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FUNCTION SPACE COMPLETIONS

BY

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Many problems in analysis require the introduction of a linear space G of functions. The next step is to equip G with a norm which is appropriate for the specific problem. This in turn leads to the necessity of constructing a complete space $\hat{G}$ (Banach space) by adjoining functions to G. We assume that the domain and range spaces remain fixed. Ordinarily the norm of a function in G will be defined in terms of the values the function assumes at points in the domain. If the same statement cannot be made for functions in $\hat{G}$, then it is unlikely that $\hat{G}$ will be of much use. The purpose of this paper is to overcome this difficulty.

We begin by describing the relationship between the norm of a function in G and the values the function assumes at points in the domain. This can always be done. The result appears as Theorem 1 of [4] and again in this paper as Corollary 5.4 to Theorem 5.3. The relationship between norm and functional values is formulated in such a way that it suggests which functions should be added to G to form the completion. The resulting normed space H has the property we seek. The norm of a function in H is defined in terms of its values at points in the domain and, moreover, the definition is derived from the manner in which norms of functions in G are related to their functional values. On the other hand, it is not always possible to obtain a completion by adjoining functions. This fact is illustrated by Example 3.3.

Our approach to the problem of obtaining function space completions parallels the treatment of Grothendieck's completion theorem given in [8]. We also view the problem as having two parts: First we obtain function spaces in which the original space is dense. Secondly, we look for complete spaces among these. Our key tool is Theorem 2.3 which generalizes the first part of Grothendieck's theorem and characterizes the additional functions and their norms. Grothendieck's theorem is only applicable to the special case in which the domain of the functions in G is the continuous dual G'. Section 3 gives several sufficient conditions for the resulting function space to be complete. The theorems are then applied to give new characterizations of L_p spaces $(1 \leq p < \infty)$ as completions of the space of continuous functions. The characterizations are totally different from any we have seen in the literature.

More must be said about the relationship of the norm to the point values of the functions. It is unrealistic to think that a norm mysteriously appears on a

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function space. Every norm must be determined by some specific properties of the functions. If this is not the case there is no need to clutter up a discussion with function spaces. The same behavior which suggests a norm will also suggest a filter. The filter will be composed of sets of the domain (or the linear space composed of finite linear combinations of points from the domain). Convergence on the filter (see below) gives the same topology as the norm. The norm can be obtained from the filter as described in Section 1 with the relationship between filter and norm detailed in Theorem 5.3 and Lemma 5.5. The convergence on a filter concept was first presented in [3] with applications in [4] and [5]. Our view is that the norm should not be regarded as the only way of obtaining a topology on a normed space, and that for some purposes a filter may be much more useful. Consider Section 4, where a filter is constructed that gives the L_p norm on $C[0, 1]$, $1 \leq p < \infty$. The filter was suggested by the Riemann integral which can be used to define the norm. This is the major reason why our representation of L_p is so different. We could have motivated our filter construction by looking at Lebesgue integration. This would have given a different representation, one more akin to the usual L_p. The desired filter always exists, but it may be a test of ingenuity to explicitly construct it.

The situation is complicated by the fact that many filters may produce the same norm. The effect on Theorem 2.3 is that distinct filters may give different spaces in which the original space is dense. It sometimes seems best to use the coarsest filter available. This is because we desire the new functions to be as much like the original functions as possible. The coarser the filter the more function properties that are preserved in the larger function space of Theorem 2.3. Filters can be constructed to preserve continuity, differentiability, integrability, and/or similar properties for the derivatives of any order. Our L_p completion of $C[0, 1]$ in Section 4 can be interpreted as those functions having a kind of Riemann integrability. The construction of a filter which preserves differentiability is suggested in a paper by R. Nielsen [9]. A filter which preserves continuity of the functions is given in [3]. It was the first application of the concept.

A coarse filter may not give the desired results. This is because the new function space will not contain enough functions. A larger function space can be obtained using a finer filter. The space obtained by means of Theorem 2.3 will now have more functions (see Lemma 5.10). It is possible that we might obtain a complete space by using a filter which is fine enough. But, we still want to keep the filter as coarse as possible in hopes of preserving properties of the original space. This trade-off is thoroughly discussed in Section 5. Theorem 5.16 gives a sufficient condition for the existence of a finer filter which will give a completion by means of Theorem 2.3. The filters constructed in Section 4 satisfy the condition. Theorem 5.16 wasn't needed there because the completions were obtained without having to refine the filter. Refining the filter may have the effect of decreasing the effective size of the domain. Other approaches to our problem have noticed a similar phenomenon [1].

Assume that there is a complete function space in which the original space is dense, in other words a completion. Theorem 5.12 tells us that the completion can be obtained by refining the filter, or the norm on the completion has no usable relationship to the manner in which the norm was defined on the original space.

1(a). Preliminaries

Throughout this section $G(S)$ denotes a space of scalar (real or complex) valued functions with domain S.

1.1 DEFINITION (Convergence on a Filter) [3, p. 287, Definition 1.2]. A filter $\mathscr{G}$ in $G(S)$ converges to a function f_0 in $G(S)$ on a filter $\mathscr{F}$ in S if for every number $\varepsilon > 0$ there is a set D in $\mathscr{G}$ such that for each f in D there is a set F_f in $\mathscr{F}$ with the property that $|f(s) - f_0(s)| < \varepsilon$ for every $s \in F_f$.

The notion of convergence on a filter was originally formulated in order to weaken the assumption of simple uniform convergence found in classical results relating convergence and continuity. The most fundamental relationship between convergence on a filter and continuous functions is given by the following theorem. Its generalization is a basic result for this paper.

1.2 THEOREM [3, p. 287, Theorem 1.3]. *Let $\mathscr{G}$ be a filter in $G(S)$ where S is a topological space. Assume there is a set D in $\mathscr{G}$ such that each g in D is continuous at a point s_0 of S. Then the filter $\mathscr{G}$ converges pointwise at s_0 to a function f_0 which is continuous at s_0 if and only if $\mathscr{G}$ converges to f_0 on the filter of neighborhoods of s_0.*

In the improved version of this theorem the filter of neighborhoods of s_0 is replaced by an arbitrary filter $\mathscr{F}$ in S. In place of functions continuous at s_0 we will consider the set $F(S, \tau, \mathscr{F})$ consisting of all $f \in \mathbf{C}^S$ such that $\lim_{\mathscr{F}(s)} f$ exists for all s in $\bigcap \{\bar{F}: F \in \mathscr{F}\}$. Here $\bar{F}$ denotes the closure of F in the τ-topology and $\mathscr{F}(s)$ is the least upper bound of $\mathscr{F}$ and the neighborhood filter $N(s)$ of s. We do not assume that $\lim_{\mathscr{F}(s)} f = f(s)$. Instead we associate with each f in $F(S, \tau, \mathscr{F})$ a new function $\bar{f}$ defined on $\bigcap \{\bar{F}: F \in \mathscr{F}\}$ by $\bar{f}(s) = \lim_{\mathscr{F}(s)} f$.

1.3 THEOREM. *Let $\mathscr{G}$ be a filter in $F(S, \tau, \mathscr{F})$ and let f_0 be a scalar valued function defined on S. Let $\bar{\mathscr{G}}$ denote the filter base $\{\{\bar{f}: f \in D\}: D \in \mathscr{G}\}$, and assume that every ultrafilter containing $\mathscr{F}$ converges to a point of $\bigcap \{\bar{F}: F \in \mathscr{F}\}$. Then $\mathscr{G}$ converges to f_0 on $\mathscr{F}$ if and only if f_0 is in $F(S, \tau, \mathscr{F})$ and $\bar{\mathscr{G}}$ converges to $\bar{f}_0$ uniformly on $\bigcap \{\bar{F}: F \in \mathscr{F}\}$.*

Proof. Suppose that $\mathscr{G}$ does not converge to f_0 on the filter $\mathscr{F}$. Then there exists $\varepsilon > 0$ such that for every D in $\mathscr{G}$ there exists $d \in D$ such that for every $F \in \mathscr{F}$ there exists $x \in F$ with $|d(x) - f_0(x)| > \varepsilon$. Find such a d, and let

$$F_d = \{x \in F: |d(x) - f_0(x)| \geq \varepsilon\}.$$

The filter base $\{F_d: F \in \mathscr{F}\}$ contains $\mathscr{F}$ and is itself contained in an ultrafilter $\mathscr{U}$. By assumption, $\mathscr{U}$ converges to a point $a \in \bigcap \{\bar{F}: F \in \mathscr{F}\}$. Thus

$$|\bar{d}(a) - \bar{f}_0(a)| = |\lim_{\mathscr{U}} d - \lim_{\mathscr{U}} f_0| = |\lim_{\mathscr{U}} (d - f_0)| \geq \varepsilon,$$

and $\bar{\mathscr{G}}$ does not converge to $\bar{f}_0$ uniformly on $\bigcap \{\bar{F}: F \in \mathscr{F}\}$.

We now proceed to obtain the implication in the opposite direction. Assume that $\mathscr{G}$ converges to f_0 on $\mathscr{F}$, and let s be an arbitrary point in $\bigcap \{\bar{F}: F \in \mathscr{F}\}$. Since $\mathscr{F}(s)$ contains $\mathscr{F}$, $\mathscr{G}$ converges to f_0 on $\mathscr{F}(s)$. Therefore, given $\varepsilon > 0$ there exists $D \in \mathscr{G}$, $f \in D$ and $F \in \mathscr{F}(s)$ such that

$$|f(x) - f_0(x)| < \varepsilon/3 \quad \text{for all } x \in F$$

and

$$|f(x') - f(x'')| < \varepsilon/3 \quad \text{for all } x', x'' \in F.$$

The latter inequality uses the fact that f is in $F(S, \tau, \mathscr{F})$. It follows that

$$|f_0(x') - f_0(x'')| < \varepsilon \quad \text{for all } x', x'' \in F.$$

Since the filter base $\{f_0(E): E \in \mathscr{F}(s)\}$ is therefore Cauchy, $\lim_{\mathscr{F}(s)} f_0$ exists and $f_0 \in F(S, \tau, \mathscr{F})$. The remainder of the result is obtained by again letting $\varepsilon > 0$ and knowing there exists $D \in \mathscr{G}$ such that for each $d \in D$ there exists $F_d \in \mathscr{F}$ such that

$$|d(x) - f_0(x)| < \varepsilon \quad \text{for all } x \in F_d.$$

Let a be an arbitrary element of $\bigcap \{\bar{F}: F \in \mathscr{F}\}$. Then

$$|\bar{d}(a) - \bar{f}_0(a)| = |\lim_{\mathscr{F}(a)} d - \lim_{\mathscr{F}(a)} f_0| \leq \varepsilon.$$

Thus $\bar{\mathscr{G}}$ converges to $\bar{f}_0$ uniformly on $\bigcap \{\bar{F}: F \in \mathscr{F}\}$. ∎

1(b). The topology of convergence on a family of filters

Let S be a set and let G be a subspace of $\mathbf{C}^S$, the space of all functions from S into the field of complex numbers. We use V^* to denote the algebraic dual of a vector space V and V' to denote the topological dual when V has a linear topology. Denote by Φ a collection of filters in S and for each $\mathscr{F} \in \Phi$ let $U(\varepsilon, \mathscr{F})$ denote the set

$$\{g \in G: \text{there exists } F_g \in \mathscr{F} \text{ such that } |g(x)| < \varepsilon \text{ for all } x \in F_g\}.$$

When S is replaced by a linear space X, denote by $B(X, \Phi)$ the set consisting of all $f \in X^*$ such that for every $\mathscr{F} \in \Phi$ there exists $F \in \mathscr{F}$ such that f is bounded on F.

