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FORMAL AND COFORMAL SPACES

BY

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1. Introduction

In this paper, we apply minimal algebras and minimal Lie algebras to the study of rational homotopy theory. Sullivan has introduced the concept of a formal space [2]. A space is formal if its rational homotopy type is a formal consequence of its cohomology algebra. We give an equivalent definition in terms of perturbations in the differential of a minimal Lie algebra model. These perturbations are related to Massey products, but they have several advantages. For example, perturbations are always defined and they are well defined once we have chosen generators for our model. Take a minimal rational CW complex. Roughly speaking, a perturbation is the deviation that the attaching maps for cells have from being quadratic.

We also introduce a concept which is dual to formality. A space is coformal if its rational homotopy type is a formal consequence of its homotopy Lie algebra. Equivalently, a space is coformal if the k invariants in its rational Postnikov system are quadratic.

The main theorems in this paper are:

COROLLARY 5.2. *Let k be a field of characteristic zero. Two simply connected finite complexes have the same rational homotopy type if there is a k homotopy equivalence such that the induced cohomology isomorphism is rational.*

PROPOSITION 4.4. *Every n connected compact m-dimensional manifold with cohomology of rank > 3 and $m \leq 3n + 1$, $n \geq 1$, is both formal and coformal.*

PROPOSITION 4.6. *Every simply connected compact manifold of dimension ≤ 6 is formal.*

As a corollary of 5.2, we get a result announced by Sullivan [14] and Halperin-Stasheff [16]. A simply connected finite complex is formal over the rationals if it is formal over an extension field. Deligne, Griffiths, Morgan, and Sullivan [2] have shown that compact Kähler manifolds are formal over the reals. Hence, compact simply connected Kähler manifolds are formal over the rationals. (The restriction to simply connected spaces is actually unnecessary [16].)

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Section 2 of this paper is devoted to a description of minimal algebra, minimal coalgebra, and minimal Lie algebra models. Perturbations in minimal algebras and minimal Lie algebras are defined there. Most of Section 2 is a summary of results in [7], [2], and [10].

In Section 3, we give several equivalent definitions of formality and coformality. Section 4 contains applications of perturbations to show that various spaces are or are not formal or coformal.

When we study a homotopy type over an arbitrary field of characteristic zero, we are faced with whether or not we can descend down to the rationals. Corollaries 5.3 and 5.7 show that a homotopy equivalence descends to the rationals if the induced maps in homology or in homotopy descend to the rationals.

2. Minimal models

Rational homotopy theory may be studied in several categories. Among these are the categories of spaces, of commutative associative differential graded algebras, of commutative associative differential graded coalgebras, and of differential graded Lie algebras. For nilpotent spaces with rational homology of finite type, the possibility of using differential graded algebras to study rational homotopy type was developed by Sullivan [2]. (See also Bousfield-Gugenheim [1].) For simply connected spaces, the possibility of using differential graded coalgebras and differential graded Lie algebras was noticed by Quillen [10]. In this section, we recall the relevant aspects of Sullivan's and Quillen's work and show how to combine them to study the rational homotopy type of any nilpotent space with finite type rational homology. Full details appear in [7].

Let X be a connected space. Sullivan [2] has defined the PL de Rham forms $\mathscr{E}(X)$ whenever X is a simplicial complex. This definition extends to any simplicial set [1], [15], and hence to all spaces by composition with the singular functor. $\mathscr{E}(X)$ defines a contravariant functor from the category of connected spaces to the category of commutative associative differential graded algebras. There is a natural isomorphism of graded algebras $H^*\mathscr{E}(X) \cong H^*(X; \mathbf{Q})$.

Sullivan goes on to define minimal algebras. A differential graded algebra A defined over a field k of characteristic zero is called minimal if it is constructible by a succession of decomposable elementary extensions starting from the ground field [2]. A decomposable elementary extension of a differential graded algebra B is a differential graded algebra A, denoted $A = B \otimes_d S[V]$, where: (1) If we forget the differential, then $A = B \otimes S[V]$ where $S[V]$ is a symmetric algebra. (2) The differential d on A extends that on B and $d(V)$ is contained in $\bar{B} \cdot \bar{B}$.

It follows that any minimal algebra is a symmetric algebra when we forget the differential and that the differential of a minimal algebra is decomposable, that is, $d(\bar{A})$ is contained in $\bar{A} \cdot \bar{A}$. In fact, these properties may be used to define minimal algebras in the simply connected case, $A^0 = k$, $A^1 = k$.

If X is a connected space, there is a minimal algebra M_X, defined over the rationals, which admits a homomorphism $M_X \to \mathscr{E}(X)$ such that $H^*M_X \to H^*\mathscr{E}(X)$ is an isomorphism. M_X is unique up to isomorphism and is called the minimal algebra model for X.

Let A be a minimal algebra and let $Q(A) = \bar{A}/\bar{A} \cdot \bar{A}$ denote the module of indecomposables. Pick a basis $\{x_i\}$ for $Q(A)$ and a splitting $Q(A) \to A$ so that we may regard $\{x_i\}$ as a minimal set of algebra generators in A. Since the differential d in A is decomposable, we may write $dx_i = \bar{W}(x_i) + P(x_i)$ where $\bar{W}(x_i) = \sum c^i_{jk} x_j x_k$ and $P(x_i)$ is a sum of monomials in the generators of length 3 or greater. $P(x_i)$ is called the perturbation of the differential. It is not well defined. $P(x_i)$ depends on the choice of the basis and the splitting.

