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POLYNOMIAL IDENTITIES OF NONASSOCIATIVE RINGS PART II: FINE POINTS OF THE STRUCTURE THEORY

BY

LOUIS HALLE ROWEN

Introduction

In Part I we have given a general structure theory of nonassociative Ω-rings, based on using central polynomials to obtain correspondence between ideals of a ring and its center. In this part, we investigate several aspects of the theory in detail, obtaining results which are of intrinsic interest but which would have diverted attention from the program of Part I. Specifically, we are interested here in the following questions: (1) What sentences pass formally from a Ω-ring from R to central extensions? (2) If Nil $(R) = 0$, what can one say about $R[\lambda]$? (3) What is the nature of universal PI-rings (defined in [10, Section 1A])? (4) If $BM(R) = 0$, what can one say about $Z(R)$? Each of these questions arise naturally in the course of Part I, and will be considered in an individual section. Although the results are not quoted for the most part, in the applications in Part III, one can readily see how they apply to alternative and Jordan rings.

Notation and definitions are taken from Part I. In particular, R will always denote a Ω-ring with center Z.

1. Sentences passing from Ω-rings to their central extensions

In [10, Section 10], R-stable identities (and central polynomials) were defined, and were characterized as those identities which pass from R to $R[\lambda]$. Two questions naturally arise: (1) When are all identities of R stable? (2) Under what conditions do sentences in the first order logic pass from R to $R[\lambda]$?

The first question already received some treatment in [10, Section 10], where it was observed that if Ω contains an infinite field then every identity of R is R-stable. We give another example, based more intrinsically on the structure of R.

THEOREM 1.1. *Suppose R can be embedded in a semiprime Ω-ring R' with $JR \subseteq R$ and* $\mathrm{Ann}_{R'} J = 0$, *where* $J = \mathrm{Jac}\,(Z(R'))$. *Then every identity of R is R-stable.*

Proof. As shown in [10, Section 1C], it is enough to prove that every identity of R is a sum of completely homogeneous identities of R. Suppose an

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identity $f(X_1, \ldots, X_m)$ of R is not homogeneous in X_1, and let f_i be the sum of those monomials of degree i in X_1. Clearly $f = \sum f_i$; we prove that each f_i is an identity of R, and the theorem will follow by iteration of this procedure on each indeterminate. Let d be the degree of X_1 in f.

Choose $r_1, \ldots, r_m$ in R arbitrarily, and let $y_i = f_i(r_1, \ldots, r_m)$, $0 \leq i \leq d$. For any c in J, for any j, we have $\sum_{i=0}^{d} c^{ji} y_i = f(c^j r_1, r_2, \ldots, r_m) = 0$. (Here we have used the hypothesis that $JR \subseteq R$.) Thinking of y_i as variables, $0 \leq i \leq d$, one can apply the associative Vandermonde argument on this system of $(d+1)$ equations to get $g(c)y_i = 0$ for all i, where $g(c)$ is a product of terms of the form $c^p - c^q$, $p < q$. Let $g(c) = c^t g_1(c)$, where $g_1(c)$ is a polynomial in c having constant term 1. Since $c \in J$, $g_1(c)$ is invertible, so $c^t y_i = 0$ for all i. Hence $(c\langle y_i \rangle)^t = 0$, where $\langle y_i \rangle$ is the ideal of R' generated by y_i. Since R' is semiprime, $c\langle y_i \rangle = 0$ for all i, and for all c in J. Hence $\langle y_i \rangle \subseteq \mathrm{Ann}_{R'} J = 0$, for all i, implying $y_i = 0$. Hence, each f_i is an identity of R, as claimed. Q.E.D.

THEOREM 1.2. *If R is prime, then either Z is a finite field or every identity of R is R-stable.*

Proof. If Z is an infinite field, then every identity of R is R-stable, by [10, Remark 1.8]. Suppose Z is not a field. Then there is a nonzero maximal ideal P; passing to R_P, which is equivalent to R by [10, Corollary 2.1], we may assume that $\mathrm{Jac}(Z) \neq 0$. $(\mathrm{Ann}_R \mathrm{Jac}(Z))(R \mathrm{Jac}(Z)) = 0$ is a product of ideals of R; since $\mathrm{Jac}(Z) \neq 0$, we conclude that $\mathrm{Ann}_R \mathrm{Jac}(Z) = 0$, and apply Theorem 1.1. Q.E.D.

Theorem 1.2 is quite interesting, because it indicates that the situation is nicer when Z is *not* a field. We will interpret this curiosity in Section 3.

An identity of an Ω-ring can be viewed as an atomic universal sentence

$$(\forall x_1, \ldots, x_m)(f(x_1, \ldots, x_m) = 0)$$

in logic (with constant symbols taken from Ω). When we study universal sentences in Section 3, we shall be interested in lifting sentences of the form

$$(\forall x_1, \ldots, x_m)(f_1(x_1, \ldots, x_m) = 0 \vee \cdots \vee f_t(x_1, \ldots, x_m) = 0)$$

from R to $R[\lambda]$. Such sentences will be called *conjunctive identities*, and we shall now generalize results of [10, Section 1C] to conjunctive identities.

Say R is *identity-separated* if, whenever

$$(\forall x_1, \ldots, x_m)(f_1(x_1, \ldots, x_m) = 0 \vee \cdots \vee f_t(x_1, \ldots, x_m) = 0)$$

holds in R, then some f_i is an identity of R. Two Ω-rings R_1 and R_2 are *strongly equivalent* if they satisfy the same *conjunctive identities*. Of course, strongly equivalent Ω-rings are equivalent.

