$ but $u = u_x = 0$ on this set which is obviously impossible since $\eta \neq 0$.

LEMMA 1.2. *There is a unique continuous function $n(t)$, $t \in [0, T^+)$ such that*

(i) $u(n(t), t) > 0$ *and*

(ii) $\{(x, t) \mid 0 \leq t < T^+, u_x(x, t) = 0$ *and* $(x, t) \in \Omega\} = \{(x, t) \mid 0 \leq t < T^+, x = n(t)\}$.

Thus $(n(t), t)$ is the curve along which $u_x = 0$ and $u > 0$ on this curve.

Proof. Take $\{\eta_i\}_{i=1}^{\infty}$ a sequence of regular values of $u_x(x, t)$ such that $-\eta_i$ are also regular values and $\eta_i \searrow 0$ as $i \to \infty$. Since $u = u_x = 0$ on the set $\partial\Omega \backslash \{(x, t) \mid t = 0\}$ it follows that for any t_0 such that $0 < t_0 < T^+$ if i is sufficiently large then

$$\eta_i, -\eta_i \in \{\delta \mid u_x(x, t_0) = \delta, x \in \mathbf{R}\}. \tag{1.16}$$

Let (x_0, t_0) be a point in Ω such that

$$u_x(x_0, t_0) = 0. \tag{1.17}$$

Since u_x is analytic in x for t fixed we may assume without loss of generality that

$$\begin{aligned} u_x(x, t_0) > 0 \quad &\text{if } x_0 - \varepsilon < x < x_0 \\ u_x(x, t_0) < 0 \quad &\text{if } x_0 < x < x_0 + \varepsilon \end{aligned} \tag{1.18}$$

for some $\varepsilon > 0$.

Choose curves $x^-_{-\eta_i}(t)$, $x^+_{\eta_i}(t)$ as in (1.11) with

$$x^-_{-\eta_i}(t_0) \searrow x_0 \text{ as } i \to \infty, \qquad x^+_{\eta_i}(t_0) \nearrow x_0 \text{ as } i \to \infty.$$

Clearly, for $0 < t < t_0$ $x^-_{-\eta_i}(t)$ is decreasing in i and $x^+_{\eta_i}(t)$ is increasing in i. Let

$$x^-(t) = \lim_{i \to \infty} x^-_{-\eta_i}(t), \qquad x^+(t) = \lim_{i \to \infty} x^+_{\eta_i}(t).$$

We have that $x^-(t)$ is upper semicontinuous and $x^+(t)$ is lower semicontinuous. Furthermore, $x^-(t) \geq x^+(t)$ and

$$u_x(x^{\pm}(t), t) = 0 \quad \text{if } 0 \leq t \leq t_0. \tag{1.19}$$

In particular $u_x(x^{\pm}(0), 0) = 0$; so $x^{\pm}(0) = x^*$ and $x^-_{-\eta_i}(0) \to x^*$ and $x^+_{\eta_i}(0) \to x^*$.
Using this and the maximum principle we conclude that

$$\lim_{i\to\infty} \sup_{\Omega_i} |u_x| = 0 \tag{1.20}$$

where $\Omega_i \equiv \{(x, t) \,|\, 0 \le t \le t_0,\ x^+_{\eta_i}(t) \le x \le x^-_{-\eta_i}(t)\}$. Therefore,

$$u(x, t) = 0 \quad \text{for } 0 \le t \le t_0,\ x^+(t) \le x \le x^-(t).$$

Since $u(x, t)$ is analytic in x for t fixed we conclude that $x^-(t) = x^+(t)$ and so the curve $x^+(t)$ is continuous. Given t_0, if there exists another curve say $x = \hat{x}(t)$ $(0 \le t \le t_0)$ along which $u_x = 0$, then by the above proof $\hat{x}(0) = x^+(0) = x^*$ and therefore $\hat{x}(t)$ will have to intersect one of the curves $x = x_{\pm\eta_i}(t)$. This is clearly impossible since $u_x \ne 0$ on the curves $x = x_{\pm\eta_i}(t)$.

Since t_0 can be taken arbitrarily close to T^+ this proves the existence and uniqueness of the curve $n(t)$ with the properties stated in Lemma 1.2.

LEMMA 1.3. *$(d/dt)(u(n(t), t)) \le -1$ (in distribution sense).*

Proof. Let $0 < t_0 < T^+$; by the proof of Lemma 1.2 there exist curves

$$n_j(t) \in C^0([0, t_0]) \cap C^\infty((0, t_0))$$

which converge to $n(t)$ monotonically and such that $u_x(n_j(t), t) = \mu_j$, where $\mu_j \to 0$ as $j \to \infty$. For any smooth ψ with support in $(0, t_0)$

$$\begin{aligned} -\int_0^{t_0} u(n_j(s), s)\frac{d}{ds}\psi(s)\,ds &= \int_0^{t_0} \frac{d}{ds}(u(n_j(s), s))\psi(s)\,ds \\ &= \int_0^{t_0} (-1 + u_{xx}(n_j(s), s))\psi(s)\,ds + O(\mu_j). \end{aligned}$$

Letting $j \to \infty$ and using $u_{xx}(n(s), s) \le 0$ we get

$$-\int_0^{t_0} u(n(s), s)\frac{d}{ds}\psi(s)\,ds \le \int_0^{t_0} -\psi(s)\,ds,$$

and the proof is complete.