PROPOSITION 2.1. *The quadratic term $\bar{W}(x_i) = \sum c^i_{jk} x_j x_k$ is well defined and the composition*

$$x_i \to \sum c^i_{jk} x_j x_k \to \sum \tfrac{1}{2} c^i_{jk}(x_j \otimes x_k + (-1)^{\deg x_j \deg x_k} x_k \otimes x_j)$$

gives a well defined map W: $Q(A) \to Q(A) \otimes Q(A)$.

The reader should refer to [7] or [9] for an alternate definition of W which is clearly well defined and functorial.

Let X be a simply connected space. Via the Whitehead product

$$\pi_n(X) \otimes \pi_m(X) \to \pi_{n+m-1}(X),$$

$\pi(X)$ has the structure of a Whitehead algebra [9].

Remark. $\pi_n(X) \cong \pi_{n-1}(\Omega X)$ and, under this isomorphism, the Whitehead product corresponds to the Samelson product in $\pi(\Omega X)$. $\pi(\Omega X) \otimes \mathbf{Q}$ is a graded Lie algebra.

PROPOSITION 2.2 [2]. *If X is a simply connected space with rational homology of finite type and M_X is its minimal algebra model, then*

$$Q(M_X) \cong \text{Hom}\ (\pi(X), \mathbf{Q})$$

and the map W in 2.1 is dual to the Whitehead product.

If X is nilpotent with rational homology of finite type, then the above isomorphism holds in dimensions > 1 and

$$Q^1(M_X) \cong \text{Hom}\ (l(\pi_1(X) \otimes \mathbf{Q}), \mathbf{Q})$$

where $l(\pi_1(X) \otimes \mathbf{Q})$ is the Lie algebra of the Malcev completion [2], [10].

Let C be a connected differential graded coalgebra. C is called minimal if the following three conditions are satisfied: (1) If we forget the differential, C is a symmetric coalgebra. (2) The differential d of C is zero when restricted to the module of primitives

$$PC = \text{kernel } \bar{C} \to \bar{C} \otimes \bar{C}.$$

(3) PC is nilpotently complete as a Whitehead algebra with bracket $PC \otimes PC \to PC$ defined in a manner dual to 2.1 (see [7]).

If X is a nilpotent space with finite type rational homology, then its minimal algebra model M_X is of finite type and the dual of $Q(M_X)$ is a nilpotent Whitehead algebra. Let $C_X \cong$ the dual coalgebra to M_X. Then C_X is a minimal coalgebra and $PC_X \cong \mathrm{Hom}\,(Q(M_X), \mathbf{Q})$. C_X is unique up to isomorphism and is called the minimal coalgebra model for X. Of course, $HC_X = H(X; \mathbf{Q})$ as coalgebras and, if X is simply connected, $PC_X = \pi(X) \otimes \mathbf{Q}$ as Whitehead algebras [7].

Let C be a commutative associative connected differential graded coalgebra which is defined over a field k of characteristic zero. Quillen [10] has defined a functor $\mathscr{L}(C)$ which takes values in differential graded Lie algebras. If we forget the differential, $\mathscr{L}(C) = F[s^{-1}\bar{C}] =$ the free graded Lie algebra on the desuspended module $s^{-1}\bar{C}$. Let $s^{-1}c$ be a generator and suppose that

$$\Delta(c) = c \otimes 1 + 1 \otimes c + \sum c_i \otimes c_j + (-1)^{\deg c_i \deg c_j} c_j \otimes c_i.$$

Then $d(s^{-1}c) = -s^{-1}\,dc - \sum (-1)^{\deg c_i}[s^{-1}c_i, s^{-1}c_j]$.

A differential graded Lie algebra L is called minimal if the following conditions hold: (1) If we forget the differential, L is a free graded Lie algebra $F[V]$. (2) The differential d is decomposable, that is, $d(L)$ is contained in $[L, L]$ (see [7]).

Given a nilpotent space X with finite type rational homology, we can consider $\mathscr{L}(C_X)$ where C_X is the minimal coalgebra model for X. Suppose X is simply connected. Up to isomorphism, there is a unique minimal Lie algebra L_X which admits a homomorphism $L_X \to \mathscr{L}(C_X)$ such that $HL_X \to H\mathscr{L}(C_X)$ is an isomorphism [7]. L_X is called the minimal Lie algebra model for X.

Let L be a minimal Lie algebra and let $Q(L) = L/[L, L]$ denote the abelianization. What follows is formally identical to 2.1 for minimal algebras. Pick a basis $\{x_i\}$ for $Q(L)$ and a splitting $Q(L) \to L$ so that we may regard $\{x_i\}$ as a minimal set of Lie algebra generators in L. We may write $dx_i = \bar{D}(x_i) + P(x_i)$ where $\bar{D}(x_i) = \sum c^i_{jk}[x_j, x_k]$ and $P(x_i)$ is a sum of brackets in the generators of length 3 or greater. $P(x_i)$ is called the perturbation of the differential. It is not well defined. It depends on the choice of basis and the splitting.

Proposition 2.3. *The quadratic term* $\bar{D}(x_i) = \sum c^i_{jk}[x_j, x_k]$ *is well defined and the composition*

$$x_i \to \sum c^i_{jk}[x_i, x_k] \to \sum c^i_{jk}(x_j \otimes x_k - (-1)^{\deg x_j \deg x_k} x_k \otimes x_j)$$

gives a well defined map $D\colon Q(L) \to Q(L) \otimes Q(L)$.

Refer to [7] for an alternate definition of D which is clearly well defined and functorial.