THEOREM 1.3. *If Z contains an infinite domain Z′ and R is torsion-free over Z′ (i.e., every nonzero element of Z′ is regular), then R is identity-separated.*

Proof. Assume that there is a counterexample; i.e., there exist polynomials $f_1(X_1, \ldots, X_m), \ldots, f_t(X_1, \ldots, X_m)$, not identities of R, such that

$$(\forall x_1, \ldots, x_m)(f_1(x_1, \ldots, x_m) = 0 \vee \cdots \vee f_t(x_1, \ldots, x_m) = 0)$$

holds in R. Choose such a counterexample with t minimal. Given $x_1, \ldots, x_m$, there is some i such that $f_i(\alpha x_1, x_2, \ldots, x_m) = 0$ for infinitely many values of α in Z'. The usual Vandermonde argument shows that, for each homogeneous component $f_{i\mu}$ of f_i in X_1, $f_{i\mu}(x_1, \ldots, x_m) = 0$. Applying this argument for each m-tuple $(x_1, \ldots, x_m)$ in R^m, we see that, for each possible homogeneous (in X_1) component $f_{i\mu_i}$ of f_i,

$$(\forall x_1, \ldots, x_m)(f_{1\mu_1}(x_1, \ldots, x_m) = 0 \vee \cdots \vee f_{t\mu_t}(x_1, \ldots, x_m) = 0).$$

On the other hand, for each i, some $f_{i\mu_i}$ is *not* an identity of R. Replacing f_i by $f_{i\mu_i}$, we may assume that f_i is homogeneous in X_1. Repeating this procedure for each X_k, we may assume that each polynomial f_i is completely homogeneous.

Since t is minimal, for each i there exist $x_{i1}, \ldots, x_{in}$ such that

$$f_i(x_{i1}, \ldots, x_{im}) = 0 \quad \text{and} \quad f_{i'}(x_{i1}, \ldots, x_{im}) \neq 0 \text{ for all } i' \neq i.$$

Now, for some i, $f_i(x_{11} + \alpha x_{21}, \ldots, x_{1m} + \alpha x_{2m}) = 0$ for an infinite number of values of α in Z'. Let d_j be the degree of f_i in X_j, and let r_k be the coefficient of λ^k in the expression

$$f_i(x_{11} + \lambda x_{21}, \ldots, x_{1m} + \lambda x_{2m}) \in R[\lambda],$$

for $0 \leq k \leq d_1 + \cdots + d_m$. But $f_i(x_{11} + \alpha x_{21}, \ldots, x_{1m} + \alpha x_{2m}) = 0$ implies $\sum_k \alpha^k r_k = 0$; since we have an infinite number of such α, the standard Vandermonde argument yields $r_k = 0$, for all k. In particular,

$$0 = r_0 = f_i(x_{11}, x_{12}, \ldots, x_{1m}),$$

implying $i = 1$. On the other hand, with $d = d_1 + \cdots + d_n$,

$$0 = r_d = f_i(x_{21}, \ldots, x_{2m}),$$

implying $i = 2$. This contradiction shows that there cannot be a counterexample to the assertion. Q.E.D.

COROLLARY 1.4. *If Z contains an infinite field then R is identity-separated.*

PROPOSITION 1.5. (i) *For any Ω-ring R, $R[\lambda]$ is identity-separated.*

(ii) *R and $R[\lambda]$ are strongly equivalent iff R is identity-separated and every identity of R is R-stable.*

Proof. (i) Use a simplified version of the proof of Theorem 1.3 (using λ instead of an infinite number of α in Z').

(ii) Suppose R and $R[\lambda]$ are strongly equivalent. Then R and $R[\lambda]$ are equivalent, so every identity of R is stable (cf. [10, Proposition 1.3]). Also, by (i), $R[\lambda]$ is identity-separated, implying R is identity-separated (because of the strong equivalence of R and $R[\lambda]$).

Conversely, suppose that R is identity-separated, and every identity of R is R-stable. If

$$\mathscr{L} = (\forall x_1, \ldots, x_n)(f_1(x_1, \ldots, x_n) = 0 \vee \cdots \vee f_t(x_1, \ldots, x_n) = 0)$$

holds in R then, by supposition, some f_i is an identity of R, hence of $R[\lambda]$, so $\mathscr{L}$ holds in $R[\lambda]$. On the other hand, $R \subseteq R[\lambda]$, so every universal sentence of $R[\lambda]$ holds in R. Therefore, R and $R[\lambda]$ are strongly equivalent. Q.E.D.

PROPOSITION 1.6. *Let $\lambda_1, \ldots, \lambda_u$ be associative, commutative indeterminates over R. If R and $R[\lambda_1]$ are strongly equivalent, then R and $R[\lambda_1, \ldots, \lambda_u]$ are strongly equivalent.*

Proof. By [10, Theorem 1.6], R and $R[\lambda_1, \ldots, \lambda_u]$ are equivalent. Suppose

$$\mathscr{L} = (\forall x_1, \ldots, x_m)(f_1(X_1, \ldots, X_m) = 0 \vee \cdots \vee f_t(X_1, \ldots, X_m) = 0)$$

holds in R. By Proposition 1.5 (ii), R is identity-separated, so some f_i is an identity of R, hence of $R[\lambda_1, \ldots, \lambda_u]$; therefore $\mathscr{L}$ holds in $R[\lambda_1, \ldots, \lambda_u]$. Q.E.D.

We are now ready to present an interesting example of the above concepts.

THEOREM 1.7. *If Z contains an infinite domain over which R is torsion free, then R and $R[\lambda_1, \ldots, \lambda_t]$ are strongly equivalent.*

Proof. By [10, Remark 1.8], every identity of R is R-stable. Moreover, by Theorem 1.3, R is identity-separated. Thus, by Proposition 1.5, R and $R[\lambda_1]$ are strongly equivalent, so we are done by Proposition 1.6. Q.E.D.

2. The Jacobson-Smiley radical of $R[\lambda]$

In [10, Section 3] we would very much have liked to show in general that Nil $(R) = 0$ implies Jac $(R[\lambda]) = 0$, a well-known theorem of Amitsur in the associative case. Incidentally, using [2], one can characterize Jac (R) as follows: Say an element r of R is *left quasiregular* if $1 - r$ is not contained in a proper left ideal of R; an ideal of R is (*left*) *quasiregular* if each element is (left) quasiregular. The sum of two quasiregular ideals is quasiregular, as one can see without difficulty. Hence there is a unique maximal quasiregular ideal, which turns out to be Jac (R).

Using some of the ideas in the proof of Amitsur's theorem given in [4], we shall generalize Amitsur's theorem to all power-associative rings, but shall use a different generalization of the Jacobson radical. Call an element r in R *left* (*resp. right*) *quasiinvertible* if $(1 - r)$ has a left (resp. right) inverse in R, and let $J(R)$ be the sum of all ideals of left quasiinvertible elements, called the Jacobson-Smiley radical [9]. Clearly $J(R)$ is a quasiregular ideal, so $J(R) \subseteq$ Jac (R), and equality holds when R is associative (or alternative cf. Zhevlakov [11], [12], [13]). Some interesting work on $J(R)$ has been done by McCrimmon [7].