1.4 PROPOSITION [4, Proposition 1]. (i) *The collection*

$$\{U(\varepsilon, \mathscr{F}): \varepsilon > 0, \mathscr{F} \in \Phi\}$$

is a local subbasis at the zero-function for a topology on G called the Φ-topology, or the $\mathscr{F}$-topology when Φ consists of a single filter $\mathscr{F}$.

(ii) *A filter in G converges to g_0 for the Φ-topology if and only if it converges to g_0 on each filter from Φ.*

(iii) *The Φ-topology is a linear topology for G if and only if for each g in G and each $\mathscr{F} \in \Phi$ there exists an F in $\mathscr{F}$ such that g is bounded on F.*

(iv) *The set $B(X, \Phi)$ is the largest linear subspace of X^* on which the Φ-topology is linear.*

We observe that the concept of a Φ-topology generalizes the more familiar notion of a $\mathfrak{S}$-topology, that is, the topology of uniform convergence on a family $\mathfrak{S}$ of sets in S [7]. The class of $\mathfrak{S}$-topologies on a function space is very limited. For example, under a $\mathfrak{S}$-topology, the evaluation at any $x \in \bigcup \{U: U \in \mathfrak{S}\}$ is a continuous map. On the other hand, the class of Φ-topologies is very general. The space G with the Φ-topology is denoted by (G, Φ).

1.5 THEOREM [4, Theorem 1]. *Let X be a vector spcce and let G be a subspace of X^*. Each locally convex topology on G can be obtained as a Φ-topology, where Φ is a family of filters in X.*

This proposition is a corollary (5.4) to the discussion of minimal filters in Section 5. A stronger result can be proved (see Proposition I.6, page 9 of [11]): If $\mathfrak{S}$ is any collection of subsets of G^* (not necessarily $\sigma(G^*, G)$-bounded), then the $\mathfrak{S}$-topology on G can be obtained as a Φ-topology for Φ a family of filters in X.

1.6 PROPOSITION [4, Proposition 2]. *Let Φ be a family of filters in the set S and assume $G \subset C^S$ is a locally convex space under the Φ-topology. Then the Φ-topology and the $\mathfrak{S}$-topology coincide if $\mathfrak{S}$ is the family of all subsets*

$$\bigcap \{\overline{e(F)}: F \in \mathscr{F}\}$$

of G^*, formed as $\mathscr{F}$ ranges through Φ. Here $e: S \to G^*$ is the evaluation map on G and the closures are taken in the weak topology of pointwise convergence on G.

It is convenient to list a few conventions which will be observed in the sequel. Suppose $G \subset \mathbf{C}^S$, $\mathscr{F}$ is a filter in S and $e: S \to G^*$ is the natural evaluation map. Then $\overline{e(F)}$, for $F \in \mathscr{F}$, will always denote the closure, in G^*, of $e(F)$ in the weak topology of pointwise convergence on G. Let U be a subset of V, a vector space; then $\langle U \rangle$ will denote the linear span of U in V, and $\Gamma(U)$ will denote the balanced convex hull of U. With each filter $\mathscr{F}$ which generates a locally convex topology on G we associate a pseudo-norm p defined by

$$p(f) = \sup \{|f(x)|: x \in \bigcap \{\overline{e(F)}: F \in \mathscr{F}\}\}.$$

Alternatively, $p(f) = \lim\sup_{\mathscr{F}} |f|$ for each f in G. (Our notation involving limits of filters follows Bourbaki. In particular, see II.7.3 and IV.5.6 of [2].) In Section 5, Theorem 5.3 and Lemma 5.5, it is shown that the pseudo-norm topology and $\mathscr{F}$-topology coincide. We will always mean the pseudo-norm defined above when we speak of the pseudo-norm associated with the filter $\mathscr{F}$.

2. The closure of G in (X^*, Φ)

This section is devoted to a generalization of Grothendieck's theorem. Let the vector spaces E and H form a pairing, and let $\mathfrak{S}$ denote a collection of $\sigma(H, E)$-closed, $\sigma(H, E)$-bounded, balanced and convex sets in H directed by $\supset$. Let T: $E \to H^*$ denote the canonical map and let $\tilde{E}$ denote the space of all linear functionals in H^* which are $\sigma(H, E)$-continuous on each member of $\mathfrak{S}$. We then have the fact that $T(E)$ is dense in $\tilde{E}$ relative to the $\mathfrak{S}$-topology on H^* (see Theorem 16.9 of [8].) The application of this result is limited to those topologies on E which can be obtained via uniform convergence on a family of $\sigma(H, E)$-bounded sets in H. On the other hand, every locally convex topology on E can be obtained via convergence on a family of filters Φ in H (Theorem 1.5) and we desire a theorem which is applicable in this new situation.

2.1 DEFINITION. *Let X be a vector space, G a subspace of X^* and Φ a family of filters in X.*

(i) $\Psi(X, G, \Phi) = \{\mathscr{N}: \mathscr{N}$ *is a filter in* X, $\mathscr{N} \supset \mathscr{F}$, *for some* $\mathscr{F} \in \Phi$, *and* $\lim_{\mathscr{N}} g$ *exists for every* $g \in G\}$.

(ii) $C(X, G, \Phi) = \{f \in X^*: \lim_{\mathscr{N}} f$ *exists for each* $\mathscr{N} \in \Psi(X, G, \Phi)\}$.

When $\Phi = \{\mathscr{F}\}$ is singleton, we write $C(X, G, \mathscr{F})$ for $C(X, G, \Phi)$. If the Φ-topology is linear on G, it is also linear on $C(X, G, \Phi)$. In any event we will always regard $C(X, G, \Phi)$ as being equipped with the Φ-topology. It is an important fact, and one to which we will refer frequently in the sequel, that $C(X, G, \Phi)$ is a closed subspace of (X^*, Φ). A formal statement follows and a proof is contained in a small part of the argument which yielded Theorem 1.3.

2.2 PROPOSITION. *Let X be a linear space with G a subset of X^* and let Φ denote a collection of filters in X. Then $C(X, G, \Phi)$ is a closed subset of X^* for the Φ-topology.*

Consider a point $a \in G^*$. We denote the filter of $\sigma(G^*, G)$-neighborhoods of a by $\mathscr{N}(a)$. Let $\mathscr{F}$ denote an arbitrary filter in X; $\mathscr{F}(a)$ denotes the least upper bound (provided it exists) of $\mathscr{N}(a)$ and the filter with base $e(\mathscr{F})$ in the set of all filters in G^*.

$$A(\mathscr{F}) = \bigcap \{\overline{e(F)}^{\sigma(G^*,G)}: F \in \mathscr{F}\} \quad \text{and} \quad \mathfrak{S}(\Phi) = \{A(\mathscr{F}): \mathscr{F} \in \Phi\}.$$

We say that Φ is directed by $\subset$ if for every pair of filters $\mathscr{F}_1$ and $\mathscr{F}_2$ in Φ there exists a filter $\mathscr{F} \in \Phi$ such that $\mathscr{F} \subset \mathscr{F}_1$, $\mathscr{F} \subset \mathscr{F}_2$. Note that when Φ is directed by $\subset$, $\mathfrak{S}(\Phi)$ is directed by $\supset$, and that if $\mathscr{F}$ has a filter base consisting of balanced and convex sets, then $A(\mathscr{F})$ is balanced and convex.

2.3 THEOREM. *Let X be a vector space, G a subspace of X^* and Φ a family of filters in X inducing a linear topology on G. Suppose also that each filter $\mathscr{F} \in \Phi$ has a filter base consisting of balanced and convex sets, and that Φ is directed by $\subset$. Then $C(X, G, \Phi)$ is the closure of G in (X^*, Φ).*

Proof. We have already observed that $C(X, G, \Phi)$ is closed in (X^*, Φ). In order to prove that G is dense in $C(X, G, \Phi)$ we first consider the case in which $\Phi = \{\mathscr{F}\}$ is a singleton set. Note that $\mathscr{F}(a)$ belongs to $\Psi(X, G, \mathscr{F})$ for each a in $A(\mathscr{F})$. Thus for each $f \in C(X, G, \mathscr{F})$, the $\lim_{\mathscr{F}(a)} f$ exists and we define $\hat{f}(a) = \lim_{\mathscr{F}(a)} f$. The map $\hat{f}$ is defined only on $A(\mathscr{F})$ and we must show that it can be extended linearly over G'. The new linear map will be denoted by $\bar{f}$. Since G' is the union of all scalar multiples of $A(\mathscr{F})$ it is natural to define $\bar{f}(x) = r\hat{f}(a)$ where $x \in G'$ and $x = ra$, $r \in \mathbf{C}$, and $a \in A(\mathscr{F})$ [7, p. 204, Proposition 3]. The map $\bar{f}$ will be single valued and homogeneous once we show that $r\hat{f}(a) = \hat{f}(ra)$ for all $r \in \mathbf{C}$ and $a \in A(\mathscr{F})$ such that $ra \in A(\mathscr{F})$. The additivity of $\bar{f}$ follows after we show that $\hat{f}(a+b) = \hat{f}(a) + \hat{f}(b)$ for all a, $b \in A(\mathscr{F})$ such that $a + b \in A(\mathscr{F})$.