Proposition 2.4 [7]. *If X is simply connected with finite type rational homo-*

logy and L_X is its minimal Lie algebra model, then $Q(L_X) \cong s^{-1}\bar{H}(X; Q)$ and, under this isomorphism,

$$\bar{D}(s^{-1}x) = -\sum (-1)^{\deg x_i'}[s^{-1}x_i', s^{-1}x_i'']$$

where

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum (x_i' \otimes x_i'' + (-1)^{\deg x_i' \deg x_i''} x_i'' \otimes x_i').$$

PROPOSITION 2.5 [7]. *If X is a simply connected space with finite type rational homology, then $HL_X \cong \pi(\Omega X) \otimes Q$ as Lie algebras.*

2.5 is a consequence of the isomorphism $H\mathscr{L}(C_X) \cong s^{-1}PC_X$. This does not require simple connectivity. Hence, if X is nilpotent with finite type rational homology and if we replace L_X by $\mathscr{L}(C_X)$, the isomorphism in 2.5 is valid in dimensions > 0 and $H_0\,\mathscr{L}(C_X)$ is isomorphic to the Lie algebra of the Malcev completion $\pi_0(\Omega X) \otimes Q$ (see [7]).

Given nilpotent spaces X and Y with finite type rational homology, let $X \otimes Q$ and $Y \otimes Q$ denote their respective rational homotopy types. The next proposition extends a result of Sullivan [2].

PROPOSITION 2.6 [7]. *The following five conditions are equivalent:*

(1) *$X \otimes Q$ and $Y \otimes Q$ are the same rational homotopy type.*
(2) *The minimal algebras M_X and M_Y are isomorphic.*
(3) *The minimal coalgebras C_X and C_Y are isomorphic.*
(4) *There is a map $\mathscr{L}(C_X) \to \mathscr{L}(C_Y)$ such that $H\mathscr{L}(C_X) \to H\mathscr{L}(C_Y)$ is an isomorphism.*
(5) *If X and Y are simply connected, the minimal Lie algebras L_X and L_Y are isomorphic.*

Let k be an extension field of Q. 2.6 suggests that we say that X and Y have the same k homotopy type if any of the following equivalent conditions hold: (a) $M_X \otimes k = M_Y \otimes k$. (b) $C_X \otimes k = C_Y \otimes k$. (c) There is a homology isomorphism $\mathscr{L}(C_X) \to \mathscr{L}(C_Y)$. (d) If X and Y are simply connected, $L_X \otimes k = L_Y \otimes k$.

Quillen [10] has defined a functor $\mathscr{C}$ from differential graded Lie algebras to commutative associative differential graded coalgebras. Let L be a differential graded Lie algebra. If we forget the differential, then $\mathscr{C}(L) \cong S'[sL]$ is the symmetric coalgebra generated by the suspended module sL. Rather than describe the differential in $\mathscr{C}(L)$, we will assume that L is of finite type and describe the differential d^* in the dual symmetric algebra $\mathscr{C}^*(L) = S[sL^*]$.

Choose a basis $\{y_i\}$ for L^* and suppose that

$$y_i \to \sum c_{jk}^i (y_j \otimes y_k - (-1)^{\deg y_j \deg y_k} y_k \otimes y_j)$$

is the dual of the Lie bracket. Then

$$d^*(sy_i) = -s\, d^*y_i - \sum (-1)^{\deg y_j} c_{jk}^i (sy_j)(sy_k).$$

The functor $\mathscr{C}$ is adjoint to the functor $\mathscr{L}$ and the adjunction maps $C \to \mathscr{C}\mathscr{L}(C)$ and $\mathscr{L}\mathscr{C}(L) \to L$ induce isomorphisms in homology [7] (or see [10] for the simply connected case).

3. Formality and coformality

In this section we give several equivalent definitions of Sullivan's concept [2] of a rational homotopy type being a "formal consequence of its cohomology algebra." Such a space is called formal or Q formal. We also introduce the concept of a rational homotopy type being a "formal consequence of its homotopy Lie algebra." Such a space is called coformal or Q coformal.

Throughout this section all spaces will be nilpotent with finite type rational homology. A weak equivalence between differential graded objects means a homomorphism which induces an isomorphism in homology or cohomology.

DEFINITION 3.1. Over a field k of characteristic zero, a minimal algebra A (free Lie algebra L) is called formal (respectively, coformal) if there is a weak equivalence $A \to H^*A$ (respectively, $L \to HL$). A space X is called k formal (k coformal) if its minimal algebra model $M_X \otimes k$ (respectively, Lie algebra model $\mathscr{L}(C_X) \otimes k$) is formal (respectively, coformal).

If X is simply connected, then X is k coformal if and only if $L_X \otimes k$ is coformal.

Remark. If X is Q formal (Q coformal), then X is k formal (k coformal) for any field k of characteristic zero. In this case, we say that X is formal (coformal).

PROPOSITION 3.2. *For a simply connected space X, the following conditions are equivalent:*

(a) *$M_X \otimes k$ is formal.*

(b) *There is a weak equivalence $H(X; k) \to C_X \otimes k$ where C_X is the minimal coalgebra model.*

(c) *$L_X \otimes k$ is isomorphic to $\mathscr{L}(H(X; k))$.*

(d) *$L_X \otimes k$ is isomorphic to a minimal Lie algebra whose differential has zero perturbation.*

PROPOSITION 3.3. *For a space X, the following conditions are equivalent:*

(a) *$\mathscr{L}(C_X) \otimes k$ is coformal.*

(b) *$C_X \otimes k$ is isomorphic to $\mathscr{C}(H\mathscr{L}(C_X)) \otimes k$.*

(c) *$M_X \otimes k$ is isomorphic to $\mathscr{C}^*(H\mathscr{L}(C_X)) \otimes k$.*

(d) *$M_X \otimes k$ is isomorphic to a minimal algebra whose differential has zero perturbation.*

Proofs of 3.2 *and* 3.3. The equivalence of 3.2(a) and 3.2(b) is simple dualization. Assume 3.2(b). Then $\mathscr{L}(H(X; k)) \to \mathscr{L}(C_X \otimes k) = \mathscr{L}(C_X) \otimes k$ is a weak equivalence [10]. Since $H(X; k)$ has zero differential, $\mathscr{L}(H(X; k))$ is a minimal Lie algebra. By the uniqueness of minimal models, $\mathscr{L}(H(X; k)) = L_X \otimes k$. Hence 3.2(b) implies 3.2(c).