PROPOSITION 2.1. *Suppose R is power-associative and B is an ideal of* $R[\lambda]$. *For any nonzero* $p(\lambda) = \sum_{i=0}^{t} r_i \lambda^i$ *in B such that the number of nonzero coefficients (of* λ*) is minimal, the ring A (with* 1*) generated by these* r_i *is commutative and associative.*

Proof. We shall prove that every multilinear identity of **Z** is a multilinear identity of A. Since $[X_1, X_2]$ and $[X_1, X_2, X_3]$ are multilinear identities of **Z**, the assertion will follow immediately. Suppose $h(X_1, \ldots, X_m)$ is a multilinear identity of **Z**. We must show, for all x_{ij} in $\{r_0, \ldots, r_t\}$ and for all $f_i(X_1, \ldots, X_{u_i})$, that

$$h(f_1(x_{11}, \ldots, x_{1u_1}), \ldots, f_m(x_{m1}, \ldots, x_{mu_m})) = 0.$$

Viewing the f_i as a sum of completely homogeneous components, we may assume without loss of generality that the f_i are completely homogeneous. Moreover, suppose f has degree d_k in the indeterminate X_k, for each k. Replacing the d_k occurrences of X_k in each monomial of f by k distinct indeterminates, we can construct a multilinear polynomial $\hat{f}_i$ from f; clearly $A(f_i) \subseteq A(\hat{f}_i)$, so we may assume that each f_i is multilinear. Let $r = x_{mu_m}$. The subring of A generated by 1 and r is associative and commutative, and thus satisfies every multilinear identity of **Z**; in particular,

$$h(f_1(r, \ldots, r), \ldots, f_m(r, \ldots, r)) = 0.$$

Hence

$$h(f_1(p(\lambda), r, \ldots, r), f_2(r, \ldots, r), \ldots, f_m(r, \ldots, r))$$

is an element of B, with fewer nonzero coefficients than $p(\lambda)$, and thus must be 0. In particular, $h(f_1(x_{11}, r, \ldots, r), f_2(r, \ldots, r), \ldots) = 0$. Then

$$h(f_1(x_{11}, p(\lambda), r, \ldots, r), f_2(r, \ldots, r), \ldots)$$

is an element of B, with fewer nonzero coefficients than $p(\lambda)$, and is therefore 0; hence

$$h(f_1(x_{11}, x_{12}, r, \ldots, r), f_2(r, \ldots, r), \ldots) = 0.$$

Continuing in this manner, we conclude that

$$h(f_1(x_{11}, x_{12}, \ldots, x_{1u_1}), \ldots, f_m(x_{m1}, \ldots, x_{mu_m}) = 0,$$

proving the assertion. Q.E.D.

We now use an idea of Herstein [4].

COROLLARY 2.2. *With notation as in Proposition* 3.1, *assume* $\lambda p(\lambda)$ *is right quasiinvertible, i.e.,* $(1 - \lambda p(\lambda))q(\lambda) = 1$ *for some* $q(\lambda)$. *Then* $q(\lambda) \in A[\lambda]$, $q(\lambda) \times (1 - \lambda p(\lambda)) = 1$, *and A is a nilpotent ring.*

Proof. Comparing coefficients of λ^0 in $q(\lambda) - \lambda p(\lambda)q(\lambda) = 1$ shows $q(\lambda) = 1 + \lambda q_1$, for some q_1 in $R[\lambda]$. Write p for $p(\lambda)$. Then

$$\lambda q_1 - (\lambda p)(1 + \lambda q_1) = 0,$$

so $q_1 = p + \lambda p q_1$. Hence $q_1 = p + \lambda p^2 + \lambda^2 p(pq_1)$; continuing in this way, we have

$$q_1 = \sum_{i=1}^{t} \lambda^{i-1} p^i + \lambda^t p(p(p \cdots (pq_1) \cdots)).$$

Choosing $t > \deg q_1$ (as a polynomial in λ), we see that all the coefficients of q_1 are coefficients of $\sum_{i=1}^{t} \lambda^{i-1} p^i \in A[\lambda]$, so $q(\lambda) \in A[\lambda]$. Since $A[\lambda]$ is commutative, we see that $q(\lambda)(1 - \lambda p(\lambda)) = 1$.

For any prime homomorphic image $\bar{A}$ of A, we have $(\bar{1} - \overline{\lambda p(\lambda)})\overline{q(\lambda)} = \bar{1}$; comparing degrees, since $\bar{A}$ is an integral domain, we have $\overline{\lambda p(\lambda)} = 0$. Thus, $\lambda p(\lambda) \in (\text{Nil}\,(A))[\lambda]$; in particular, $r_0, \ldots, r_t$ are all nilpotent. Suppose $r_t^k = 0$. Then $p(\lambda)r_t^{k-1}$ has fewer nonzero coefficients than p_t, so $p(\lambda)r_t^{k-1} = 0$; in particular, $x_1 r_t^{k-1} = 0$ for any x_1 in A. Continuing in this way, we see that $x_1 \cdots x_k = 0$ for all x_i in A; i.e., $A^k = 0$. Q.E.D.

THEOREM 2.3. *If R is power-associative and* Nil $(R) = 0$, *then $R[\lambda]$ has no nonzero left or right quasiinvertible ideals. In particular, $J(R[\lambda]) = 0$.*

Proof. Suppose there is a nonzero ideal J of right quasiinvertible elements, and choose nonzero $p(\lambda) = \sum_{i=0}^{t} r_i \lambda^i$ in J, such that the number of nonzero coefficients r_i is minimal. (For convenience, assume that $r_t \neq 0$.) Let

$$I = \left\{ \sum_i r_i' \lambda^i \in J \,\middle|\, r_i' = 0 \text{ if } r_i = 0 \right\} \quad \text{and} \quad I_t = \left\{ r_t' \,\middle|\, \sum_{i=0}^{t} r_i' \lambda^i \in I \right\}.$$

Clearly I_t is an ideal of R, and every element of I_t is nilpotent, by Corollary 2.2. Hence $I_t = 0$, contrary to $0 \neq r_t \in I_t$. Hence there are no nonzero right quasiinvertible ideals.

An analogous proof shows that there are no left quasiinvertible ideals, so $J(R[\lambda]) = 0$. Q.E.D.

