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ON THE MULTIPLICATIVE STRUCTURE OF THE DE RHAM COHOMOLOGY OF INDUCED FIBRATIONS

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For a space X three types of "de Rham complex" over a field k will be considered in this paper:

(i) Classical de Rham theory; "space" means C^∞-manifold, $k =$ the real numbers.
(ii) Sullivan "PL" de Rham theory; "space" means simplicial set or simplicial complex; cf. [1], [3], or [5], $k =$ any field of characteristic 0.
(iii) Chen's de Rham theory of "differential spaces"; cf. [2] or [7], $k =$ the real numbers.

In each case we denote by $\mathscr{T}$ the category of "spaces" and by $\mathscr{A}$ the category of nonnegatively graded commutative differential algebras. The de Rham complex is a contravariant functor $\Lambda^*: \mathscr{T} \to \mathscr{A}$ or $\mathscr{T}_0 \to \mathscr{A}_0$ where $\mathscr{T}_0$, $\mathscr{A}_0$ are the appropriate pointed categories (i.e., with basepoint and augmentation respectively). The Stokes map is a transformation of functors $\rho^*: \Lambda^* \to C^*$ where C^* is the smooth normalized cochain functor; the word "smooth" having the empty meaning in case (ii). $\rho = H(\rho^*)$ is multiplicative (as follows from the existence of P_0^* below); and, in cases (i) and (ii), ρ is an isomorphism. In favorable cases the Eilenberg-Moore theorem applies, i.e., $H(B\Lambda^*X)$, where B is the "bar" construction, is the cohomology $H^*(\Omega X, k)$ of the loop-space ΩX. Since Λ^*X is commutative, $B\Lambda^*X$ has the structure of an algebra. We shall prove that if the Eilenberg-Moore theorem applies at all, this is precisely the cup-product structure of $H^*(\Omega X, k)$.

Chen has proved a theorem expressing $H^*(\Omega X, k)$ as $H(\int A^*)$ where $\int A^*$ is an "algebra of iterated integrals"; cf. [2] or [7]. If one wants to use the above result to prove that this is an isomorphism of *algebras*, one has to burden the theorem with an extra hypothesis which seems hard to verify, see the remark after Proposition 6. For this reason, we give a second proof of the multiplicativity of the appropriate map, which does *not* depend on the Eilenberg-Moore theorem; see Proposition 5 below.

We shall deal not merely with the loop-space, but with the general case of an

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induced fibration. We introduce appropriate notations. Let the diagram

$$\begin{array}{ccc} E & \xrightarrow{g''} & X'' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle f''} \\ X' & \xrightarrow[f']{} & X \end{array}$$

be a pull-back diagram in the category $\mathscr{T}_0$. In case (iii) (Chen's theory) the space E is turned into a "differential space" by the requirement that $\alpha\colon U \to E$ is to be a "plot" if and only if $g'\alpha, g''\alpha$ are "plots" in X', X'' respectively; cf. [2] or [7].

Eilenberg and Moore, [4], have introduced a map, the dualization of which we denote by

$$\theta_c\colon \operatorname{Tor}_{C^*X}(C^*X', C^*X'') \to HC^*E = H^*(E, k)$$

(or $\theta_c(X', X, X'')$); cf. [8] or [11] for the present cohomological case. The map θ_c is induced by the chain-map θ_c^* which is the compositions

$$B(C^*X', C^*X, C^*X'') \xrightarrow{a} B(C^*E, C^*E, C^*E) \xrightarrow{\varepsilon} C^*E.$$

Here, B stands for the "two sided bar construction," cf. [8] or [9], a is induced by the maps g', g'' and $f'g' = f''g''$, and ε is the "augmentation map" $e_1[\ \ ]e_2 \to e_1 \cup e_2$. In an entirely analogous way we define the map

$$\theta_\Lambda\colon \operatorname{Tor}_{\Lambda^*X}(\Lambda^*X', \Lambda^*X'') \to H\Lambda^*E$$

induced by a chain map θ_Λ^*.

Using the Künneth theorem and diagonal maps, Eilenberg and Moore introduced a natural product ϕ_C in $\operatorname{Tor}_{C^*X}(C^*X', C^*X'')$ (cf. [8] or [11] for the present, cohomological case) and they proved:

PROPOSITION 1. *$\cup\,(\theta_C \otimes \theta_C) = \theta_C\,\phi_C$ where $\cup$ is the cup-product.*

Remark. Originally, ϕ_C was not defined by a chain map because one of the Eilenberg-Zilber maps used in its construction is in the wrong direction. Using maps in DASH, however, one can obtain a natural chain map

$$\begin{array}{c} B(C^*X', C^*X, C^*X'') \otimes B(C^*X', C^*X, C^*X'') \\ \downarrow{\scriptstyle \phi_{C^*}} \\ B(C^*X', C^*X, C^*X''). \end{array}$$

One begins with the case where $X' = X''$ is a point to obtain

$$\phi\colon B(C^*X \otimes C^*X) \to B(C^*X);$$

cf. 4.2 in [9]. Then one uses 3.7_* and 3.5_* of that paper and appropriate shuffle maps to obtain ϕ_C^*.