Assume 3.2(c). Then there is a weak equivalence $\mathscr{L}(H(X; k)) \to \mathscr{L}(C_X \otimes k)$ and hence a weak equivalence $\mathscr{C}\mathscr{L}(H(X; k)) \to \mathscr{C}\mathscr{L}(C_X \otimes k)$ (see [10]). $C_X \otimes k$ is fibrant in the sense of [10] so that there is a splitting $\mathscr{C}\mathscr{L}(C_X \otimes k) \to C_X \otimes k$. This splitting is also a weak equivalence. Composition gives a weak equivalence $H(X; k) \to C_X \otimes k$. Hence, 3.2(c) implies 3.2(b).

The differential in $\mathscr{L}(H(X; k))$ has zero perturbation; hence, 3.2(c) implies 3.2(d).

Assume 3.2(d). Then we can assume $L_X \otimes k$ to have a differential with zero perturbation. By Proposition 2.4, $L_X \otimes k$ is isomorphic to $\mathscr{L}(H(X; k))$. Therefore, 3.2(d) implies 3.2(c).

Proposition 3.3 is proved in the same way as Proposition 3.2 but with the roles of $\mathscr{C}$ and $\mathscr{L}$ interchanged. Instead of using Proposition 2.4, we use an algebraic version of Proposition 2.2. It asserts that $s^{-1}PC_X \cong H\mathscr{L}(C_X)$ as Lie algebras. See [7, Proposition 6.3]. In any case, 2.2 is sufficient to prove that 3.3(d) implies 3.3(b) when X is simply connected. ∎

We cannot resist giving geometric interpretations of formality and coformality. A complete justification would take us off our path; therefore, we will be brief and let the suspicious reader regard them as mere guides to his intuition. Sullivan has shown that the differentials on generators for a minimal algebra model can be regarded as cochain representatives for the k-invariants of the rational Postnikov system [2]. By 3.3(d), a space X is Q coformal if and only if $X \otimes Q$ can be constructed by a Postnikov system in which the k-invariants have quadratic cochain representatives. In a dual fashion, the differentials on generators for a minimal Lie algebra model can be regarded as chain representatives for attaching maps for cells in a minimal rational CW decomposition of $X \otimes Q$. By 3.2(d), a simply connected space X is Q formal if and only if $X \otimes Q$ has a minimal rational CW decomposition where the attaching maps of cells have quadratic chain representatives.

In terms of the identifications $\pi_n(X) \otimes Q = P_n C_X = H_{n-1} L_X$, the Hurewicz map has two descriptions [2], [10]:

(1) $PC_X \to C_X$ induces the Hurewicz map $PC_X \to HC_X = H(X; Q)$ and
(2) $L_X \to Q(L_X)$ induces the Hurewicz map $HL_X \to Q(L_X) = s^{-1}\bar{H}(X; Q)$.

As a trivial consequence of 3.2(c) and 3.3(b), we get:

PROPOSITION 3.4. *Let X be a simply connected space. If X is formal, then the Hurewicz map, $\pi(X) \otimes Q \to PH(X; Q)$, is surjective onto the primitives. If X is* *coformal, then the quotient of the Hurewicz map, $Q(\pi(X)\otimes Q)\to H(X;Q)$, is injective.*

The first part of 3.4 is proved by another method in [9]. If X is both formal and coformal, then $Q(\pi(X)\otimes Q)=PH(X;Q)$. Hence:

COROLLARY 3.5. *If X is a formal, coformal, and simply connected space, then there is an epimorphism of Lie algebras $F[s^{-1}PH(X;Q)]\to\pi(\Omega X)\otimes Q$.*

3.5 may be thought of as a weak upper bound on the rational homotopy.

4. Examples

In this section, we give some examples of spaces which are or are not formal or coformal.

LEMMA 4.1. *If X and Y are simply connected spaces with finite type rational homology, then $X\vee Y$ and $X\times Y$ are formal (coformal) if X and Y are formal (coformal).*

Proof. By [7, Lemma 8.5], the minimal coalgebra model for $X\times Y$ is $C_X\otimes C_Y$. If X and Y are formal, there are weak equivalences $H(X;\mathbf{Q})\to C_X$ and $H(Y,\mathbf{Q})\to C_Y$ by 3.2. Tensoring gives a weak equivalence

$$H(X\times Y;\mathbf{Q})\to C_X\otimes C_Y.$$

Thus $X\times Y$ is formal. If X and Y are coformal, then

$$C_X\cong\mathscr{C}(\pi(\Omega X)\otimes\mathbf{Q})\quad\text{and}\quad C_Y\cong\mathscr{C}(\pi(\Omega Y)\otimes\mathbf{Q})$$

by 3.2. Hence,

$$C_X\otimes C_Y\cong\mathscr{C}(\pi(\Omega X)\otimes\mathbf{Q}\times\pi(\Omega Y)\otimes\mathbf{Q})\cong\mathscr{C}(\pi(\Omega(X\times Y))\otimes\mathbf{Q}).$$

Hence, $X\times Y$ is coformal.