Additional results can be obtained by considering associative subrings of R. Let $N_1(R) = \{r \in R \mid [r, R, R] = [R, R, r] = 0\}$, an associative subring of R. Let us state a sample result without proof.

PROPOSITION 2.4. *If* Nil $(N_1(R)) = 0$ *and if every nonzero ideal of R intersects $N_1(R)$ nontrivially, then* Jac $(R[\lambda]) \neq 0$.

Obviously at this point we are interested in situations for which $J(R) =$ Jac (R) (and in particular when $J(R[\lambda]) =$ Jac $(R[\lambda])$), especially since this is clearly true when R is associative. Our results reduce this question to cases when R is purely nonassociative.

PROPOSITION 2.5. *Suppose R is semiprime with $R_1 = R/\text{Ann}\, U(R)$ and $R_2 = R/U(R)$, as in* [10, *Proposition* 5.4]. *Let A be an ideal of R, and let A_i be the canonical homomorphic image of A in R_i. We have $A \subseteq J(R)$ (resp. $A \subseteq \text{Jac}\,(R)$) iff $A_i \subseteq J(R_i)$ (resp. $A \subseteq \text{Jac}\,(R_i)$) for $i = 1, 2$.*

Proof. Clearly if $A \subseteq J(R)$ (resp. $A \subseteq \text{Jac}\,(R)$) then $A_i \subseteq J(R_i)$ (resp. $A_i \subseteq \text{Jac}\,(R_i)$), since the image of a left quasiinvertible (resp. left quasiregular) element is left quasiinvertible (resp. left quasiregular). Conversely, if each A_i is left quasiinvertible then, for each a in A, we can find y_i in R_i such that $y_i(1 - a_i) = 1$ in R_i. Letting y_i' be the preimage of y_i in A, we have $1 - y_1'(1 - a) \in \text{Ann}\, U(R)$ and $1 - y_2'(1 - a) \in U(R)$, so

$$\begin{aligned} 0 &= (1 - y_2'(1 - a))(1 - y_1'(1 - a)) \\ &= 1 - y_2'(1 - a) - y_1'(1 - a) + (y_2'(1 - a))(y_1'(1 - a)) \\ &= 1 - (y_2' + y_1' - (y_2'(1 - a))y_1')(1 - a) \end{aligned}$$

since $y_2'(1 - a) \in N(R)$. This proves a is left quasiinvertible, for all a in A, so $a \subseteq J(R)$. The proof that each $A_i \subseteq \text{Jac}\,(R_i)$ implies $A \subseteq \text{Jac}\,(R)$ is even more immediate. Q.E.D.

Since R_1 is associative, we know $J(R_1) = \text{Jac}\,(R_1)$.

COROLLARY 2.6. *Suppose R is semiprime, notation as in Proposition* 2.5.
(i) $J(R) = 0$ *iff* $J(R_1) = 0$ *and* $J(R_2) = 0$.
(ii) $\text{Jac}\,(R) = 0$ *iff* $\text{Jac}\,(R_1) = 0$ *and* $\text{Jac}\,(R_2) = 0$.

Proof. We work with $J(\ )$; the proof for $\text{Jac}\,(\ )$ is the same. Suppose $J(R) \neq 0$; then by Proposition 2.5, the images of $J(R)$ in R_1 and R_2 are left quasiinvertible and obviously cannot both be 0. Thus, if $J(R_1) = 0$ and $J(R_2) = 0$, we have $J(R) = 0$, by the contrapositive.

Conversely, suppose $J(R_1) \neq 0$ or $J(R_2) \neq 0$, and let J_1, J_2 be the respective preimages (in R) of $J(R_1)$ and $J(R_2)$. Then either $\text{Ann}\, U(R) \subset J_1$ or $U(R) \subset J_2$. Then $J_1 \cap U(R)$ or (respectively) $J_2 \cap \text{Ann}\, U(R)$ is a nonzero left quasiinvertible ideal of R. (Proof. If $\text{Ann}\, U(R) \subset J_1$ then

$$0 \neq J_1 U(R) \subseteq J_1 \cap U(R),$$

and $J_1 \cap U(R)$ is left quasiinvertible, by Proposition 2.5; a similar proof holds for $J_2 \cap \text{Ann}\, U(R)$ if $U(R) \subset J_2$.) This proves $J(R) \neq 0$. Q.E.D.

COROLLARY 2.7. *Suppose* $J(R) = 0$. *We have* $\text{Jac}\,(R) = 0$ *iff* $\text{Jac}\,(R/U(R)) = 0$.

Proof. By Corollary 2.6, $J(R_1) = 0$. But R_1 is associative, so $\text{Jac}\,(R_1) = 0$. Hence the result follows immediately from Corollary 2.6 (ii). Q.E.D.

Thus, we have reduced the question of $J(R) = \text{Jac}\,(R)$ in many cases to the situation where R is purely nonassociative. Actually, we should change $J(R)$ as follows.

Given a successor ordinal number μ, define $J_\mu(R)$ as the preimage of $J(R/J_{\mu-1}(R))$; define $J_0(R)=0$, and for limit ordinals μ, define $J_\mu(R)=\bigcup_{\beta<\mu} J_\beta(R)$. Then define $J'(R)=J_\mu(R)$ where μ is the ordinal of R (as a set). For any R, $J(R/J'(R))=0$. Often $J(R)=J'(R)$; for example, this is immediate when R is associative, and is easy when R is alternative (cf. Smiley [11]). If $J(R)=0$ then $J'(R)=0$, so, in particular, Theorem 2.3 says for power-associative strongly semiprime rings R, $J'(R[\lambda])=0$. The correct question is:

Question 2.8. (a) For what Ω-rings does Jac $(R[\lambda])=J'(R[\lambda])=0$?
(b) For what Ω-rings does Jac $(R)=J'(R)$?

PROPOSITION 2.9. Jac $(R[\lambda])=0$ *iff* Jac $(R/U(R))[\lambda]=0$.

Proof. Obviously $U(R[\lambda])=U(R)[\lambda]$, so the result follows from Proposition 2.6. Q.E.D.

Thus to prove a central variety of Ω-rings is Kaplansky, we need check only the purely nonassociative members.