Let $\{\delta_i\}_{i=1}^{\infty}$ be a sequence of regular values of $u(x, t)$ such that $\delta_i \searrow 0$ as $i \to \infty$. Set $\Gamma_i \equiv \{(x, t) \,|\, u(x, t) = \delta_i\}$.

LEMMA 1.4. *If (x, t), $(y, \tau) \in \Gamma_i$ and $u_x(x, t) = u_x(y, \tau) = 0$ then $x = y$ and $t = \tau$. (That is, the curve $x = n(t)$ meets the curve Γ_i in at most one point. Further, since $u_x(x, t) > 0$ if $x < n(t)$ and $u_x(x, t) < 0$ if $x > n(t)$ it is clear that Γ_i consists of two components $x = S_j^-(t)$ and $x = S_j^+(t)$.)*

Proof. Since $u_x(x, t) = u_x(y, \tau) = 0$ we have $x = n(t)$ and $y = n(\tau)$. By Lemma 1.3, $s \to u(n(s), s)$ is a strictly monotone function. Thus $t = \tau$ and the proof is complete.

We shall denote by τ_i the unique value of t which gives $u(n(\tau_i), \tau_i) = \delta_i$. Thus $(n(\tau_i), \tau_i)$ is the "top" of the curve Γ_i.

LEMMA 1.5. *$\{\tau_i\}_{i=1}^{\infty}$ is strictly increasing and $\tau_i \to T^+$ as $i \to \infty$.*

Proof. The strict monotonicity follows from Lemma 1.3 since $u(n(\tau_i), \tau_i) = \delta_i$ and $\delta_i \searrow$. Let $\tau_0 = \lim_{i\to\infty} \tau_i$. If $\tau_0 < T^+$ then $u(n(s), s)$ is strictly decreasing in (τ_0, T^+) which is impossible since $u(n(\tau_0), \tau_0) = 0$.

As stated previously Γ_i consists of two components $x = S_j^-(t)$ and $x = S_j^+(t)$. We have $S_j^+(t), S_j^-(t) \in C^\infty((0, \tau_i)) \cap C([0, \tau_i])$ and

$$S_i^+(t) \geq S_i^-(t), \tag{1.21}$$

$$S_i^+(\tau_i) = S_i^-(\tau_i), \tag{1.22}$$

$$u(S_i^\pm(t), t) = \delta_i. \tag{1.23}$$

LEMMA 1.6. *$|u_t/u_x|$ is bounded on*

$$d\Omega_i \equiv (\Gamma_{i+1} \cup \{(x, 0) \mid 0 < h(x) \leq \delta_{i+1}\}) \cap \{(x, t) \mid 0 \leq t \leq \tau_i\}$$

by a positive constant B_i.

Proof. It is clear that u_t is bounded on Γ_{i+1}. We now consider u_x.

For $(x, t) \in \Gamma_{i+1}$ and $t \leq \tau_i$, by Lemma 1.4, $u_x \neq 0$. Since this set is compact we have in fact $|u_x| \geq C > 0$ on this set. On $\{(x, 0) \mid 0 \leq h(x) \leq \delta_{i+1}\}$, u_t/u_x is bounded by (1.8).

2. Smoothness of the free boundary

Set

$$\Omega_i^+ \equiv \{(x, t) \mid 0 < t \leq \tau_i,\ S_{i+1}^+(t) < x < S^+(t)\},$$

$$\Omega_i^- \equiv \{(x, t) \mid 0 < t \leq \tau_i,\ S^-(t) < x < S_{i+1}^-(t)\}.$$

THEOREM 2.1. *Suppose $S^-(t), S^+(t) \in C^\infty((0, t_0)) \cap C^1([0, t_0))$ and $t_0 < \tau_{i_0}$. Then $|\dot{S}^-(t)| \leq B_{i_0}$ for all $t \in [0, t_0)$ and $|\dot{S}^+(t)| \leq B_{i_0}$ for all $t \in [0, t_0)$.*

Proof. Since $\dot{S}^+ \in C^\infty((0, t_0))$ we get by differentiating $u(S^+(t), t) = 0$ that

$$u_t(S^+(t), t) = 0. \tag{2.1}$$

Let us define $w^\varepsilon(x, t)$ for $\varepsilon > 0$, $(x, t) \in \Omega_{i_0}^+$ by

$$w^\varepsilon(x, t) \equiv \frac{u_t(x, t)}{u_x(x, t) - \varepsilon} \quad \text{for } (x, t) \in \Omega_{i_0}^+.$$