By [7, Lemma 8.6], the minimal Lie algebra model for $X\vee Y$ is $L_X\vee L_Y$ (= free product). If X and Y are formal, then

$$L_X\vee L_Y\cong\mathscr{L}(H(X;\mathbf{Q}))\vee\mathscr{L}(H(Y;\mathbf{Q}))\cong L(H(X\vee Y;\mathbf{Q})).$$

Hence, $X\vee Y$ is formal. If X and Y are coformal, then there is a weak equivalence

$$L_X\vee L_Y\to\pi(\Omega X)\otimes\mathbf{Q}\vee\pi(\Omega Y)\otimes\mathbf{Q}\cong\pi(\Omega(X\vee Y))\otimes\mathbf{Q}.$$

Hence, $X\vee Y$ is coformal. ∎

Let I be an ideal in a graded symmetric algebra $S=S[x_1,\dots,x_k]$. I is called a Borel ideal if it has a set of ideal generators $y_1,\dots,y_l$ decomposable in S such that $H^*(A, d) \cong S/I$ where

$$A = S[x_1, \ldots, x_k, s^{-1}y_1, \ldots, s^{-1}y_l]$$

and d is a derivation such that $dx_i = 0$ and $ds^{-1}y_i = y_i$ (see [13]).

Let J be an ideal in a free graded Lie algebra $F = F[z_1, \ldots, z_k]$. J is called a Borel ideal if it has a set of ideal generators $w_1, \ldots, w_l$ decomposable in F such that $H(B, d) \cong F/J$ where

$$B = F[z_1, \ldots, z_k, s^{-1}w_1, \ldots, s^{-1}w_k]$$

and d is a derivation such that $dz_i = 0$ and $ds^{-1}w_i = w_i$.

LEMMA 4.2. *If X is a nilpotent space such that $H^*(X; \mathbf{Q}) \cong S/I$ where S is a graded symmetric algebra and I is a Borel ideal, then X is formal. If the generators for I can be chosen to be quadratic, then X is also coformal.*

LEMMA 4.3. *If X is a simply connected space such that $\pi(\Omega X) \otimes \mathbf{Q} \cong F/J$ where F is a free graded Lie algebra and J is a Borel ideal, then X is coformal. If the generators for J can be chosen to be quadratic, then X is also formal.*

We will prove 4.3 and leave 4.2 as an exercise.

Proof. Let B be the graded Lie algebra in the definition of a Borel ideal. There is a weak equivalence $B \to F/J$ given by $z_i \to \bar{z}_i$, $s^{-1}w_i \to 0$. On the other hand, there is a weak equivalence $B \to L_X$ given by $z_i \to a_i$, $s^{-1}w_i \to b_i$ where a_i is a cycle representative for z_i and db_i is the image of w_i.

Since B is a minimal Lie algebra, $B \cong L_X$ and there is a weak equivalence $L_X \to F/J = \pi(\Omega X) \otimes \mathbf{Q}$. Hence, X is coformal. The second statement in 4.3 follows from 3.2(d). ∎

From 4.2 and 4.3, it follows that a wedge of spheres $\bigvee_\alpha S^{n_\alpha}$, $n_\alpha \geq 2$, and a product of Eilenberg-MacLane spaces $\times_\alpha K(\pi_\alpha, n_\alpha)$, $n_\alpha \geq 1$, are both formal and coformal. In the first case,

$$\pi\left(\Omega \bigvee_\alpha S^{n_\alpha}\right) \otimes \mathbf{Q} \cong F[x_\alpha]$$

where degree $x_\alpha = n_\alpha - 1$. In the second case,

$$H^*\left(\times_\alpha K(\pi_\alpha, n_\alpha); \mathbf{Q}\right) \cong S\left[\bigoplus_\alpha \pi_\alpha^*\right]$$

where $\pi_\alpha^* = \operatorname{Hom}(\pi_\alpha, \mathbf{Q})$ is concentrated in dimension n_α. Since any connected Lie group G has the rational homotopy type of a product of odd spheres (these are rational Eilenberg-MacLane spaces), it follows that G is both formal and coformal.

PROPOSITION 4.4. *Let M be a compact n-connected m-dimensional manifold with $m \leq 3n+1$, $n \geq 1$. Then M is formal and, if* rank $PH(M; \mathbf{Q}) \geq 2$, *M is coformal.*

Proof. It follows from Poincare duality that

$$H(M; \mathbf{Q}) = \mathbf{Q} \oplus PH(M; \mathbf{Q}) \oplus H_m(M; \mathbf{Q})$$

where $H_m(M; \mathbf{Q})$ is generated by the fundamental class μ_M. Consider the minimal Lie algebra model L_M. By 2.4, L_M is isomorphic to the free graded Lie algebra generated by $s^{-1}\bar{H}(M; \mathbf{Q})$. For dimension reasons, the differential on $s^{-1}\bar{H}(M; \mathbf{Q})$ can have no perturbation. Hence, M is formal by 3.2.

Therefore, $L_M \cong \mathscr{L}(H(M; \mathbf{Q}))$ and d is zero on $s^{-1}PH(M; \mathbf{Q})$. By [8],

$$\pi(\Omega M) \times \mathbf{Q} \cong F/J$$

where $F = F[s^{-1}PH(M; \mathbf{Q})]$ and J is the ideal generated by $ds^{-1}\mu_M$. Since $HL_M \cong F/J$, J is a Borel ideal. Hence, M is coformal by 4.3. ∎

In order to study more subtle examples of formal spaces, it is helpful to know a basis for a free graded Lie algebra defined over $\mathbf{Q}$. In the ungraded case, a basis is given by a Hall family. We refer the reader to [Serre, 12] for the definition of the nonassociative monomials which make up a Hall family.