3. Universal PI-rings

One of the main features of the associative PI-theory is the "ring of generic matrices," which is the universal PI-ring with respect to the identities of $M_n(\mathbf{Z})$. This ring is absolutely prime, and its ring of central quotients is a division ring (of dimension n^2 over its center), which is an example of utmost importance in the theory of division rings (cf. [1]). This fact is motivation enough for a detailed study of universal Ω-rings, defined in [10, Section 1A]; also, one can often learn more about an Ω-ring R by studying its universal Ω-ring. (This happens in particular in the study of alternative rings). Thus, we shall spend this section examining universal Ω-rings.

Recall that if $\tilde{\mathscr{S}}$ is the set of identities of R, then we can form the ring $\Omega\{X\}/\tilde{\mathscr{S}}$, which has the following universal property. Let $\bar{X}_i$ be the canonical image of X_i in $\Omega\{X\}/\tilde{\mathscr{S}}$. Given any elements $r_1, \ldots, r_2$ in R, there is a homomorphism $\Omega\{X\}/\tilde{\mathscr{S}}\to R$ so that $\bar{X}_i\mapsto r_i$ for all i. We call $\Omega\{X\}/\tilde{\mathscr{S}}$ the *universal* Ω-*ring* of R, written $\mathscr{U}(R)$. A Ω-ring of the form $\mathscr{U}(R)$ (for suitable R) is called *universal.*

Some examples of universal Ω-rings are the "free" associative ring, the "free" commutative, associative ring, the "free" Jordan ϕ-algebra, the "free" alternative ϕ-algebra, and Amitsur's ring of generic $n\times n$ matrices (cf. [1]). We shall encounter other examples in Part III.

An obvious question is: Which homomorphic images of $\Omega\{X\}$ are universal (with respect to a suitable Ω-ring)? The answer is given in [10, Section 1A]: If A is an ideal of $\Omega\{X\}$, then $\Omega\{X\}/A$ is universal iff, for every endomorphism ϕ of

$\Omega\{X\}$, $\phi(A) \subseteq A$. Accordingly, call an ideal A of a Ω-ring R a *T-ideal* if $\phi(A) \subseteq A$ for every endomorphism ϕ of R.

Remark 3.1. If W is a universal Ω-ring and A is a T-ideal of W, then W/A is a universal Ω-ring. (Proof. Write $W = \Omega\{X\}/B$, for B a suitable T-ideal of $\Omega\{X\}$, and write $A = A'/B$. For any endomorphism ϕ of $\Omega\{X\}$, ϕ induces an endomorphism $\bar{\phi}$ of $W = \Omega\{X\}/B$ and thus $\bar{\phi}(A) \subseteq A$; it follows that $\phi(A') \subseteq A'$, so $W/A \approx \Omega\{X\}/A'$ is universal.

Motivated by Remark 3.1, we shall look at more ways of obtaining T-ideals of universal Ω-rings. In general, W will be a universal Ω-ring, and, under the canonical surjection $\Omega\{X\} \to W$, the images of X_i will be written as $\bar{X}_i$; we shall still talk of the elements of W as polynomials in the $\bar{X}_i$, by a slight abuse of language.

LEMMA 3.2. *Suppose W is a universal ring and $f_1, \ldots, f_t \in W$. If ϕ is an endomorphism of W then there is an epimorphism $\psi\colon W \to W$ such that $\psi(f_i) = \phi(f_i)$, $1 \leq i \leq t$, and $\psi(T) = T$ for every T-ideal T of W.*

Proof. Suppose the indeterminates $\bar{X}_1, \ldots, \bar{X}_m$ occur in $f_1, \ldots, f_t$. Define ψ by $\psi(\bar{X}_i) = \phi(\bar{X}_i)$, $1 \leq i \leq m$, and $\psi(\bar{X}_i) = \bar{X}_{i-m}$ for $i > m$. Clearly $\psi(f_i) = \phi(f_i)$, $1 \leq i \leq t$. For any $f(\bar{X}_1, \ldots, \bar{X}_k)$ in T, we have

$$f(\bar{X}_1, \ldots, \bar{X}_k) = \psi(f(\bar{X}_{m+1}, \ldots, \bar{X}_{m+k}),$$

so $\psi(T) = T$. Q.E.D.

One point of Lemma 3.2 is that to check that an ideal A of W is a T-ideal, it suffices to check that $\psi(A) \subseteq A$ for every *onto* endomorphism ψ of W. But, in this case, $\psi(A)$ is an ideal. So, heuristically, if A is the "largest" ideal having a certain property which can be expressed in terms of the ring operations, $\psi(A)$ will have this property, and we will conclude $\psi(A) \subseteq A$. For example, if $A = \text{Nil}\ (W)$ then every element of A is nilpotent, so every element of $\psi(A)$ is nilpotent, and we conclude that $\psi(A) \subseteq A$. Let us formalize this argument.

An *atomic formula in* $x_1, \ldots, x_m$ has the form $f(x_1, \ldots, x_m) = 0$ where f is a polynomial, and we write formally x_i instead of X_i. We are interested in an expression of the form $(Q_2 x_{\pi 2}) \cdots (Q_m x_{\pi m})(F_1 \wedge \cdots \wedge F_k)$, where each Q_i is a quantifier, π is a permutation of $(2, \ldots, m)$, and each F_i is a formula in $x_1, \ldots, x_m$. (Note that all occurrences of x_1 are free, and all occurrences of all other x_i are bounded.) Such an expression we call an *atomic condition*, written as $L(x_1)$.

DEFINITION 3.3. An ideal B is *atomically defined* by a set $\mathcal{L}$ of atomic conditions if, for each b in B, we can find some atomic condition L in $\mathcal{L}$ such that $L(b)$ holds, and if B contains all ideals having this property.

THEOREM 3.4. *If B is an atomically defined ideal in a universal Ω-ring W, then B is a T-ideal.*

Proof. For every onto endomorphism ψ of W, and for every atomic condition L of $\mathscr{L}$, clearly $L(b)$ implies $L(\psi(b))$. Moreover $\psi(B)$ is an ideal; so, by definition, $\psi(B) \subseteq B$. By Lemma 3.2, B is a T-ideal. Q.E.D.

To apply Theorem 3.4, we need only show certain ideals are atomically defined.