In an analogous way the exterior product leads to a product ϕ_Λ in $\mathrm{Tor}_{\Lambda^*X}(\Lambda^*X', \Lambda^*X'')$ induced by the chain map ϕ^*_Λ which is the composition

$$\begin{array}{c} B(\Lambda^*X', \Lambda^*X, \Lambda^*X'') \otimes B(\Lambda^*X', \Lambda^*X, \Lambda^*X'') \\ \downarrow \gamma \\ B(\Lambda^*X' \otimes \Lambda^*X', \Lambda^*X \otimes \Lambda^*X, \Lambda^*X'' \otimes \Lambda X'') \\ \downarrow \mu \\ B(\Lambda^*X', \Lambda^*X, \Lambda^*X'') \end{array}$$

where γ, again, is defined by the evident shuffles and μ is induced by the (commutative!) product.

Analogously to Proposition 1, one easily proves the following result by making an appropriate diagram.

PROPOSITION 2. $\Lambda(\theta_\Lambda \otimes \theta_\Lambda) = \theta_\Lambda \phi_\Lambda$ *where* Λ *denotes the exterior product.*

In [1], [6], and [7] it was proved that the natural map

$$\rho^*: \Lambda^*X \to C^*X$$

could be "extended" to a natural map

$$P_0^*: \Lambda^*X \Rightarrow C^*X$$

of DASH, i.e., a map of coalgebras $B(\Lambda^*X) \to B(C^*X)$.

Thus, our pull-back diagram leads to a commutative diagram

$$\begin{array}{ccccc} \Lambda^*X' & \leftarrow & \Lambda^*X & \to & \Lambda^*X'' \\ \Downarrow P_0^* & & \Downarrow P_0^* & & \Downarrow P_0^* \\ C^*X' & \leftarrow & C^*X & \to & C^*X'' \end{array}$$

of DASH. Using Theorem 3.7.2$_*$ of [9] we thus obtain a natural map

$$\mathbf{P}_0: \mathrm{Tor}_{\Lambda^*X}(\Lambda^*X', \Lambda^*X'') \to \mathrm{Tor}_{C^*X}(C^*X', C^*X'')$$

namely $\mathbf{P}_0 = \mathrm{Tor}_{P_0}(P_0, P_0; 0, 0)$ in the notation of that theorem.

PROPOSITION 3. *If* $\rho = H(\rho^*)$ *is the morphism induced by the Stokes map,* $\theta_C \mathbf{P}_0 = \rho\theta_\Lambda$.

Proof. We consider the diagram

$$\begin{array}{ccccc} B(\Lambda^*X', \Lambda^*X, \Lambda^*X'') & \xrightarrow{a'} & B(\Lambda^*E, \Lambda^*E, \Lambda^*E) & \xrightarrow{\varepsilon'} & \Lambda^*E \\ \downarrow P_0^* & ① & \downarrow P_0^* & ② \;\; \overset{i'}{\leftarrow} \; \underset{i}{\leftarrow} & \downarrow \rho^* \\ B(C^*X', C^*X, C^*X'') & \xrightarrow[a]{} & B(C^*E, C^*E, C^*E) & \xrightarrow[\varepsilon]{} & C^*E \end{array}$$

in which $\mathbf{P}_0^*$ denotes the chain-map inducing $\mathbf{P}_0$. The construction of this map from P_0^* is explained in the proofs of 3.5_* and $3.7.2_*$ of [9]. The important fact is that this chain-map itself is natural, and hence ① is commutative.

We next prove that ② is chain-homotopy commutative: First, observe that ε and ε' are homology-isomorphisms and have the homology inverse i and i' (dotted arrows) given by $e \to e[\ \ ]1$ where $e, 1 \in \Lambda^*E$ or C^*E. Now, using the explicit definition of $\mathbf{P}_0^*$ in [9], we see that $\mathbf{P}_0^*(e[\ \ ]1) = \rho^*e[\ \ ]1$. Hence, $\mathbf{P}_0^* i' = i\rho^*$. Hence, $\rho^*\varepsilon'$ and $\varepsilon\mathbf{P}_0^*$ are chain homotopic, and we are done.

PROPOSITION 4. *If* θ_C: $\mathrm{Tor}_{C^*X}(C^*X', C^*X'') \to H^*(E, k)$ *is a monomorphism (e.g., the "Eilenberg-Moore theorem" applies) then the morphism*

$$\mathbf{P}_0 : \mathrm{Tor}_{\Lambda^*X}(\Lambda^*X', \Lambda^*X'') \to \mathrm{Tor}_{C^*X}(C^*X', C^*X'')$$

is multiplicative.

Proof. This is immediate from Proposition 3 since θ_Λ, ρ and θ_C are multiplicative.