PROPOSITION 4.5. *If F is a rational graded Lie algebra which is free on graded generators $x_1, \ldots, x_k$, then F has a graded basis $A \cup B$ where A is a Hall family and $B = \{[x_\alpha, x_\alpha]$: $x_\alpha \in A$ and* degree *x_α is odd$\}$.*

Since we use this proposition only for simply connected spaces, we shall assume that degree $x_i \geq 1$ for all generators x_i. There is a longer algebraic proof which does not require this assumption.

Proof. Let $A = \{x_\alpha\}$ be a Hall family and let degree $x_\alpha = n_\alpha$. Let $X = \bigvee_{i=1}^k S^{n_i+1}$. By the Hilton-Milnor theorem [3], [6] there is a homotopy equivalence $\Omega Y \to \Omega X$ where $Y = \times_\alpha S^{n_\alpha+1}$. Hence,

$$F \cong \pi(\Omega X) \otimes \mathbf{Q} \cong \pi(\Omega Y) \otimes \mathbf{Q} = \prod_\alpha \pi(\Omega S^{n_\alpha+1}) \otimes Q = \prod_\alpha F[x_\alpha].$$

If degree x_α is even, then $F[x_\alpha]$ has a basis of one element x_α. If degree x_α is odd, then $F[x_\alpha]$ has a basis of two elements x_α and $[x_\alpha, x_\alpha]$. ∎

Using 4.5, we can prove:

PROPOSITION 4.6. *Every compact simply connected manifold M of dimension $m \leq 6$ is formal.*

Proof. It follows from 4.4 that M is formal if $m \leq 4$. The cases $m = 5$ and

$m = 6$ are very similar. We will treat the case $m = 6$ and leave $m = 5$ to the reader.

By Poincaré duality, $\bar{H}^*(M; \mathbf{Q})$ has a graded basis $x_1^*, \ldots, x_k^*$(degree 2); $z_1^*, \ldots, z_l^*, w_1^*, \ldots, w_l^*$(degree 3); $y_1^*, \ldots, y_k^*$(degree 4); and μ^*(degree 6) such that $x_i^* y_j^* = \delta_{ij}\mu^*$, $z_i^* w_j^* = \delta_{ij}\mu^*$. Let x_i, z_i, w_i, y_i, μ be the dual basis for $\bar{H}(M; \mathbf{Q})$.

By 2.4, the minimal Lie algebra model L_M is generated by $s^{-1}x_i, s^{-1}z_i, s^{-1}w_i$, $s^{-1}y_i$, $s^{-1}\mu$ and, on these generators, the quadratic term of the differential is given by the comultiplication. The only possible perturbation of the differential is $P(s^{-1}\mu)$. If we look at the basis in 4.5, we can write $P(s^{-1}\mu) = \sum [s^{-1}x_i, a_i]$ where degree $a_i = 3$. Define an isomorphism f: $L_M \to \mathscr{L}(H(M; \mathbf{Q}))$ by

$$f(s^{-1}y_i) = s^{-1}y_i - a_i (i = 1, \ldots, k)$$

and f equals the identity on all other generators. By 3.2, M is formal. ∎

The rest of this section is devoted to giving examples of spaces which are not formal or not coformal.

Example 4.7. Consider complex projective space CP^n. Since

$$H^*(\mathbf{C}P^n; \mathbf{Q}) \cong \mathbf{Q}[u]/u^{n+1} = 0,$$

4.2 implies that $\mathbf{C}P^n$ is formal but it is not coformal if $n \geq 2$ because u^{n+1} is not quadratic.

Example 4.8. Consider the CW complex

$$X = S^2 \vee S^2 \bigcup_{\alpha} e^5$$

where α: $S^4 \to S^2 \vee S^2$ is the triple Whitehead product $[x, [x, y]]$ and x, y is a basis of $\pi_4(S^2 \vee S^2) \otimes \mathbf{Q}$. The minimal Lie algebra model for X is

$$F[s^{-1}x, s^{-1}y, s^{-1}\mu]$$

where degree $\mu = 5$ and $d(s^{-1}\mu) = [s^{-1}x, [s^{-1}x, s^{-1}y]]$, $d(s^{-1}x) = d(s^{-1}y) = 0$. Since there is a (nonremovable) perturbation into cubic terms, X is not formal.

As Example 4.8 hints, perturbations of the Lie algebra differential are related to nontrivial Massey products. In fact, perturbations are better than Massey products because perturbations are always defined and well defined once generators are chosen.

Example 4.9. Let L be a differential graded Lie algebra. Suppose that α, β, γ are homology classes in HL (of degrees r, s, t) such that $[\beta, \gamma] = [\gamma, \alpha] = [\alpha, \beta] = 0$. There is a higher order Lie product $[\alpha, \beta, \gamma]$ which is well defined in

$Q(HL) = HL/[HL, HL]$. Let a, b, c be representative cycles and let $dx = bc$, $dy = ca$, $dz = ab$. Then $[\alpha, \beta, \gamma]$ is represented by the cycle

$$(-1)^{\varepsilon_1}[x, a] + (-1)^{\varepsilon_2}[y, b] + (-1)^{\varepsilon_3}[z, c]$$

where $\varepsilon_1 = tr - t$, $\varepsilon_2 = rs - r$, $\varepsilon_3 = ts - s$.