Remark 3.5. Nil (R) is atomically defined. Just take

$$\mathscr{L} = \{x_1 = 0; x_1^2 = 0; x_1(x_1x_1) = 0; (x_1x_1)x_1 = 0; \ldots\}.$$

Remark 3.6. $U(R)$ is atomically defined. Just take

$$\mathscr{L} = \{(\forall x_2)(\forall x_3)([x_1, x_2, x_3] = 0 \wedge [x_2, x_1, x_3] = 0 \wedge [x_2, x_3, x_1] = 0)\}.$$

Remark 3.7. If f is a polynomial (in $\Omega\{X\}$) then $\langle W(f)\rangle$, the ideal generated by $W(f)$, is atomically defined. (This is because, for any w in W, every element of $\langle w\rangle$ can be written as a sum of products each of which contain w.) In particular, $D(R)$ is atomically defined (taking $f = [X_1, X_2, X_3]$).

Remark 3.8. $J(R)$ is atomically defined. Take

$$\mathscr{L} = \{(\exists x_2)((1 - x_2)(1 - x_1) = 1)\}.$$

Remark 3.9. Jac (R) is atomically defined, seen by using the characterization of Jac (R) as the largest left quasiregular ideal of R, stated in Section 2.

Remark 3.10. In [3], Brown and McCoy define $BM(R)$ as the largest "F-regular" ideal of R (and later show $BM(R) = \bigcap$ {maximal ideals of R}). Using the Brown-McCoy definition, one sees easily that $BM(R)$ is atomically defined.

COROLLARY 3.11. *If W is universal then* Nil (W), $U(W)$, $D(W)$, $J(W)$, Jac (W), *and* $BM(W)$ *are T-ideals of W.*

Recall that for additive subgroups A, B of R, $(B:A)$ is defined as the largest ideal of R such that $(B:A)A \subseteq B$.

PROPOSITION 3.12. *If A, B are T-ideals of a universal Ω-ring W, then $(B:A)$ is also a T-ideal of W.*

Proof. By Lemma 3.2, we need $\psi(B:A) \subseteq (B:A)$ for every epimorphism ψ of W such that $\psi(A) = A$ and $\psi(B) = B$. Since $\psi(B:A)$ is an ideal, we need only show that $\psi(f) \in (B:A)$ for every element f of $(B:A)$. But

$$\psi(f)B = \psi(f)\psi(B) = \psi(fB) \subseteq \psi(A) = A,$$

so $\psi(f) \in (B:A)$, as desired. Q.E.D.

Having a wide range of T-ideals at our disposal, we are ready to prove some facts about universal Ω-rings. Our motivation is from [8]. The first step is to pass information from R to $\mathscr{U}(R)$.

THEOREM 3.13. *Suppose B is a nonzero ideal of $\mathscr{U}(R)$, atomically defined by a set $\mathscr{L}$ of atomic conditions (cf. Definition 3.3). Then there is a nonzero ideal of R which is atomically defined by $\mathscr{L}$.*

Proof. Take some nonzero $f(X_1, \ldots, X_m)$ in B. By Theorem 3.4, B is a T-ideal of $\mathscr{U}(R)$. Hence there exist elements $r_1, \ldots, r_m$ in R such that $f(r_1, \ldots, r_m) \neq 0$. It is easy to see $f(r_1, \ldots, r_m)$ is in the ideal of R atomically defined by $\mathscr{L}$, which is therefore nonzero. Q.E.D.

COROLLARY 3.14. *If $U(R)$, $D(R)$, Nil (R), $J(R)$, Jac (R), or $BM(R)$ is 0, then (respectively) $U(\mathscr{U}(R))$, $D(\mathscr{U}(R))$, $J(\mathscr{U}(R))$, Jac $(\mathscr{U}(R))$, or $BM(\mathscr{U}(R))$ is 0.*

Proof. Use Corollary 3.13 and Remarks 3.5–3.10. Q.E.D.

At this stage we can piece together information using the obvious fact that if R_1 and R_2 are equivalent then $\mathscr{U}(R_1) = \mathscr{U}(R_2)$.

PROPOSITION 3.15. *If R is power-associative and every identity of R is R-stable and if* Nil $(R) = 0$, *then $J(\mathscr{U}(R)) = 0$.*

Proof. Let $W = \mathscr{U}(R)$. By Theorem 2.3, $J(R[\lambda]) = 0$. But $R[\lambda]$ is equivalent to R, so $W = \mathscr{U}(R[\lambda])$, implying $J(W) = 0$, by Corollary 3.14. Q.E.D.

In a Kaplansky class, we can strengthen Proposition 3.15 quite a bit, as we see after an easy lemma.

LEMMA 3.16. *If R is semiprime and $\mathscr{P}$ is a class of prime ideals of intersection 0, then for every ideal B of R,* $\operatorname{Ann}_R B = \bigcap \{P \in \mathscr{P} \mid B \nsubseteq P\}$.

Proof. For any prime ideal $P \nsupseteq B$, we have $(\operatorname{Ann}_R B)B = 0 \subseteq P$, implying $\operatorname{Ann}_R B \subseteq P$. Thus, for $A = \bigcap \{P \in \mathscr{P} \mid B \nsubseteq P\}$, we conclude $\operatorname{Ann}_R B \subseteq A$. Conversely,

$$AB \subseteq A \cap B \subseteq \bigcap \{P \in \mathscr{P}\} = 0. \qquad \text{Q.E.D.}$$

THEOREM 3.17. *For every universal Ω-ring W in a Kaplansky class,* Nil $(W) = BM(W)$.

Proof. Clearly Nil $(W) \subseteq BM(W)$. Passing to the universal Ω-ring $W/\text{Nil}\,(W)$, we may assume W is strongly semiprime and need prove $BM(W) = 0$. Since $BM(W) =$ Jac (W), by [10, Proposition 3.22], we need only prove $J_1 = 0$, where $J_1 =$ Jac (W).

So suppose $J_1 \neq 0$. Let $\{P_\gamma \mid \gamma \in \Gamma\}$ be a set of strongly prime ideals of W with zero intersection, such that W/P_γ is in our Kaplansky class for each γ in Γ (cf. [10, Definition 3.18]). Let $\Gamma_1 = \{\gamma \in \Gamma \mid J_1 \nsubseteq P_\gamma\}$. For each γ in Γ_1, let $W_\gamma = W/P_\gamma$; $(J_1 + P_\gamma)/P_\gamma$ is a nonzero left quasiregular ideai of W_γ, implying

Jac $(W_\gamma) \neq 0$, so

$$0 \neq Z \cap \text{Jac}\,(W_\gamma) \subseteq \text{Jac}\, Z(W_\gamma).$$

Hence, by Theorem 1.1, every identity of W_γ is W_γ-stable. But

$$\text{Ann}\, J_1 = \bigcap \{P_\gamma \mid \gamma \in \Gamma_1\},$$

by Lemma 3.16; it follows immediately that every identity of $W/\text{Ann}\, J_1$ is $W/\text{Ann}\, J_1$-stable. Let $\bar{W} = W/\text{Ann}\, J_1$; we conclude that $\bar{W}$ is equivalent to $\bar{W}[\lambda]$.