The most important special case arises when X' is a point, $X'' = PX$ the path-space and $E = \Omega X$ the loop space. Then one considers the commutative diagram of spaces

$$\begin{array}{ccccc} \text{point} & \longrightarrow & X & \xleftarrow{\text{end point}} & PX \\ \| & & \| & & \uparrow \text{(constant path)} \\ \text{point} & \longrightarrow & X & \longleftarrow & \text{point} \end{array}$$

which induces the commutative diagram

$$\begin{array}{ccc} B(k, \Lambda^*X, \Lambda^*PX) & \xrightarrow{P_0^*} & B(k, C^*X, C^*PX) \\ \downarrow & & \downarrow \\ B(\Lambda^*X) & \xrightarrow[P_0^*]{} & B(C^*X) \end{array}$$

where the vertical morphisms are multiplicative due to the naturality of the chain-maps inducing the products; also they are homology isomorphisms. Hence we obtain as a corollary of Proposition 4:

PROPOSITION 5. *The morphism* $P_0 = H(P_0^*)$: $HB(\Lambda^*X) \to HB(C^*X)$ *is multiplicative.*

Proof. We have only proved Proposition 5 on the hypothesis that θ_C (point, X, PX) is a monomorphism. In fact, however, we can omit this hypothesis. Quite independently one can prove that the diagram

$$\begin{array}{ccc} B(\Lambda^*X) \otimes B(\Lambda^*X) & \xrightarrow{P_0^* \otimes P_0^*} & B(C^*X\ \ ) \otimes B(C^*X) \\ \downarrow \phi_{\Lambda^*} & & \downarrow \phi_{C^*} \\ B(\Lambda^*X) & \xrightarrow[P_0^*]{} & B(C^*X) \end{array}$$

is homotopy commutative in the category of coalgebras; cf. 3.2 in [9]. First, one examines the corresponding diagram in the unpointed categories, replacing P_0^* by P^* and B by $\mathbf{B}$ (cf. [7]). Then, calling the two compositions involved U and V, let $\bar{U} = \tau U$, $\bar{V} = \tau V$ where τ: $B(C^*X) \to C^*X$ is the twisting function. Then we have to find

$$\bar{W}\colon B(\Lambda^*X) \otimes B(\Lambda^*X) \to C^*X$$

such that $\bar{W}[\ \ ] = 1$ and $D\bar{W} = \bar{U} \cup \bar{W} - \bar{W} \cup \bar{V}$ (cf. 3.2.1$_*$ in [9]). This can now be accomplished by the same acyclic models argument by which the existence of P was established in [1], [6]. We omit further details.

I have been unable to obtain an analogous proof of Proposition 4 without hypothesis on θ_C. The difficulties are of the kind described in Section 9 of [10].

We now apply this result to a theorem of Chen; we are in the context (iii) so that C^*X is the "smooth" cochain functor, which we shall denote by C_s^*X for the moment, so that C^*X can stand for the usual singular functor. Now suppose:

(i) $A^* \subset \Lambda^*X$ is a subalgebra such that $\rho^* \mid A^*\colon A^* \to C_s^*X$ is a homology isomorphism, and such that $dA^0 = A^1 \cap d\Lambda^0X$.

(ii) The restriction map $C^*X \to C_s^*X$ is a homology isomorphism (e.g., X is a C^∞-manifold).

(iii) θ_C(point, X, PX) is an isomorphism for C^*, i.e., the Eilenberg-Moore theorem applies.

Then we consider the following sequence of chain maps: (cf. [7] for I_0 and $\int A^*$)

$$\begin{array}{ccccc} \int A^* \xleftarrow{I_0} B(A^*) \xrightarrow{\rho_0^*} & B(C_s^*X) & & & \\ & \nearrow & & & \\ B(C^*X) & \leftarrow & B(R,\, C^*X,\, C^*PX) & \xrightarrow{\theta_{C^*}} & C^*\Omega X. \end{array}$$

Each morphism induces an isomorphism of *algebras* in homology and hence we have:

PROPOSITION 6 (Chen's Theorem). *Under the above hypotheses $H(\int A^*)$ and $H^*(\Omega X, R)$ are isomorphic as algebras.*

Remark. Note that in the above we used the full strength of Proposition 5 as stated. If we had relied on Proposition 4 we would have needed the following additional hypothesis:

(iv) θ_C(point, X, P_sX) for C_s^* is a monomorphism into $HC_s^*\Omega_sX$, where P_sX, Ω_sX are the "smooth" path and loop-space respectively.

Alternatively we could *replace* (iii) by (iii)$_s$, namely, θ_C(point, X, $P_s X$) for C_s^* is an isomorphism into $H^*(\Omega_s X, R)$.

But then, our result would be about $\Omega_s X$ and not about ΩX. The relationship between $\Omega_s X$ and ΩX appears to be obscure. If (iii), (iii)$_s$ are both true, $H^*(\Omega X, R)$ and $H^*(\Omega_s X, R)$ are isomorphic algebras.

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