If X is a coformal space, all higher order Lie products must be zero in $Q(L_X)$. For example, if $Y = S^2 \times S^2 \times S^2$ – the top cell, then $[\alpha, \beta, \gamma] \neq 0$ where α, β, γ generate $H_1 L_Y$. Hence, Y is not coformal.

5. Descent

Throughout this section, let k be a field of characteristic zero.

If X and Y are nilpotent spaces with finite type rational homology, then X and Y may have the same k homotopy type but different rational homotopy types. For example, let X and Y be two compact $(2n-1)$-connected $4n$-dimensional manifolds such that rank $H_{2n}(X; Q) =$ rank $H_{2n}(Y; Q)$. Then X and Y have the same complex homotopy type but will have different rational homotopy types if the quadratic forms on H^{2n} are not rationally isomorphic. In this section, we give sufficient criteria for a k equivalence to descend to a rational equivalence. As a corollary, we get that k formal (respectively, k coformal) is equivalent to Q formal (respectively, Q coformal).

Let V and V' be graded vector spaces defined over Q. A k linear map

$$f\colon V' \otimes k \to V \otimes k$$

is said to be defined over Q if there is a Q linear map $g\colon V' \to V$ such that $f = g \otimes 1$.

PROPOSITION 5.1. *Let L' and L be minimal Lie algebras defined over Q such that $Q(L')$ and $Q(L)$ are finite dimensional and $L'_0 = L_0 = 0$. Suppose that there is an isomorphism $f\colon L' \otimes k \to L \otimes k$ such that*

$$Q(f)\colon Q(L') \otimes L \to Q(L) \otimes k$$

is defined over Q. Then there exists an isomorphism $g\colon L' \to L$ such that $Q(g) \otimes 1 = Q(f)$.

If we apply 5.1 to minimal Lie algebra models, we get the following result.

COROLLARY 5.2. *If X and Y are simply connected spaces with finite dimensional rational homology, then X and Y have the same rational homotopy type if and only if any of the following equivalent conditions hold:*

(a) *There exists an isomorphism $f\colon L_X \otimes k \to L_Y \otimes k$ such that $Q(f)$ is defined over Q.*

(b) *There exists an isomorphism* $g\colon C_X \otimes k \to C_Y \otimes k$ *such that*

$$H(g)\colon H(X; k) \to H(Y; k)$$

is defined over Q.

(c) *There exists an isomorphism* $h\colon M_X \otimes k \to M_Y \otimes k$ *such that*

$$H^*(h)\colon H^*(Y; k) \to H^*(X; k)$$

is defined over Q.

COROLLARY 5.3. *If* X *and* Y *are simply connected spaces with finite dimensional rational homology, then an isomorphism of rational cohomology algebras is induced by a map* $X \otimes Q \to Y \otimes Q$ *if and only if it is induced by a map* $M_Y \otimes k \to M_X \otimes k$.

COROLLARY 5.4. *If* X *is a simply connected space with finite dimensional rational homology, then* X *is* k *formal if and only if* X *is* Q *formal.*

Given 5.1, only Corollary 5.4 is not trivial.

Proof of 5.4. Suppose X is k formal. By 3.2, there is an isomorphism

$$f\colon L_X \otimes k \to \mathscr{L}(H(X; k)).$$

By 2.4, $Q(f)$ corresponds to an automorphism of the coalgebra $H(X; k)$. If g is any automorphism of $H(X; k)$, then $\mathscr{L}(g)$ is an automorphism of $\mathscr{L}(H(X; k))$ and $Q\mathscr{L}(g)$ corresponds to g under the isomorphism

$$Q(\mathscr{L}(H(X; k))) \cong s^{-1}\bar{H}(X; k).$$

Hence, there exists an automorphism h of $\mathscr{L}(H(X; k))$ such that $Q(hf)$ is defined over $\mathbf{Q}$. By 5.1 applied to hf, there exists an isomorphism $L_X \to \mathscr{L}(H(X; \mathbf{Q}))$. Hence, X is $\mathbf{Q}$ formal. ∎

As a corollary of the proof of 5.4, we get:

COROLLARY 5.5 [14]. *If* X *is simply connected with finite type rational homology and* X *is formal, then any automorphism of* $H^*(X; \mathbf{Q})$ *is induced by a self map of* X.

In the proofs of 5.1 to 5.5, we can replace minimal Lie algebras by minimal algebras. If we do so, we get the following result.

PROPOSITION 5.6. *If* X *and* Y *are simply connected spaces with finite dimensional rational homotopy, then* X *and* Y *have the same rational homotopy type if and only if any of the following conditions hold:*

(a) *There exists an isomorphism* $f\colon M_Y \otimes k \to M_X \otimes k$ *such that* $Q(f)$ *is defined over* $\mathbf{Q}$.

(b) *There exists an isomorphism* $g\colon C_X \otimes k \to C_Y \otimes k$ *such that* $P(g)$ *is defined over* $\mathbf{Q}$.

(c) *There exists an isomorphism* $h\colon L_X \otimes k \to L_Y \otimes k$ *such that*

$$H(h)\colon \pi(\Omega X) \otimes k \to \pi(\Omega Y) \otimes k$$

is defined over $\mathbf{Q}$.