But $\bar{W}$ is universal, so $\bar{W}$ is the universal Ω-ring of $\bar{W}[\lambda]$, implying Jac $\bar{W} = 0$, by Corollary 3.14. (Recall that Jac $\bar{W}[\lambda] = 0$, by [10, Definition 3.18].) Hence $J_1 \subseteq \text{Ann}\, J_1$, implying $J_1 = 0$. Q.E.D.

The point of the complicated argument of Theorem 3.17 is to bypass using W-stable identities. Our next goal is to show that $\mathscr{U}(R)$ is prime if R is prime. For this result, we need to assume stability of the defining identities, even in the presence of central polynomials, as shown in the following example.

Example 3.18. If R has an identity which is *not* R-stable and if R satisfies a central polynomial, then $\mathscr{U}(R)$ is not prime. (Indeed, $\mathscr{U}(R)$ has an infinite center; if $\mathscr{U}(R)$ were prime then all identities of $\mathscr{U}(R)$ would be $\mathscr{U}(R)$-stable, by Theorem 1.2, contrary to the assumption on R). In particular, if F is a finite field, $\mathscr{U}(F)$ is not prime, for, if n is the order of F, then $X^n - X$ is an identity of F which is *not* F-stable. (Incidentally, in $\mathscr{U}(F)$, $0 = \bar{X}_1(\bar{X}_1^{n-1} - 1)$ yielding an explicit way of seeing that $\mathscr{U}(F)$ is not an integral domain.)

THEOREM 3.19. *Suppose R is a Ω-ring in which every identity is R-stable. If R is prime then $\mathscr{U}(R)$ is prime.*

Proof. Suppose R is prime and $\mathscr{U}(R)$ is not prime. Then there exist nonzero ideals A, B of $\mathscr{U}(R)$, such that $AB = 0$. Then A and B have respective nonzero elements $f(\bar{X}_1, \ldots, \bar{X}_m)$ and $g(\bar{X}_1, \ldots, \bar{X}_n)$; adding dummy indeterminates (if necessary) to f or g, we may assume $m = n$. Pick $r_1, \ldots, r_m, r'_1, \ldots, r'_m$ in R, such that $f(r_1, \ldots, r_m) \neq 0$ and $g(r'_1, \ldots, r'_m) \neq 0$.

Passing to $R[\lambda]$, where λ is a commuting, associating indeterminate over R, we have

$$f(r_1\lambda + r'_1(1-\lambda), \ldots, r_m\lambda + r'_m(1-\lambda)) \neq 0$$

and

$$g(r_1\lambda + r'_1(1-\lambda), \ldots, r_m\lambda + r'_m(1-\lambda)) \neq 0,$$

seen by respectively specializing $\lambda \mapsto 1$ and $\lambda \mapsto 0$. However, $\mathscr{U}(R) = \mathscr{U}(R[\lambda])$ since R is equivalent to $R[\lambda]$ by hypothesis, whereas

$$\langle f(r_1\lambda + r'_1(1-\lambda), \ldots, r_m\lambda + r'_m(1-\lambda))\rangle \times \langle g(r_1\lambda + r'_1(1-\lambda), \ldots, r_m\lambda + r'_m(1-\lambda))\rangle \neq 0$$

since $R[\lambda]$ is prime. But clearly, for any nonzero element x of

$$\langle f(r_1\lambda + r_1'(1-\lambda), \ldots)\rangle\langle g(r_1\lambda + r_1'(1-\lambda), \ldots)\rangle,$$

there is a homomorphism $\mathscr{U}(R) \to R[\lambda]$ sending $\bar{X}_j$ to $r_j\lambda + r_j'(1-\lambda)$, $1 \leq j \leq m$, whose range contains x; this says that $x \neq 0$ is in the image of $AB = 0$, a contradiction. Thus we conclude $AB = 0$ implies $A = 0$ or $B = 0$, so R is prime. Q.E.D.

COROLLARY 3.20. *If R is strongly prime and every identity of R is R-stable, then $\mathscr{U}(R)$ is strongly prime.*

Proof. By Theorem 3.19, $\mathscr{U}(R)$ is prime, and Nil $(\mathscr{U}(R)) = 0$ by Corollary 3.14. Q.E.D.

COROLLARY 3.21. *If R is strongly prime and every identity of R is R-stable, then $\mathscr{U}(R)$ is strongly prime and $J(\mathscr{U}(R)) = 0$.*

At this point, we may wish to see what other properties would pass from R to $\mathscr{U}(R)$. This seems to be quite an interesting area, and we only treat a small part of it, namely those logically elementary sentences passing from R to $\mathscr{U}(R)$. In other words, we are going in a different direction from Corollary 3.14. This is inspired by the fact that sentences of this sort are used by Amitrsur [1] in proving that the universal PI-ring of an associative central division **Q**-algebra is an order in an associative central division algebra (of the same dimension) which often fails to have a maximal subfield which is Galois over the center.

We proceed in a formal manner (cf. [6]). The language will be a "first-order" language, whose atomic formulas have the form "$f = g$" where $f, g \in \Omega\{X\}$. (Thus, our list of "constants" includes the elements of Ω.) All formulas are built inductively from atomic formulas, via the unary operation $\sim$ and the binary operations $\wedge$, $\vee$, and $\rightarrow$. Using the laws of associativity of $\wedge$ and $\vee$, we can often remove parentheses, without ambiguity, and we also use other set-theoretic properties, without giving justifications. (In particular, the formula $P_1 \rightarrow P_2$ can be replaced by $P_2 \vee (\sim P_1)$.) Quantification is done in the usual way, with $\forall$ or $\exists$.