Since $u_x(x, t) \leq 0$ for $(x, t) \in \Omega_{i_0}^+$, Lemma 1.6 implies that $|w^\varepsilon| < B_{i_0}$ on the part of the boundary of $\Omega_{i_0}^+$ which belongs to $d\Omega_{i_0}$. By (2.1), $w^\varepsilon = 0$ on $x = S^+(t)$ the remaining part of the boundary of $\Omega_{i_0}^+$. Therefore

(2.2) $$\sup_{\mathscr{B}_{i_0}^+} |w^\varepsilon| \leq B_{i_0} \quad \text{where } \mathscr{B}_{i_0}^+ \equiv \partial\Omega_{i_0}^+ \cap \{(x, t) | 0 \leq t < t_0\}.$$

Since

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)(w^\varepsilon(x, t)(u_x(x, t) - \varepsilon)) = 0,$$

we find that

$$(u_x(x, t) - \varepsilon)(-w_{xx}^\varepsilon(x, t) + w_t^\varepsilon(x, t)) - 2u_{xx}(x, t)w_x^\varepsilon(x, t) = 0,$$

or equivalently

$$-w_{xx}^\varepsilon - 2\left(\frac{u_{xx}}{u_x - \varepsilon}\right)w_x^\varepsilon + w_t^\varepsilon = 0.$$

Therefore we may apply the maximum principle to w^ε and use (2.2) to conclude that

$$\sup_{\Omega_{i_0}^+ \cap \{(x,t)|0 \leq t < t_0\}} |w^\varepsilon(x, t)| \leq B_{i_0}.$$

Letting $\varepsilon \to 0$ we get

(2.3) $$\sup_{\Omega_{i_0}^+ \cap \{(x,t)|0 \leq t < t_0\}} \left|\frac{u_t}{u_x}\right| \leq B_{i_0}.$$

Now, for $j > i_0$, $(S_j^+(t), t) \in \Omega_{i_0}^+$ for $0 \leq t < t_0$. Therefore, by (2.3), $|\dot{S}_j^+(t)| \leq B_{i_0}$ for $0 \leq t < t_0$. It is also clear that $S_j^+ \to S^+$ as $j \to \infty$ for $0 \leq t < t_0$. It then follows that $|\dot{S}^+(t)| \leq B_{i_0}$ for $0 \leq t < t_0$. By similar reasoning we get $|\dot{S}^-(t)| \leq B_{i_0}$ for $0 \leq t < t_0$.

THEOREM 2.2. $S^+(t), S^-(t) \in C^\infty((0, T^+))$.

Proof. By [2] we get:

(2.4) If $\sup_x |u_{xt}(x, t_1)| < K$ then there is an ε depending only on K such that $S^+(t)$ and $S^-(t)$ are in $C^{1,\alpha}(\alpha > 0)$ in $[t_1, t_1 + \varepsilon]$.

By [4] it then follows that $S^\pm(t) \in C^\infty((t_1, t_1 + \varepsilon])$. Thus, for $t_1 = 0$,

$$S^+(t), S^-(t) \in C^\infty((0, \varepsilon]) \quad \text{for some } \varepsilon > 0.$$

Since $u_{xt}(S^\pm(t), t) = -\dot{S}^\pm(t)$ and $|\dot{S}^\pm(t)| \leq B_{i_0}$ for $0 \leq t \leq t_{i_0}$ (by Theorem 2.1) the maximum principle applied to u_{xt} gives the a priori bound

$$|u_{xt}(x, t)| \leq K_1, S^-(t) < x < S^+(t) \quad \text{and} \quad t_1 \leq t \leq t_1 + \varepsilon,$$

with K_1 depending only on K and B_{i_0}.

We can now proceed step by step (start with $t_1 = 0$) to show that $S^+(t), S^-(t)$ are in $C_\infty((0, t_0])$. Since t_0 can be any number smaller than T^+, the proof is complete.

COROLLARY 2.3. *Theorem 2.2 is still valid on* $(0, T^+)$ *if we replace* (1.6), (1.7) *and* (1.8) *by*

(1.6)* $h \in C^2([x_1, x_2])$,
(1.7)* *h' changes sign once.*

Proof. Under these assumptions it is proved in [1] that for some $\varepsilon > 0$,

$$S^- \in C^\infty((0, \varepsilon)) \quad \text{and} \quad S^+ \in C^\infty((0, \varepsilon)).$$

Apply Theorem 2.2 to the problem with initial data given by $u(x, \varepsilon/2)$ on $[S^-(\varepsilon/2), S^+(\varepsilon/2)]$.

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