COROLLARY 5.7. *If X and Y are simply connected spaces with finite dimensional rational homotopy, then an isomorphism of homotopy Lie algebras*

$$\pi(\Omega X) \otimes \mathbf{Q} \to \pi(\Omega Y) \otimes \mathbf{Q}$$

is induced by a map $X \otimes \mathbf{Q} \to Y \otimes \mathbf{Q}$ if and only if it is induced by a map

$$L_X \otimes k \to L_Y \otimes k.$$

COROLLARY 5.8. *If X is a nilpotent space with finite dimensional rational homotopy, then X is k coformal if and only if X is $\mathbf{Q}$ coformal. If X is $\mathbf{Q}$ coformal, then any automorphism of the rational homotopy Lie algebra is induced by a self map of $X \otimes \mathbf{Q}$.*

The remainder of this paper is devoted to the proof of 5.1.

The first part of the proof is devoted to showing that we can assume that k is a Galois extension of $\mathbf{Q}$. In fact, we can assume that $k = \bar{\mathbf{Q}}$, the field of algebraic numbers.

Let $f\colon L' \otimes k \to L \otimes k$ be an isomorphism of minimal Lie algebras satisfying the hypotheses of 5.1. Consider the map

$$f_1\colon L' \to L' \otimes k \to L \otimes k.$$

Since $Q(L')$ is finite dimensional, the image of f_1 is contained in $L \otimes A$ where A is a subring of k which is finitely generated as an algebra over $\mathbf{Q}$. By [Lang, 4, p. 256, Corollary 2], there exists a homomorphism $\phi\colon A \to \bar{\mathbf{Q}}$ of algebras over $\mathbf{Q}$. The composition $(1 \otimes \phi)f_1\colon L' \to L \otimes A \to L \otimes \bar{\mathbf{Q}}$ extends to a map $f_2\colon L' \otimes \bar{\mathbf{Q}} \to L \otimes \bar{Q}$ which is an isomorphism because $Q(f_2)$ is an isomorphism and L and L' are free graded Lie algebras. Note that $Q(f_2)$ is an extension of the original rational map $Q(L') \to Q(L)$. Therefore, we can assume that $k = \bar{\mathbf{Q}}$.

From now on, k is a Galois extension of $\mathbf{Q}$. Suppose L is a minimal Lie algebra defined over k. A differential graded automorphism $\alpha\colon L \to L$ is called a restricted automorphism if $Q(\alpha)\colon Q(L) \to Q(L)$ is the identity map. The group of restricted automorphisms will be denoted by $A(L)$. It is easy to see that $A(L)$ is a unipotent algebraic group defined over k.

A degree 0 derivation $\beta\colon L \to L$ is called a restricted derivation if β commutes with the differential and $Q(\beta)\colon Q(L) \to Q(L)$ is the zero map. Defining $[\beta, \beta'] = \beta\beta' - \beta'\beta$ for all β and β' makes the vector space of restricted derivations into a

Lie algebra. This Lie algebra will be denoted by $D(L)$. It is easy to see that $D(L)$ is a nilpotent Lie algebra defined over k.

The exponential map maps $D(L)$ bijectively onto $A(L)$. That is, if β is a restricted derivation, then $\exp(\beta) = \sum_{k=0}^{\infty} (1/k!)\beta^k$ is a restricted automorphism which is well defined because β is nilpotent. If α is a restricted automorphism, write $\alpha = 1 + \gamma$ where γ is nilpotent. Then

$$\log(\alpha) = \log(1 + \gamma) = \sum_{k=1}^{\infty} (-1)^{k-1}(1/k)\gamma^k$$

defines the inverse of the exponential map.

$A(L)$ is a connected unipotent algebraic group because $A(L) \cong D(L)$ as algebraic varieties.

Let $f\colon L' \otimes k \to L \otimes k$ be an isomorphism satisfying the hypotheses of 5.1 with k a Galois extension of $\mathbf{Q}$. Let G be the Galois group of automorphisms of k over $\mathbf{Q}$. Then G acts on the group $A(L \otimes k)$ of restricted automorphisms of $L \otimes k$ by the left action

$$g\alpha = (1 \otimes g)\alpha(1 \otimes g^{-1})$$

for all g in G and α in $A(L \otimes k)$.

Recall the definition of the Galois cohomology $H^1(G; A(L \otimes k))$ (see [11]). A cocycle is a continuous function $\phi\colon G \to A(L \otimes k)$ such that $\phi(gh) = \phi(g) \times (g\phi(h))$ for all g, h in G. Two cocycles ϕ and ψ are cohomologous if there exists an element α in $A(L \otimes k)$ with $\phi(g) = \alpha^{-1}\psi(g)(g\alpha)$. $H^1(G; A(L \otimes k))$ is the quotient of the set of cocycles under the equivalence relation of cohomology.

The isomorphism $f\colon L' \otimes k \to L \otimes k$ defines a cocycle ϕ_f by

$$\phi_f(g) = f(1 \otimes g)f^{-1}(1 \otimes g^{-1})$$

for all g in G. The coefficient group is the group of restricted automorphisms because $Q(f)$ is defined over $\mathbf{Q}$ and commutes with the action of G.

Because $A(L \otimes k)$ is a connected unipotent algebraic group, [Serre, 11, Chapter III, Proposition 6] implies that $H^1(G; A(L \otimes k) = 0$. Therefore, ϕ_f is equivalent to the trivial element; that is, there exists α in $A(L \otimes k)$ such that $f(1 \otimes g)f^{-1}(1 \otimes g^{-1}) = \alpha^{-1}(g\alpha)$ for all g in G. This is equivalent to $(\alpha f) \times (1 \otimes g) = (1 \otimes g)(\alpha f)$. Because αf commutes with the action of the Galois group, it follows that αf is defined over $\mathbf{Q}$. This proves 5.1. ∎

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