Consider the case $0 < |r| \leq 1$ with $a \in A(\mathscr{F})$. The filter $r\mathscr{F}(a)$ is a refinement of $\mathscr{F}(ra)$ and thus $\hat{f}(ra) = \lim_{\mathscr{F}(ra)} = f \lim_{\mathscr{F}(a)} rf = r\hat{f}(a)$. When $|r| > 1$ and $ra \in A(\mathscr{F})$, the filter $\mathscr{F}(ra)$ is a refinement of $r\mathscr{F}(a)$ and the result is again obtained. For the next verification consider a, $b \in A(\mathscr{F})$ such that $a + b \in A(\mathscr{F})$. Observe that the filter $\mathscr{I}$ with a filter basis

$$\{(F_1 \cap U) + (F_2 \cap V) \colon F_1, F_2 \in \mathscr{F},\ U \in \mathscr{N}(a) \text{ and } V \in \mathscr{N}(b)\}$$

is a refinement of the filter $\mathscr{H}$ with basis

$$\{2F \cap 2W \colon F \in \mathscr{F},\ W \in \mathscr{N}(\tfrac{1}{2}(a + b))\}.$$

Thus $\hat{f}(a) + \hat{f}(b) = \lim_{\mathscr{I}} f = \lim_{\mathscr{H}} f = \hat{f}(a + b)$.

We turn to the general Φ. For each $a \in \bigcup \{A(\mathscr{F}) \colon \mathscr{F} \in \Phi\}$ define $\hat{f}(a) = \lim_{\mathscr{F}(a)} f$ where $a \in A(\mathscr{F})$. Since Φ is directed by $\subset$, we may use the result just established for singleton Φ to show that $\hat{f}$ is singled valued and possesses a linear extension $\bar{f}$ defined on G'.

It is now crucial to show that $\hat{f}$ or the restriction of $\bar{f}$ to $A(\mathscr{F})$ is $\sigma(G', G)$ continuous for every $\mathscr{F}$ from Φ. Consider an arbitrary $a \in A(\mathscr{F})$ and let $\varepsilon > 0$. There is a $U \in \mathscr{F}(a)$ such that $|f(x) - \hat{f}(a)| < \varepsilon/2$ for all x in U. Assume without harm that U is of the form $F \cap V$ where V is an open $\sigma(G', G)$ neighborhood of the point a and $F \in \mathscr{F}$. Let y be an arbitrary member of $V \cap A$. Note that $F \cap V$ is a member of $\mathscr{F}(y)$ and contains a point z such that $|f(z) - \hat{f}(y)| < \varepsilon/2$. Thus

$$|\hat{f}(y) - \hat{f}(a)| \leq |\hat{f}(y) - f(z)| + |f(z) - \hat{f}(a)| < \varepsilon,$$

and the desired continuity is obtained.

Choose arbitrary $A(\mathscr{F})$ for some $\mathscr{F}$ from Φ, and let $\varepsilon > 0$. The approximation theorem [8, p. 145, 16.8] presents us with a function g in G such that $|\bar{f}(x) - g(x)| < \varepsilon$ for all $x \in A(\mathscr{F})$. It follows that there is a filter in G converging to $\bar{f}$ uniformly on every $A(\mathscr{F})$. The filter also converges to f for the Φ-topology as is seen by applying Theorem 1.3 with the filter having basis $e(\mathscr{F})$ in the role of $\mathscr{F}$, $S = G^*$, and $\tau = \sigma(G^*, G)$. Thus $C(X, G, \Phi)$ is the closure of G in (X^*, Φ). ∎

2.4 *Remark.* Let G be a locally convex Hausdorff space. Let $X = G'$ and let Φ be the family of all filters in X having, for a base, a convex equicontinuous subset of G'. The Φ-topology is the same as uniform convergence on the family of convex equicontinuous sets, and thus is the original topology on G. The linear space $C(X, G, \Phi)$ has now become all linear forms on G' whose restrictions to equicontinuous subsets are $\sigma(G', G)$-continuous. Completeness of $C(X, G, \Phi)$ is a familiar result in general topology (see Remark 1, p. 249 of [7]). Thus we may regard 2.3 as a generalization of the Grothendieck completion theorem, which says, in this special case, that $C(X, G, \Phi)$ is a completion of G.

3. Complete $C(X, G, \Phi)$

While $C(X, G, \Phi)$ yields a closure of G in X^*, it may not provide a completion. In fact, it may be impossible to find a complete linear space H such that $G \subset H \subset X^*$ (Example 3.3). In this section we restrict our attention to singleton $\Phi = \{\mathscr{F}\}$ and examine two situations in which $C(X, G, \mathscr{F})$ is complete. The results are applied to obtain a completion of the space $C[0, 1]$ under the L_p norm, $1 \leq p < \infty$.

3.1 THEOREM. *Let $(G, \mathscr{F})$ be a seminormed space, $G \subset X^*$, $\mathscr{F}$ a filter in X. If $\mathscr{F}$ contains a linearly independent subset of X then $C(X, G, \mathscr{F})$ is complete, and $C(X, G, \Gamma(\mathscr{F}))$ is a completion of G. Here $\Gamma(\mathscr{F})$ denotes the filter in X with a base consisting of the balanced, convex hulls of the sets in $\mathscr{F}$.*

Proof. Let $F_0 \in \mathscr{F}$ denote a linearly independent subset of X given by the hypothesis and set $H = C(X, G, \mathscr{F})$, $A = \bigcap \{\overline{e(F)}\colon F \in \mathscr{F}\}$. Let $\{h_n\}$ denote an arbitrary Cauchy sequence in H; we will exhibit an element $h \in H$ such that $h_n \to h$ on $\mathscr{F}$. By Proposition 1.6 the sequence $\{h_n\}$ converges uniformly on A to a $\sigma(H^*, H)$-continuous function $h_0\colon A \to \mathbf{C}$. Since A is a compact subset of the completely regular Hausdorff space $(H^*, \sigma(H^*, H))$ it is, in particular, a closed subset of the Stone-Cech compactification $p(H^*)$; the Tietze extension theorem guarantees the existence of a $\sigma(H^*, H)$-continuous extension $\tilde{h}$ of h_0 to H^*. Define, for each $x \in F_0$, $h(x) = \tilde{h}(e(x))$. Since F_0 is linearly independent h can be extended to a map in X^*, which we will also denote by h. We claim that $h_n \to h$ on $\mathscr{F}$. To show this we apply Theorem 1.3 with $S = H^*$, $\tau = \sigma(H^*, H)$ and with $e(\mathscr{F})$ in the role of $\mathscr{F}$. Note that $\tilde{h}_n(a) = h_n(a)$ and $\tilde{h}(a) = h_0(a)$ for every $a \in A$. Thus $h_n \to \tilde{h}$ on $e(\mathscr{F})$. Since $\mathscr{F}$ contains F_0 this implies that $h_n \to h$ on $\mathscr{F}$. Since

$C(X, G, \mathscr{F})$ is closed, $h \in C(X, G, \mathscr{F})$. The assertion about $C(X, G, \Gamma(\mathscr{F}))$ is now an immediate consequence of Theorem 2.3. ∎

Note that we do not assert that $C(X, G, \mathscr{F})$ is a completion of G. The trouble is that Theorem 2.3 only applies when the filters involved possess bases of balanced, convex sets.

3.2 THEOREM. *Let $(G, \mathscr{F})$ be a normed space where $G \subset X^*$, $X \subset G^*$ and the filter $\mathscr{F}$ in X has a base consisting of balanced and convex sets. Then $C(X, G, \mathscr{F})$ is complete if there exists a linear map T: $X \to G'$ with the following properties:*

(i) *For every $\varepsilon > 0$ and $g \in G$ there exists a set $F = F(\varepsilon, g)$ in $\mathscr{F}$ such that $|(Tv)(g) - v(g)| < \varepsilon$ for all $v \in F$.*

(ii) *The filter base $\{TF: F \in \mathscr{F}\}$ contains an equicontinuous subset of G'.*

Proof. Let $A = \bigcap \{\bar{F}: F \in \mathscr{F}\}$. Since G is dense in $C(X, G, \mathscr{F})$ it suffices to show that an arbitrary Cauchy sequence $\{f_n\}$ in G has a limit in $C(X, G, \mathscr{F})$. Choose $f_1 \in (G')^*$ such that $f_n \to f_1$ uniformly on A. We will use f_1 and T to define a candidate f for the limit of f_n in $C(X, G, \mathscr{F})$: For each $x \in X$ set $f(x) = f_1(Tx)$. For convenience in applying Theorem 1.3 we may assume f has been defined (arbitrarily) on $G^* \backslash X$. Letting $(S, \tau) = (G^*, \sigma(G^*, G))$ we need to show that $f \in F(S, \tau, \mathscr{F})$ and that $\bar{f}_n \to \bar{f}$, uniformly on A. Let a be an arbitrary element of A and let $\mathscr{U}$ be an ultrafilter containing $\mathscr{F}(a)$. We claim that $\lim_{\mathscr{U}} f = f_1(a)$. In order to prove this we need the fact that the filter base $\{U - TU: U \in \mathscr{U}\}$ converges weakly to 0. To see this let $\varepsilon > 0$ be given along with a function $g \in G$. Choose $F = F(\varepsilon, g)$ in $\mathscr{F}$ so that $|(Tv)(g) - v(g)| < \varepsilon$ for all $v \in F(\varepsilon, g)$. Since $\mathscr{F}$ induces a linear topology on G, it contains a $\sigma(G^*, G)$-bounded set and consequently $\mathscr{U}$ is a Cauchy filter. Thus there is a set $U_1 \in \mathscr{U}$ such that $|u(g) - v(g)| < \varepsilon$ for all pairs $u, v \in U_1$. Choose $U_2 \in \mathscr{U}$ such that $U_2 \in \mathscr{F}$ and let $U = U_1 \cap U_2$. If u and v belong to U,

$$\begin{aligned} |(Tu)(g) - v(g)| &= |(Tu)(g) - u(g) + u(g) - v(g)| \\ &\leq |(Tu)(g) - u(g)| + |u(g) - v(g)| \\ &\leq 2\varepsilon. \end{aligned}$$

This establishes the fact that the filter base $\{U - TU: U \in \mathscr{U}\}$ converges weakly to 0. It follows that TU converges weakly to a. Since TU is, by (ii), eventually in some multiple of A, the $\sigma(G', G)$-continuity of f_1 on A implies that $\lim_{\mathscr{U}} f = \lim_{T\mathscr{U}} f_1 = f_1(a)$. Thus $f \in F(S, \tau, \mathscr{F})$ and, in fact, $\bar{f}(a) = f_1(a)$ for each $a \in A$. We can now invoke Theorem 1.3 to assert that $f_n \to f$ on $\mathscr{F}$. Since $C(X, G, \mathscr{F})$ is closed, $f \in C(X, G, \mathscr{F})$. ∎

We close this section with an example of a vector space X and a function space $G \subset X^*$ for which no complete $H \subset X^*$ can be found containing G as a subspace.

3.3 *Example.* Let S be the set of positive integers and let $G_1 = G_1(S)$ be the space of finite (complex) sequences. Let $X = \langle e(S)\rangle \subset G_1^*$ and regard G_1 as a space of linear functionals on X. Let G_2 denote a subspace of X^* complementary to G_1. We will define a norm on $G = X^*$ such that the resulting normed space is incomplete. Let $\| \ \|_1$ denote the sup norm on G_1, i.e., $\|g\|_1 = \sup_{n\in S} |g(n)|$. Let $\| \ \|_2$ denote an arbitrary norm of G_2; such a norm can be obtained via a Hamel basis argument. For $g \in X^*$ define $\|g\| = \|g_1\|_1 + \|g_2\|_2$ where $g = g_1 + g_2$ is the unique decomposition of g in $X^* = G_1 \oplus G_2$. The Cauchy sequence

$$h_n = (1, 1/2, 1/3, \ldots, 1/n, 0, 0, \ldots)$$

has no limit in $(X^*, \| \ \|)$. To see this, suppose $h_n \to h$ in X^*. Let $h = f + g$ where $f \in G_1$, $g \in G_2$. Since each h_n belongs to G_1 and $\{h_n\}$ does not converge in G_1, there exists an $\varepsilon > 0$ such that $\limsup \|h_n - f\|_1 > \varepsilon$. Now $h_n - h = h_n - f - g$. Hence

$$\limsup \|h_n - h\| = \limsup (\|h_n - f\|_1 + \|g\|_2) > \varepsilon,$$

a contradiction.

4. An application

Let $G = C[0, 1]$ and let $X = \langle e([0, 1])\rangle \subset G^*$ where $e\colon [0, 1] \to G^*$ denotes the evaluation map. As usual, we regard G as a space of linear forms on X. In this section we shall construct, for each real p, $1 \le p < \infty$, a filter $\mathscr{F}_p$ in X inducing the L_p-topology on $C[0, 1]$. We shall then show that the spaces $C(X, G, \mathscr{F}_p)$ are complete, and thus L_p-completions of $C[0, 1]$.

We will denote the conjugate index of p by p'; that is, $p' = p/(p-1)$ if $p \in (1, \infty)$ and $p' = \infty$ if $p = 1$. The L_p-norm on $C[0, 1]$ is denoted by $\| \ \|_p$.

4.1 Definition. Let $p \in (1, \infty)$; $\mathscr{F}_p$ is the filter in X with base

$$\{D_k\colon k = 1, 2, \ldots\}$$

where $D_k \subset X$ is defined by

$$D_k = \left\{\frac{1}{n}\sum_{i=1}^{n} a_{i,n} e(t_{i,n})\colon n \ge k,\ a_{1,n}, a_{2,n}, \ldots, a_{n,n} \text{ scalars such that } \sum_{i=1}^{n} |a_{i,n}|^{p'} \le n,\ t_{i,n} = \frac{i-1}{n} + \frac{1}{2n}\right\}.$$

To obtain $\mathscr{F}_1$ we replace the above condition on the scalars $a_{i,n}$ by $\sup_{1\le i\le n} |a_{i,n}| \le 1$.

4.2 *Remark.* Note that if

$$z = \frac{1}{n}\sum_{i=1}^{n} a_{i,n} e(t_{i,n})$$

and θ_z denotes the step function defined by

$$\theta_z: x \to a_{i,n}, \quad x \in \left[\frac{i-1}{n}, \frac{i}{n}\right), i = 1, 2, \ldots, n,$$

then

$$\|\theta_z\|_{p'} = \left(\int_0^1 |\theta_z(x)|^{p'} dx\right)^{1/p'} = \left(\frac{1}{n} \sum_{i=1} |a_{i,n}|^{p'}\right)^{1/p'}.$$

In particular, if $z \in D_k$ then $\|\theta_z\|_{p'} \leq 1$. The mapping $\theta: z \to \theta_z$ is clearly linear. It is also clear that if g is any function on [0, 1] which is constant on each interval $[(i - 1/n), (i/n))$, $i = 1, 2, \ldots, n$ and satisfies $\|g\|_{p'} \leq 1$, then

$$g = \theta_z \quad \text{for } z = \frac{1}{n} \sum_{i=1}^{n} g(t_{i,n})e(t_{i,n}).$$

In this case $\sum_{i=1}^n |g(t_{i,n})|^{p'} \leq n$.

Let $z = (1/n) \sum_{i=1}^n a_{i,n} e(t_{i,n})$ and let $f \in G$. Then $\langle f, z\rangle$ will denote the scalar

$$\frac{1}{n} \sum_{i=1}^{n} a_{i,n} f(t_{i,n}).$$

4.3 LEMMA. *Let $p' \in (0, \infty]$ and let $f \in C[0, 1]$, $\varepsilon > 0$. Suppose that $N \in \mathbf{N}$ is chosen so that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < 1/N$. Then*

$$\left|\int_0^1 f(x)\theta_z(x)\, dx - \langle f, z\rangle\right| < \varepsilon$$

whenever $n \geq N$ and $z = \sum_{i=1}^n a_{i,n} e(t_{i,n})$ belongs to D_n.

Proof. Let I_i denote the interval $[(i - 1)/n, i/n)$, $i = 1, 2, \ldots, n$.

$$\begin{aligned}\left|\int_0^1 f(x)\theta_z(x)\, dx - \langle f, z\rangle\right| &= \left|\sum_{i=1}^{n} \int_{I_i} (f(x) - f(t_{i,n}))a_{i,n}\, dx\right| \\ &\leq \left|\sum_{i=1}^{n} \frac{1}{n} \varepsilon a_{i,n}\right| \leq \frac{1}{n} \varepsilon \sum_{i=1}^{n} |a_{i,n}| \\ &\leq \varepsilon,\end{aligned}$$

The last inequality is obtained from $\sum_{i=1}^n |a_{i,n}|^{p'} \leq n$ by means of Jensen's inequality [6, p. 202, 3.34]. ∎

4.4 LEMMA. *Let $p \in [1, \infty)$. The $\mathscr{F}_p$ and L_p topologies coincide on $C[0, 1]$.*

Proof. Let $\varepsilon > 0$ be arbitrary, and let U denote the set

$$\{f \in C[0, 1]: \|f\|_p \leq 1\}.$$

We first show that $\frac{1}{2}\varepsilon U \subset U(\varepsilon, \mathscr{F}_p)$, that is, that the L_p-topology on $C[0, 1]$ is stronger than the $\mathscr{F}_p$-topology. Let f belong to $\frac{1}{2}\varepsilon U$. Choose N sufficiently large that $|x-y| < 1/N$ implies $|f(x)-f(y)| < \varepsilon/2$. Let $n \geq N$ and let $z \in D_n(p')$. By Lemma 4.3, $|\int_0^1 f(x)\theta_z(x)\,dx - \langle f, z\rangle| < \varepsilon/2$. It follows that

$$|\langle f, z\rangle| < \left|\int_0^1 f(x)\theta_z(x)\,dx\right| + \varepsilon/2.$$

By Hölder's inequality, $|\int f(x)\theta_z(x)\,dx| \leq \|f\|_p\|\theta_z\|_{p'} \leq \varepsilon/2$. Thus $f \in U(\varepsilon, \mathscr{F}_p)$. For the other direction suppose that $f \notin \varepsilon U$. We shall show that $f \notin U(\varepsilon, \mathscr{F}_p)$. The case where $p \in (1, \infty)$ is considered first. There exists $\delta > 0$ such that $\|f\|_p > \varepsilon + \delta$. Since $C[0, 1]$ is dense in $L_{p'}[0, 1]$ there is a continuous function g, $\|g\|_{p'} < 1$ such that $\int_0^1 f(x)g(x)\,dx > \varepsilon + \delta$ (Theorem 3.14, page 68 of [10]). Since g is uniformly continuous there is a positive integer N_1 such that whenever $n > N_1$ there exists $z \in D_n(p')$ such that

$$\|g - \theta_z\|_{p'} < (2\|f\|_p)^{-1}\delta.$$

By Lemma 4.3, there exists N_2 sufficiently large such that whenever $n > N_2$ and $z \in D_n(p')$ we have $|\int_0^1 f(x)\theta_z(x)\,dx - \langle f, z\rangle| < \delta/2$. Let $N = \max\{N_1, N_2\}$ and let $n > N$. Choose $z \in D_n(p')$ such that

$$\|g - \theta_z\|_{p'} < (2\|f\|_p)^{-1}\delta.$$

Then

$$\begin{aligned}\left|\int f(x)g(x)\,dx - \langle f, z\rangle\right| &\leq \left|\int_0^1 f(x)g(x)\,dx - \int_0^1 f(x)\theta_z(x)\,dx\right| \\ &\quad + \left|\int_0^1 f(x)\theta_z(x)\,dx - \langle f, z\rangle\right| \\ &\leq \left|\int_0^1 f(x)(g(x)-\theta_z(x))\,dx\right| + \delta/2.\end{aligned}$$

By Hölder's inequality $|\int_0^1 f(x)(g(x)-\theta_z(x))\,dx| \leq \|f\|_p\|g-\theta_z\|_{p'} < \delta/2$. Consequently, $|\langle f, z\rangle| > \varepsilon$. Since every set in $\mathscr{F}_p$ contains a z with this property we see that $f \notin U(\varepsilon, \mathscr{F}_p)$.

We now consider the case $p = 1$. We show that $f \notin \varepsilon U$ implies $f \notin U(\varepsilon, \mathscr{F}_1)$. Choose δ such that $\|f\|_1 > \varepsilon + 2\delta$ and for each positive integer n let f_n denote the function defined by $f_n(x) = f(t_{i,n})$ for $x \in [(i-1)/n, i/n)$ and $i = 1, 2, \ldots, n$. Choose $N \in \mathbf{N}$ such that $|x-y| < 1/N$ implies $|f(x)-f(y)| < \delta/2$. Then whenever $n > N$, it follows that

$$\|f - f_n\|_1 = \int_0^1 |f(x) - f_n(x)|\,dx < \delta/2.$$

We now define, for each $n > N$, a step function g_n:

$$g_n(x) = 1 \quad \text{if } x \in [(i-1)/n, i/n) \text{ and } f_n(x) \geq 0,$$

$$g_n(x) = -1 \quad \text{if } x \in [(i-1)/n, i/n) \text{ and } f_n(x) < 0.$$

Let $z_n = (1/n) \sum_{i=1}^{n} g_n(t_{i,n}) e(t_{i,n})$. It is clear that $\|g\|_\infty = 1$,

$$\int_0^1 f_n(x) g_n(x)\, dx = \int_0^1 |f_n(x)|\, dx,$$

$z \in D_n(1)$ and $\langle f, z_n \rangle = \int_0^1 f_n(x) g_n(x)\, dx = \|f_n\|$. Therefore $\langle f, z_n \rangle = \|f_n\| > \varepsilon + 3\delta/2$. Since each set in $\mathscr{F}_1$ contains at least one such z_n, it follows that $f \notin U(\varepsilon, \mathscr{F}_1)$. ∎

It is now immediate from Lemmas 4.3 and Theorem 3.2 that $C(X, G, \mathscr{F}_p)$ is a completion of $G = (C[0, 1], \| \quad \|_p)$. Indeed, for the map $T: X \to G'$ we may take the map $\theta: z \to \theta_z$; conditions (i) and (ii) of 3.2 follow from Lemma 4.3 and Remark 4.2, respectively. We observe that the filter base described in 4.1 for $\mathscr{F}_1$ does not contain a linearly independent subset of X. However, it is not difficult to modify the sets D_k to obtain a filter $\mathscr{F}$ with the property, and still have the $\mathscr{F}$-topology and L_1-topology coincide on $C[0, 1]$. In this case, we can obtain an L_1-completion of $C[0, 1]$ by using Theorem 3.1.

5. Best filters and optimal $C(X, G, \mathscr{F})$

An examination of the examples of Section 3 shows that it is possible to have many filters which give the same topology on our function spaces. This complicates the situation but makes it more interesting. The space $C(X, G, \mathscr{F})$ will also change as the filter $\mathscr{F}$ changes. This raises questions as to the "best" filter for a given chore and the most suitable $C(X, G, \mathscr{F})$. Such questions will be formalized by placing partial orders on both the collection of filters and the collection of spaces of the form $C(X, G, \mathscr{F})$.

Throughout this section X will be a linear space with G a linear subspace of X^*. The filters which generate topologies on G will always have a base composed of balanced, convex subsets of X. The evaluation map of X into G^* will be denoted by the symbol e. When a linear form x in G^* is evaluated at g in G we will write $g(x)$.

5.1 Definition. Consider the collection of all filters composed of subsets of X with a base of balanced convex sets. Two such filters, $\mathscr{F}_1$ and $\mathscr{F}_2$, will be said to have the relation $\mathscr{F}_1 > \mathscr{F}_2$ when there is a number $r \geq 1$ such that $\mathscr{F}_1$ is a refinement of $r\mathscr{F}_2 = \{rF: F \in \mathscr{F}_2\}$. The filters are said to be equivalent, $\mathscr{F}_1 \approx \mathscr{F}_2$, when $\mathscr{F}_1 > \mathscr{F}_2$ and $\mathscr{F}_2 > \mathscr{F}_1$. The reader should observe that this partial ordering makes the collection of equivalence classes a lattice. The least upper bound of $\mathscr{F}_1$ and $\mathscr{F}_2$ is denoted by $\mathscr{F}_1 \vee \mathscr{F}_2$ and the greatest lower bound by $\mathscr{F}_1 \wedge \mathscr{F}_2$. One representative of the class $\mathscr{F}_1 \vee \mathscr{F}_2$ is $\mathscr{F}_1 \cap \mathscr{F}_2$. To obtain a representative for $\mathscr{F}_1 \wedge \mathscr{F}_2$ we can take the filter with base

$$\{\Gamma(F_1 \cup F_2) \mid F_1 \in \mathscr{F}_1, F_2 \in \mathscr{F}_2\}.$$

We now look for a greatest lower bound of all filters giving a fixed locally convex topology on G. A candidate is defined below.

5.2 DEFINITION. A minimal filter $\mathscr{M}$ in X relative to the function space G is determined by a subset A of G^* which is balanced, convex, $\sigma(G^*, G)$-closed and bounded. The subbasis for $\mathscr{M}$ is all sets of the form $\{x \in X \mid |g(x)| \leq a\}$ where $g \in G$ and $a > 0$ satisfy $|g(x)| < a$ for all $x \in A$. There is no need to mention A explicitly in the notation because $A = \bigcap \{\overline{e(M)} \mid M \in \mathscr{M}\}$.

5.3 THEOREM. *Let $\mathfrak{S}$ be a collection of balanced, convex, $\sigma(G^*, G)$-closed and bounded subsets of G^*. Let Φ be all minimal filters that can be constructed from members of $\mathfrak{S}$ by means of Definition 5.2. Then the $\mathfrak{S}$-topology and the Φ-topology coincide on G.*

Proof. By Proposition 1.6, the Φ-topology is uniform convergence on the collection

$$\{\{\bigcap \overline{e(M)}: M \in \mathscr{M}\} \mid \mathscr{M} \in \Phi\}.$$

This collection coincides with $\mathfrak{S}$. We see this by letting A be a member of $\mathfrak{S}$ and $\mathscr{M}$ the associated minimal filter. Observe that

$$\bigcap \{\overline{e(M)} \mid M \in \mathscr{M}\}$$
$$= \bigcap \{\{x \in G^* \mid |g(x)| \leq a\} \mid g \in G \text{ and } a > 0 \text{ satisfy } |g(x)| < a \text{ for all } x \in A\}.$$

This holds because M may be considered to be of the form

$$M = \{e^{-1}(x) \mid x \in G^* \text{ and } |g(x)| < a\},$$

and $e(M)$ is dense in $\{x \in G^* \mid |g(x)| < a\}$ for the $\sigma(G^*, G)$-topology. A corollary to the Separation Theorem (p. 119 of [8]) asserts that the above intersections are the desired set A. ∎

5.4 COROLLARY [4, Theorem 1, p. 711]. *Every locally convex topology on G can be represented as convergence on a family of filters composed of subsets of X.*

The proof follows from 5.3 by taking $\mathfrak{S}$ to be the collection of balanced, convex and weakly closed equicontinuous subsets of the dual.

5.5 LEMMA. (i) *Let $\mathscr{F}$ be a filter in X giving a pseudo-norm on G. Then there is a minimal filter with the same pseudo-norm.*

(ii) *If $\mathscr{M}$ is a minimal filter relative to G and $\mathscr{F}$ gives a smaller or equal pseudo-norm, then $\mathscr{F}$ is a refinement of $\mathscr{M}$.*

Proof. Consider a minimal filter $\mathscr{M}$ relative to G. The proof of 5.3 showed that $\bigcap \{\overline{e(M)} \mid M \in \mathscr{M}\}$ is the set A used in the definition of the minimal filter. Part (i) is obtained by constructing the minimal filter which uses $\bigcap \{\overline{e(F)} \mid F \in \mathscr{F}\}$ as the set A of Definition 5.2. The pseudo-norm p for both the filter $\mathscr{M}$ and $\mathscr{F}$ is $p(f) = \sup \{|f(x)| \mid x \in A\}$ for all f in G. This is the same as $p(f) = \lim \sup_{\mathscr{F}} |f| = \lim \sup_{\mathscr{M}} |f|$. For part (ii) we note that the set A used in the definition of $\mathscr{M}$ must have $\bigcap \{\overline{e(F)}: F \in \mathscr{F}\}$ as a subset, because of our assumption on the relative sizes of the norms. Assume $\mathscr{F}$ is not a refinement of $\mathscr{M}$. Then there is an $M \in \mathscr{M}$ such that $F \backslash M$ is not empty for all F in $\mathscr{F}$. We may assume that we have a $g \in G$ and $a > 0$ such that $M = \{x \in X \mid |g(x)| \leq a\}$ where $g(x) < a$ for all $x \in A$. Let $\mathscr{U}$ be an ultrafilter refining $\mathscr{F}$ and containing a set U such that $U \cap M = \emptyset$. Thus $|g(x)| > a$ for any $x \in U$, but $e(\mathscr{U})$ will converge to a point of $\bigcap \{\overline{e(F)} \mid F \in \mathscr{F}\}$ which is also a point of A. This contradicts the requirement that $|g(x)| < a$ for any $x \in A$. ∎

The justification for the definition of minimal filters is in the following proposition.

5.6 PROPOSITION. *Let $\mathscr{M}$ be a minimal filter relative to G.*

(i) *If the filter $\mathscr{F}$ gives a topology on G weaker than the $\mathscr{M}$-topology then $\mathscr{F} > \mathscr{M}$.*

(ii) *The minimal filter is the greatest lower bound of all filters giving the same topology on G.*

Proof. It suffices to consider statement (i). There is a number $r \geq 1$ such that r times the pseudo-norm for $\mathscr{M}$ gives a larger pseudo-norm than the one obtained from $\mathscr{F}$. This is because of the relative strength of the topologies. The new pseudo-norm is related to the filter $r\mathscr{M} = \{rM \mid M \in \mathscr{M}\}$. Note that $r\mathscr{M} \approx \mathscr{M}$. Lemma 5.5 says that $\mathscr{F}$ is a refinement of $r\mathscr{M}$, in other words, $\mathscr{F} > r\mathscr{M} \approx \mathscr{M}$. Thus $\mathscr{F} > \mathscr{M}$. ∎

5.7 LEMMA. *Consider a minimal filter $\mathscr{M}$ relative to G and a filter $\mathscr{F}$ in X whose topology on G is stronger than the $\mathscr{M}$-topology. The $\mathscr{F}$-topology need not be linear on G. Let Φ be the collection of all filters larger than $\mathscr{F}$ and giving the $\mathscr{M}$-topology on G. Then Φ contains its greatest lower bound. The g.l.b. may be taken to be a refinement of $\mathscr{M}$ giving the same pseudo-norm.*

5.8 COROLLARY. *If $\mathscr{F}$ and $\mathscr{M}$ give the same topology on G, then there is a refinement of $\mathscr{M}$ which is equivalent to $\mathscr{F}$ and has the same pseudo-norm as $\mathscr{M}$.*

Proof of 5.7. It is not hard to conjecture that the desired filter is equivalent to $\mathscr{F} \vee \mathscr{M}$. We will proceed to show this.

Observe that a fundamental system of neighborhoods of zero for the $\mathscr{F}$-

topology is all scalar multiples of the set

$$S = \{f \in G \mid \text{for every } \varepsilon > 0 \text{ there exists } F_{\varepsilon,f} \in \mathscr{F} \text{ such that } |f(x)| < 1 + \varepsilon \text{ for all } x \in F_{\varepsilon,f}\}.$$

Assuming the natural duality between G and G^*, we consider the polar S^0 of S in G^*. We shall show that $S^0 = \bigcap \{\overline{e(F)} \mid F \in \mathscr{F}\}$, and $S^{00} = S$. Let $x_0 \in S^0$ be arbitrary, and assume $x_0 \notin \bigcap \{\overline{e(F)} \mid F \in \mathscr{F}\}$, so that there exists a convex balanced set $F \in \mathscr{F}$ such that $x_0 \notin \overline{e(F)}$. By the Hahn-Banach theorem, there exists $f \in G$ such that $|f(x)| \leq 1$ for all $x \in \overline{e(F)}$, and $|f(x_0)| > 1$. This is a contradiction, and it follows that S^0 is a subset of the intersection. Now let $x_0 \in \bigcap \{\overline{e(F)} \mid F \in \mathscr{F}\}$ be arbitrary. Then for every f in S and $\varepsilon > 0$, $|f(x_0)| \leq 1 + \varepsilon$, because there exists $F \in \mathscr{F}$ with $|f(x)| < 1 + \varepsilon$ for all $x \in \overline{e(F)}$. Since ε is arbitrary, we have $|f(x_0)| \leq 1$. Thus $x_0 \in S^0$ and $S^0 = \bigcap \{\overline{e(F)}: F \in \mathscr{F}\}$. It remains to show that $S = S^{00}$. This will follow from the bipolar theorem (page 192 of [7]) once we have shown that S is $\sigma(G^*, G)$-closed. Let

$$Q = \{f \in G \mid \text{there exists } F_f \in \mathscr{F} \text{ such that } |f(x)| < 1 \text{ for all } x \in F_f\};$$

note that $S = \bigcap \{rQ \mid r > 1\}$. Thus S is the closure of Q for the strongest locally convex topology on G (Exercise F, page 125 of [8]). Since $G^* = G'$ for this topology we know that the convex set S is also $\sigma(G, G^*)$-closed.

Let A be the balanced, convex, $\sigma(G^*, G)$-closed and bounded subset of G^* related to our minimal filter as in Definition 5.2. A fundamental system of neighborhoods for the $\mathscr{M}$-topology is all scalar multiples of A^0. Since the $\mathscr{F}$-topology is stronger than the $\mathscr{M}$-topology there exists $r \geq 1$ such that $A^0 \supset (1/r)S$; consequently,

$$A \subset rS^0 = \bigcap \{\overline{e(F)} \mid F \in r\mathscr{F}\}.$$

Let $\mathscr{F}_0$ be the filter with subbase $\{F \cap M \mid F \in r\mathscr{F},\ M \in \mathscr{M}\}$. Note that $\mathscr{F}_0 \approx \mathscr{F} \vee \mathscr{M}$. We will now show that the $\mathscr{F}_0$-topology coincides on G with the $\mathscr{M}$-topology, thus exhibiting $\mathscr{F}_0$ as the filter in Φ we are seeking. We must show that $\overline{e(F \cap M)} \supset A$ for all F in $r\mathscr{F}$ and M in $\mathscr{M}$. The definition of $\mathscr{M}$ allows us to assume that $\overline{e(M)}$ contains A in its interior and

$$e(M) = e(X) \cap \overline{e(M)}^{\sigma(G^*,G)}.$$

Let $x_0 \in A$ be arbitrary and let V be a $\sigma(G^*, G)$-neighborhood of x_0. The set $V \cap \overline{e(M)}$ is also a neighborhood of x_0. Since $\overline{e(F)} \supset A$, there exists an x in F such that $e(x) \in V \cap \overline{e(M)}$. Thus

$$e(x) \in e(F) \cap \overline{e(M)} = e(F) \cap e(X) \cap \overline{e(M)} = e(F) \cap e(M).$$

It follows that $\overline{e(F \cap M)} \supset A$. ∎

The proof of the corollary results from showing that $\mathscr{F}_0$ is equivalent to $\mathscr{F}$ when the $\mathscr{F}$ and $\mathscr{M}$-topologies coincide. By Proposition 5.6, $\mathscr{F} > \mathscr{M}$. Thus $\mathscr{F} \approx \mathscr{F} \vee \mathscr{M} \approx r\mathscr{F} \vee \mathscr{M} = \mathscr{F}_0$. ∎

We now look at spaces of type $C(X, G, \mathscr{F})$. They are of major concern because of the important role they play in our generalization of Grothendieck's completion theorem (2.3). Such spaces depend on the choice of $\mathscr{F}$. As we consider refinements of $\mathscr{F}$ the function space will become larger. This enlarging can be effected without changing the topology on G. We again desire the "best" $C(X, G, \mathscr{F})$ for each particular purpose. The following partial order is placed on the spaces.

5.9 DEFINITION. Consider a fixed X and G. We define $C(X, G, \mathscr{F}_1) > C(X, G, \mathscr{F}_2)$ when the second space is a linear topological subspace of the first.

The following lemma presents much of the basic structure. The space $B(X, \mathscr{F})$ was defined in Proposition 1.4.

5.10 LEMMA. (i) *If $\mathscr{M}$ is a minimal filter relative to G then* $(G, \mathscr{M}) = C(X, G, \mathscr{M}) = B(X, \mathscr{M})$.

(ii) *If $\mathscr{F}_1 \approx \mathscr{F}_2$, then $C(X, G, \mathscr{F}_1) = C(X, G, \mathscr{F}_2)$, and $B(X, \mathscr{F}_1) = B(X, \mathscr{F}_2)$.*

(iii) *If $(G, \mathscr{F}_1) = (G, \mathscr{F}_2)$ and $\mathscr{F}_1 > \mathscr{F}_2$, then $C(X, G, \mathscr{F}_1) > C(X, G, \mathscr{F}_2)$.*

(iv) *Consider $(G, \mathscr{F}_1) = (G, \mathscr{F}_2)$ with $\mathscr{F}_2 > \mathscr{F}_1$. Let Φ be the collection of all filters $\mathscr{F}$ such that $\mathscr{F} > \mathscr{F}_1$ and $C(X, G, \mathscr{F}_2) = C(X, G, \mathscr{F})$. Then Φ contains its unique (up to equivalence) greatest lower bound.*

(v) *Consider $(G, \mathscr{F}_1) = (G, \mathscr{F}_2) = (G, \mathscr{F}_3)$ with $\mathscr{F}_2 > \mathscr{F}_1$ and $\mathscr{F}_3 > \mathscr{F}_1$. If $C(X, G, \mathscr{F}_3) > C(X, G, \mathscr{F}_2)$ and $\mathscr{F}_3$, $\mathscr{F}_2$ are the g.l.b.'s constructed in the proof of* (iv) *relative to $\mathscr{F}_1$, then $\mathscr{F}_3$ is a refinement of $\mathscr{F}_2$ giving the same pseudo-norm on $C(X, G, \mathscr{F}_2)$.*

Proof. (i) It is sufficient to show that $(G, \mathscr{M}) \supset B(X, \mathscr{M})$. Consider an arbitrary f_0 in $B(X, \mathscr{M})$. There is an M in $\mathscr{M}$ on which f_0 is bounded. We may assume that $M = \{x \in X \mid |f(x)| \leq a\}$ for some f in G and number $a > 0$, or that M is a finite intersection $\bigcap_{i=1}^{n} \{x \in X \mid |f_i(x) \leq a_1\}$ of such sets. The kernel of f_0 contains the intersection of the kernels of the f_i's which determine M, because f_0 is bounded on M. We conclude that f_0 is a linear combination of the f_i's (see Lemma, page 186 of [7]).

(ii) The verification is left to the reader.

(iii) The construction of the spaces is such as to make $C(X, G, \mathscr{F}_2)$ a subset of $C(X, G, \mathscr{F}_1)$ with the $\mathscr{F}_2$-topology on the first set being stronger than the $\mathscr{F}_1$-topology. Recall that both topologies agree on the dense subspace G. We can obtain a fundamental system of neighborhoods at zero in $C(X, G, \mathscr{F}_2)$ for either topology by taking the respective closures of the closed, balanced, convex neighborhoods of zero in G (see p. 134 of [7]). The closures are the same because the weak topology is unchanged regardless of which topology is considered, and our sets are convex. Thus the $\mathscr{F}_2$-topology and $\mathscr{F}_1$-topology are the same on $C(X, G, \mathscr{F}_2)$ making it a linear topological subspace of

$C(X, G, \mathscr{F}_1)$. It is concluded that $C(X, G, \mathscr{F}_1) > C(X, G, \mathscr{F}_2)$ according to Definition 5.9.

(iv) We first replace $\mathscr{F}_2$ by an equivalent filter giving the same pseudo-norm on G as $\mathscr{F}_1$. To do this we let $\mathscr{M}_1$ be the minimal filter relative to G giving the $\mathscr{F}_1$-pseudo-norm (5.5(i)). By 5.8 we can replace $\mathscr{F}_2$ by an equivalent filter which is a refinement of $\mathscr{M}_1$ and gives the same pseudo-norm. The symbol $\mathscr{F}_2$ will now denote this new filter.

Let $\mathscr{M}_2$ be the minimal filter relative to $C(X, G, \mathscr{F}_2)$ giving the $\mathscr{F}_2$-pseudo-norm (5.5(i)). Let Φ' be the collection of all filters larger than $\mathscr{F}_1$ which give the $\mathscr{F}_2$-topology on $C(X, G, \mathscr{F}_2)$. Lemma 5.7 says that Φ' contains its g.l.b. $\mathscr{F}_0$. The filter $\mathscr{F}_0$ can be taken as a refinement of $\mathscr{F}_2$ with the same pseudo-norm. We now have $\mathscr{F}_2 > \mathscr{F}_0 > \mathscr{M}_2$.

The collection Φ' contains the Φ of statement (iv). We must now show that $\mathscr{F}_0$ is in Φ in order to complete the proof. We let $\tilde{G}$ denote the function space $C(X, G, \mathscr{F}_2)$. From Theorem 2.3 and part (i) we can conclude that the following three spaces are identical: $C(X, G, \mathscr{F}_2)$, $C(X, \tilde{G}, \mathscr{F}_2)$ and $C(X, \tilde{G}, \mathscr{M}_2)$. The ordering of the filters and (iii) yield $C(X, \tilde{G}, \mathscr{F}_2) > C(X, \tilde{G}, \mathscr{F}_0) > C(X, \tilde{G}, \mathscr{F}_2)$. Thus $C(X, \tilde{G}, \mathscr{F}_0) = C(X, G, \mathscr{F}_2)$. By using Theorem 2.3 again we obtain $C(X, \tilde{G}, \mathscr{F}_0) = C(X, G, \mathscr{F}_0)$ because G is dense in the first space. It follows that $C(X, G, \mathscr{F}_0) = C(X, G, \mathscr{F}_2)$ and $\mathscr{F}_0$ is the g.l.b. of Φ as well as of Φ'.

(v) Assume that $\mathscr{F}_2$ and $\mathscr{F}_3$ have been obtained relative to $\mathscr{F}_1$ in the manner we constructed the $\mathscr{F}_0$ of part (iv). Thus $\mathscr{F}_2 = \mathscr{F}_1 \vee \mathscr{M}_2$ and $\mathscr{F}_3 = \mathscr{F}_1 \vee \mathscr{M}_3$ where $\mathscr{M}_i$ $(i = 2, 3)$ is the minimal filter relative to $C(X, G, \mathscr{F}_i)$ giving the $\mathscr{F}_i$-pseudo-norm on G. Our hypothesis tells us that $C(X, G, \mathscr{F}_2)$ is a linear topological subspace of $C(X, G, \mathscr{F}_3)$. The space $(G, \mathscr{F}_1)$ is dense in both spaces with all three related pseudo-norms agreeing on G. Thus $\mathscr{M}_3$ gives the same pseudo-norm on $C(X, G, \mathscr{F}_2)$ as $\mathscr{M}_2$. We can make use of 5.5 to conclude that $\mathscr{M}_3$ is a refinement of $\mathscr{M}_2$. Thus $\mathscr{F}_3 = \mathscr{F}_1 \vee \mathscr{M}_3$ is a refinement of $\mathscr{F}_2 = \mathscr{F}_2 \vee \mathscr{M}_2$. ∎

In Section 3 we were concerned with whether or not $C(X, G, \mathscr{F})$ was complete. We can broaden the question by letting $\mathscr{F}'$ be a refinement of $\mathscr{F}$ and ask when $C(X, G, \mathscr{F}')$ is complete. We now give a definition of a function space completion which will serve our purpose.

5.11 DEFINITION. Consider $(H, \mathscr{G})$ and $(G, \mathscr{F})$ as two subspaces of X^* with the $\mathscr{G}$-topology and $\mathscr{F}$-topology respectively. We call $(H, \mathscr{G})$ a completion of $(G, \mathscr{F})$ by functions when

(i) $(H, \mathscr{G})$ is complete,
(ii) $(G, \mathscr{F})$ is a dense subspace of $(H, \mathscr{G})$ and
(iii) $\mathscr{G} > \mathscr{F}$.

The definition can be generalized to cover the case where the topology is obtained from a family of filters.

We now establish the relationship of the usual notion of completion to the present one.

5.12 THEOREM. *Consider a locally convex space* $(G, \mathscr{F})$. *Assume* $(\hat{G}, p)$ *is a subspace of* X^* *which is complete for the pseudo-norm* p *and has* $(G, \mathscr{F})$ *as a dense subspace. The following statements are equivalent.*

(i) *There exists a filter* $\mathscr{F}' > \mathscr{F}$ *such that* $(\hat{G}, p) = (\hat{G}, \mathscr{F}')$ *is a completion of* $(G, \mathscr{F})$ *by functions.*

(ii) *The* $\mathscr{F}$*-topology is stronger than the* p*-topology on* $(\hat{G}, p) \cap B(X, \mathscr{F})$.

(iii) *There exists a positive real number* K *such that*

$$\limsup_{\mathscr{F}} |f(x)| \geq Kp(f)$$

for all f *in* $(\hat{G}, p)$.

(iv) *There is a filter* $\mathscr{F}' > \mathscr{F}$ *such that* $C(X, G, \mathscr{F}')$ *is a completion of* $(G, \mathscr{F})$ *by functions with* $(\hat{G}, p)$ *as a dense subspace.*

In statements (i) *and* (iv), $\mathscr{F}'$ *may be taken as the g.l.b. of the filters with the desired property. In statement* (iv), $C(X, G, \mathscr{F}')$ *may be taken as the g.l.b. of such function spaces. All filters (except* $\mathscr{F}$*) can be chosen to give the* p*-pseudo-norm on* $(\hat{G}, p)$.

Proof. We start with statement (iv) and construct a $C(X, G, \mathscr{F}_0)$ which is the required g.l.b. of such spaces. The filter $\mathscr{F}_0$ will be the g.l.b. of usable filters.

Let $\mathscr{M}$ be a minimal filter relative to $(\hat{G}, p)$ giving the p-topology (5.4 and 5.6). Let $\mathscr{F}_0 = \mathscr{F} \vee \mathscr{M}$. Note that $\mathscr{F} < \mathscr{F}'$, and thus $\mathscr{M} < \mathscr{F}_0 < \mathscr{F}'$, where $\mathscr{F}'$ is as stated in (iv). This tells us that the $\mathscr{F}_0$-topology on $\hat{G}$ is the p-topology. Recall that G is a dense subspace for this topology. We apply Theorem 2.3 to see that $C(X, \hat{G}, \mathscr{F}_0) = C(X, G, \mathscr{F}_0)$.

Let us assume that $C(X, G, \mathscr{F}'')$ is another space with the property of statement (iv). We use Proposition 5.6 to note that $\mathscr{F}'' > \mathscr{M}$ because both give the same topology on $\hat{G}$. Since $\mathscr{F}'' > \mathscr{F}$, $\mathscr{F}'' > \mathscr{F}_0 \vee \mathscr{M}$. Both filters give the same topology on G, thus $C(X, G, \mathscr{F}'') > C(X, G, \mathscr{F}_0)$ by part (iii) of Lemma 5.10.

(i)$\leftrightarrow$(iv) Since $(G, \mathscr{F}')$ is dense in $(\hat{G}, \mathscr{F}')$, their closures in $(X^*, \mathscr{F}')$, $C(X, G, \mathscr{F}')$ and $C(X, \hat{G}, \mathscr{F}')$, coincide. The implication (iv)$\rightarrow$(i) is immediate.

(iv)$\rightarrow$(ii) Since $(\hat{G}, p)$ is a subspace of $C(X, G, \mathscr{F}')$, it is a subspace of $B(X, \mathscr{F}')$, with the $\mathscr{F}'$-topology giving the p-topology on $(\hat{G}, p)$. The set $B(X, \mathscr{F})$ is a linear subspace of $B(X, \mathscr{F}')$ on which the $\mathscr{F}$-topology is stronger than the $\mathscr{F}'$-topology because $\mathscr{F}' > \mathscr{F}$. Thus the $\mathscr{F}$-topology on $(\hat{G}, p) \cap B(X, \mathscr{F})$ is stronger than the p-topology.

(ii)$\rightarrow$(iv) Consider a filter $\mathscr{I}$ in $\hat{G}$ converging to f_0 in $\hat{G}$ on the filter $\mathscr{F}$. The filter $\mathscr{I} - f_0$ converges to the zero function on $\mathscr{F}$ and thus is eventually in $B(X, \mathscr{F})$, by definition of convergence on a filter. Thus $\mathscr{I} - f_0$ converges to 0 in the p-topology on $B(X, \mathscr{I}) \cap (\hat{G}, p)$, and it follows that $\mathscr{I}$ converges to f_0 in the p-topology. We conclude that the $\mathscr{F}$-topology on $(\hat{G}, p)$ is stronger than the

p-topology. Let $\mathscr{M}$ be the minimal filter relative to $(\hat{G}, p)$ giving the p-topology. Applying Lemma 5.7 we obtain a filter $\mathscr{F}'$ which is the g.l.b. of all filters larger than $\mathscr{F}$ and $\mathscr{M}$ giving the p-topology on $\hat{G}$.

(ii)$\leftrightarrow$(iii) Recall that the pseudo-norm q related to the $\mathscr{F}$-topology is given by

$$q(f) = \limsup_{\mathscr{F}} |f(x)|, f \in B(X, \mathscr{F}).$$

Since this pseudo-norm gives a stronger topology than the pseudo-norm p, we know there is a positive number K such that

$$\limsup_{\mathscr{F}} |f(x)| > Kp(f), \quad f \in (\hat{G}, p) \cap B(X, \mathscr{F}).$$

If $f \in (\hat{G}, p)$ but $f \notin B(X, \mathscr{F})$, the inequality holds because $\limsup_{\mathscr{F}} |f(x)| = \infty$. The implication (iii)$\rightarrow$(ii) is immediate. ∎

5.13 *Example.* Here is an example where 5.12 is not applicable. The filter $\mathscr{F}$ in the example cannot be refined to obtain a completion, although a very sensible function completion exists.

Let G denote the space of polynomials on $[0, 1]$ with the usual supremum norm $\| \quad \|$. This space clearly has a function completion, for example, $C[0, 1]$. Let $X = \langle e[0, 1]\rangle \subset G^*$ and regard G as a space of linear forms on X. Let G_1 be an algebraic complement to G in X^*, so that $X^* = G \oplus G_1$ is a direct sum. Place on G_1 the trivial norm q so that $q(h) = 0$ for all $h \in G_1$. Define a pseudo-norm $||| \quad |||$ on each $f \in X^*$ by setting

$$|||f||| = \|g\| + q(h) = \|g\|$$

where $f = g + h$ is the unique decomposition of f as an element of $G \oplus G_1$. Let $\mathscr{F}$ be the minimal filter relative to X^* for this topology (Proposition 5.6). We will show that there is no refinement $\mathscr{F}'$ of $\mathscr{F}$ for which $C(X, G, \mathscr{F}')$ is complete.

Consider an arbitrary refinement $\mathscr{F}'$ of $\mathscr{F}$ which induces the same topology on G. Note that G is dense in X^* for the $\mathscr{F}'$-topology since it is dense for the stronger $\mathscr{F}$-topology. It follows that the $\mathscr{F}$ and $\mathscr{F}'$ topologies coincide on X^*. Let $\{g_n\}$ be a sequence of polynomials in G which doesn't have a limit in G for the supremum norm. Suppose there exists f in X^* such that $g_n \to f$ in the $\mathscr{F}'$-topology. Thus $f = g + h$, $g \in G$, $h \in G_1$ and

$$\begin{aligned} 0 &= \lim_n |||g_n - f||| \\ &= \lim_n |||g_n - g - h||| \\ &= \lim_n \|g_n - g\| + q(h) \\ &= \lim_n \|g_n - g\|. \end{aligned}$$

This is a contradiction.

In looking for a completion of $(G, \mathscr{F})$ by functions we must consider refinements of $\mathscr{F}$. If there is a completion of any usable kind, Theorem 5.12 asserts that it is obtained in this manner. Note that $C(X, G, \mathscr{F}'')$ is complete whenever $\mathscr{F}''$ refines an $\mathscr{F}'$ for which $C(X, G, \mathscr{F}')$ is complete. This is because $C(X, G, \mathscr{F}')$ is a dense subspace of $C(X, G, \mathscr{F}'')$ (Lemma 5.10 (ii) and Theorem 2.2) and a linear space is complete when it has a complete dense subspace.

The above remarks raise the question of how large a space of type $C(X, G, \mathscr{F}'')$ can become. If there were maximal ones and we wanted to investigate the existence of a completion, it would be best to look at these first. In fact, these maximal spaces do exist.

5.14 THEOREM. *Consider* $(G, \mathscr{F}_1)$ *and let* θ *be the collection of all spaces* $C(X, G, \mathscr{F})$ *such that* $\mathscr{F} > \mathscr{F}_1$ *and* $(G, \mathscr{F}) = (G, \mathscr{F}_1)$. *If* θ *is given the partial order of Definition* 5.9, *then it contains a maximal element.*

Proof. Let θ' denote a linearly ordered subset of θ. For each $C(X, G, \mathscr{F})$ in θ', replace the filter $\mathscr{F}$ in $C(X, G, \mathscr{F})$ by the filter whose existence is established in Lemma 5.10, (iv) and (v). Observe that the filter $\mathscr{F}_1$ in the present theorem plays the same role as the filter $\mathscr{F}_1$ of the lemma. The new filters are linearly ordered by set inclusion (Lemma 5.10 (v)). Let $\mathscr{F}_0$ be the filter containing all the new filters related to members of θ'. Then $\mathscr{F}_0$ refines each of these filters, and furthermore $(G, \mathscr{F}_1) = (G, \mathscr{F}_0) = (G, \mathscr{F})$ for each $\mathscr{F}$ related to a member of θ. Thus $C(X, G, \mathscr{F}_0) > C(X, G, \mathscr{F})$ for all such filters (Lemma 5.10 (iii)). Having exhibited an upper bound for θ', the theorem follows from Zorn's lemma. ∎

We now consider necessary conditions and sufficient conditions for $(G, \mathscr{F})$ to have a completion by functions. We have seen that this is equivalent to having a complete $C(X, G, \mathscr{F}')$ with $\mathscr{F}' > \mathscr{F}$ and $(G, \mathscr{F}') = (G, \mathscr{F})$. A readily available necessary condition is that the dimension of $\hat{G}/G$ is less than the dimension of X^*/G where $\hat{G}$ is the usual abstract completion. This condition is also sufficient when the filter is minimal (Theorem 5.15 below). A specific set of sufficient conditions is given in Theorem 5.16. This theorem is applicable to the examples of Section 4; we didn't need it as the filters $\mathscr{F}_p$ constructed there were themselves sufficiently fine that the spaces $C(X, G, \mathscr{F}_p)$ provided completions. Theorems 3.1 and 3.2 are restated in the present content as Theorems 5.17 and 5.18.

5.15 THEOREM. *Consider* $(G, \mathscr{M})$ *where* $\mathscr{M}$ *is minimal relative to* G. *If* $\dim \hat{G}/G \leq \dim X^*/G$, *where* $\hat{G}$ *is the abstract completion, then* $(G, \mathscr{M})$ *has a completion by functions.*

Proof. A Hamel basis argument is used to construct a complete space $(\tilde{G}, p)$ in X^*, where the p-pseudo-norm on G is the same as the $\mathscr{M}$-pseudo-norm. By Lemma 5.10 (i), $B(X, \mathscr{M}) = (G, \mathscr{M})$. The present result now follows from Theorem 5.12. ∎

5.16 THEOREM. *Let* $(G, \mathscr{F})$ *be a separable infinite dimensional normed space.*

If $\mathscr{F}$ has a countable base of balanced, convex sets, then $(G, \mathscr{F})$ has a completion by functions.

Proof. Denote the countable base of $\mathscr{F}$ by $\{F_i: i = 1, 2, \ldots\}$ and assume that $F_i \subset F_j$ when $i < j$. Let H be a dense subspace of G spanned by a countable subset. We will use the fact that the $\sigma(H^*, H)$-topology on H^* can be obtained from an invariant metric d (see Theorem 1, p. 111 of [7]). Let $e: X \to H^*$ denote the natural evaluation map. Note that each $e(F)$ spans an infinite dimensional subspace of H^*, for if this subspace were finite dimensional it would be $\sigma(H^*, H)$-closed and thus contain $A = \bigcap \{\overline{e(F_j)}: F_j \in \mathscr{F}\}$. This cannot happen because the linear span of A is the infinite dimensional space H^* (see Proposition 1.6 and Proposition 3, p. 204 of [7]).

Since the set A above is $\sigma(H^*, H)$-compact, we can choose in each $e(F_k)$ a finite subset $S_k = \{b_1^k, b_2^k, \ldots, b_{n_k}^k\}$ such that for each x in A there is a b_i^k such that $d(x, b_i^k) < 1/k$. We will now replace each set S_k by a subset $D_k \subset e(F_k)$,

$$D_k = \{a_1^k, a_2^k, \ldots, a_{n_k}^k\},$$

such that $d(a_i^k, b_i^k) < 1/k$ for $i = 1, 2, \ldots, n_k$ and with the important property that

$$\bigcup \{D_k: k = 1, 2, \ldots\}$$

is linearly independent. We proceed by induction. Let $V_1 = \{0\}$ and let V_k, $k \geq 2$, be the linear span of $D_1 \cup D_2 \cup \cdots \cup D_{k-1}$. Assume V_k is finite dimensional and choose, for each $i = 1, 2, \ldots, n_k$, $\lambda_i \in (0, 1)$ such that $d(\lambda_i b_i, b_i) < 1/2k$. (For simplicity of notation we omit the superscript k from the λ_i, b_i, d_i, r_i, and a_i appearing here and below.) Now choose a subset $\{d_1, d_2, \ldots, d_{n_k}\}$ of F_k such that its linear span intersects the linear span of $V_k \cup S_k$ only in the origin. This is possible because the linear span of F_k is infinite dimensional. Since the metric topology is linear we may choose, for each i, a number r_i, $0 < r_i < 1 - \lambda_i$, such that

$$d(\lambda_i b_i + r_i d_i, \lambda_i b_i) < 1/2k.$$

Each $a_i = \lambda_i b_i + r_i d_i$ belongs to $e(F_k)$ since this set is balanced and convex. Now define $D_k = \{a_1, a_2, \ldots, a_{n_k}\}$; $d(a_i, b_i) < 1/k$. Let $\mathscr{J}$ denote the filter base, in $e(X)$, composed of the balanced, convex hulls of the sets $\{\bigcup \{D_k: k \geq m\}\}$, $m = 1, 2, \ldots$. We will show that the $\mathscr{F}$-topology and the $\mathscr{J}$-topology coincide on H. Here we are considering H as a space of functions with domain X and with domain H^*, respectively. We shall, in fact, show that

$$\bigcap \{\overline{e(F)}: F \in \mathscr{F}\} = \bigcap \{\bar{J}: J \in \mathscr{J}\}$$

and invoke Proposition 1.6. Only the containment

$$\bigcap \{\bar{J}: J \in \mathscr{J}\} \supset \bigcap \{\overline{e(F)}: F \in \mathscr{F}\}$$

requires proof. Let $\varepsilon > 0$ and $x \in \bigcap \{\overline{e(F)}: F \in \mathscr{F}\}$ be arbitrary. Choose an integer k such that $1/k < \varepsilon/2$. There is a b_i in S_k and an $a_i \in D_k$ such that $d(x, b_i) < 1/k$ and $d(b_i, a_i) < 1/k$. Thus $d(x, a_i) < \varepsilon$, and it follows that $x \in \bigcap \{\bar{J}: J \in \mathscr{J}\}$.

Let $\mathscr{F}'$ denote the filter in X with base

$$\{F \cap e^{-1}(J): F \in \mathscr{F}, J \in \mathscr{J}\},$$

$\mathscr{F}'$ being a refinement of $\mathscr{F}$. Note that $\mathscr{J}$ has $e(\mathscr{F}')$ as a filter base. Thus the $\mathscr{F}'$-topology and $\mathscr{J}$-topology coincide on H.

By Theorem 3.1, $C(e(X), H, \mathscr{J})$ is complete, which is the same as $C(X, H, \mathscr{F}')$'s being complete. The space H is dense in G for both the $\mathscr{F}$-topology and $\mathscr{F}'$-topology. The topologies coincide on H. Because the $\mathscr{F}'$-topology is weaker they must also coincide on G. By Theorem 2.3 we can now conclude that $C(X, G, \mathscr{F}') = C(X, H, \mathscr{F}')$ is the desired completion of $(G, \mathscr{F})$ by functions. ∎

5.17 THEOREM. *If the filter $\mathscr{F}$ from $(G, \mathscr{F})$ has a refinement which contains a linearly independent set and gives the original topology on G, then $(G, \mathscr{F})$ has a completion by functions.*

5.18 THEOREM. *The linear topological space $(G, \mathscr{F})$ has a completion by functions if there is a filter $\mathscr{F}' > \mathscr{F}$ ($\mathscr{F}'$ having a base of balanced convex sets) and a linear map $T: X \to G'$ with the properties:*

(i) *For every $\varepsilon > 0$ and $g \in G$ there exists a set $F = F(\varepsilon, g)$ in $\mathscr{F}'$ such that $|(Tv)(g) - v(g)| < \varepsilon$ for all $v \in F$.*

(ii) *The filter base $\{T(F): F \in \mathscr{F}'\}$ contains an equicontinuous subset of G'.*

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