Given a Ω-ring R, with universal Ω-ring $W = \mathscr{U}(R)$, let $\bar{X}_i$ denote the canonical image of X_i in W. We shall also use the $\bar{X}_i$ in our language, when analyzing sentences in W, and shall call them "indeterminates" (with slight abuse of language). A sentence without indeterminates is "indeterminate-free." A sentence without any quantifiers (resp. without $\forall$, without $\exists$), will be called "quantifier-free" (resp. existential-free, universal-free). Now the fact that $f(X_1, \ldots, X_m)$ is an identity of R can be written as

$$(\forall x_1, \ldots, x_m)(f(x_1, \ldots, x_m) = 0).$$

On the other hand, the fact that $f(X_1, \ldots, X_m)$ is an identity of W can be written as either

$$(\forall x_1, \ldots, x_m)(f(x_1, \ldots, x_m) = 0), (\forall x_2, \ldots, x_m)(f(\bar{X}_1, x_2, \ldots, x_m) = 0), \ldots,$$

or $f(\bar{X}_1, \ldots, \bar{X}_m) = 0$. This suggests a procedure to transform an elementary sentence of W to a quantifier-free sentence. The method is as follows:

Given $(\forall x_i)\mathscr{A}(x_i)$, replace x_i in $\mathscr{A}(x_i)$ by some $\bar{X}_j$ not occurring in $\mathscr{A}$.

Given $(\exists x_i)\mathscr{A}(x_i)$, take some element $f(\bar{X}_1, \ldots, \bar{X}_t)$ in W for which

$$\mathscr{A}(f(\bar{X}_1, \ldots, \bar{X}_t))$$

holds, and replace x_i by $f(\bar{X}_1, \ldots, \bar{X}_t)$.

Clearly, after a finite number of steps, any given sentence $\mathscr{L}$ holding in W can be replaced by a quantifier-free sentence $\mathscr{L}'$ which holds in W. The positions of the unary and binary operations are still the same, only the atomic formulas have been modified.

We now apply this algorithm to a well-known class of sentences. Say a formula has *type n* if it has the form $P_1 \vee \cdots \vee P_n \vee \sim P_{n+1} \vee \cdots \vee \sim P_t$, each P_j atomic. A *Horn sentence* is a sentence of the form

$$(Q_1 x_1) \cdots (Q_m x_m)(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_v),$$

where the Q_i are quantifiers and each $\mathscr{A}_i$ has type 0 or type 1. Horn sentences are interesting because they are preserved in filtered products. (See [6, p. 145 and Theorem 1, p. 169].)

THEOREM 3.22. *Any universal Horn sentence holding in R, also holds in W.*

Proof. Suppose $\mathscr{L} = (\forall x_1, \ldots, x_m)(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_v)$ is a Horn sentence holding in R, and assume that $\sim \mathscr{L} = (\exists x_1 \cdots \exists x_m)(\sim \mathscr{A}_1 \vee \cdots \vee \sim \mathscr{A}_v)$ holds in W. Applying the preceding algorithm, $(\sim \mathscr{L})' = \sim \mathscr{A}'_1 \vee \cdots \vee \sim \mathscr{A}'_v$ holds in W, where each $\mathscr{A}'_i$ is quantifier-free, of type 0 or type 1; hence $\sim \mathscr{A}'_i$ holds in W, for some i. Now $\sim \mathscr{A}'_i$ has the form

$$P_1 \wedge \cdots \wedge P_t \quad \text{or} \quad \sim P_1 \wedge P_2 \wedge \cdots \wedge P_t.$$

Without loss of generality, we can read P_j as "$f_j(\bar{X}_1, \ldots, \bar{X}_u) = 0$." Then $\sim \mathscr{A}'_i$ means (respectively) "$f_j(X_1, \ldots, X_u)$ is an identity of W, for $1 \leq j \leq t$," or "$f_1(X_1, \ldots, X_u)$ is not an identity of W, and $f_j(X_1, \ldots, X_u)$ is an identity of W, for $2 \leq j \leq t$." In the first case,

$$(\forall x_1, \ldots, x_u)(\sim \mathscr{A}'_i(x_1, \ldots, x_u))$$

holds in R. In the second case, we can choose $r_1, \ldots, r_u$ such that $f_1(r_1, \ldots, r_u) \neq 0$ (since f_1 is not an identity of W); since f_j is an identity of R, $2 \leq j \leq t$, we see that

$$(\exists x_1, \ldots, x_u)(\sim \mathscr{A}'_i(x_1, \ldots, x_u))$$

holds in R. In both cases, $(\exists x_1 \cdots \exists x_m)(\sim \mathscr{A}_i)$ holds in R, contrary to the fact that $\mathscr{L}$ holds in R. Hence $\mathscr{L}$ holds in W. Q.E.D.

For example, if R has no nilpotent elements then W has no nilpotent elements, for we can express this property by $(\forall x_1)(x_1 = 0 \vee x_1^2 \neq 0)$. On the other hand, for $\Omega = \mathbf{Q}$, the following Horn sentences hold for $\mathbf{C}$, the field of complex numbers, but not for the free associative, commutative ring:

(i) $(\forall x_1\ \exists x_2)(x_1' x_2 x_1 = x_1)$ (i.e., von Neumann regular),
(ii) $(\exists x_1)(x_1^2 = 2)$,
(iii) $(\exists x_1)((x_1^3 = 1) \wedge (x \neq 1))$.

We can salvage some other Horn sentences, however. Call a Horn sentence

$$(Q_1 x_1) \cdots (Q_m x_m)(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_t)$$

special if each $\mathscr{A}_i$ has type 0.

THEOREM 3.23. *Any special Horn sentence holding in R also holds in W.*

Proof. Suppose $\mathscr{L} = (Q_1 x_1) \cdots (Q_m x_m)(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_v)$ is a special Horn sentence holding in R, and assume $\sim \mathscr{L}$ holds in W. Using the algorithm preceding Theorem 3.22,

$$(\sim \mathscr{L})' = \sim \mathscr{A}_1' \vee \cdots \vee \sim \mathscr{A}_v'$$

holds in W, where each $\mathscr{A}_i'$ is quantifier-free, of type 0; hence some $\sim \mathscr{A}_i'$ holds in W. Now $\sim \mathscr{A}_i'$ has the form $P_1 \wedge \cdots \wedge P_t = 0$. Reading P_j as

$$f_j(\bar{X}_1, \ldots, \bar{X}_u) = 0,$$

we see that, for $1 \leq j \leq t, f_j(X_1, \ldots, X_u)$ is an identity of W, hence of R. But this implies $\sim ((Q_1 x_1) \cdots (Q_m x_m)\mathscr{A}_i)$ holds in R, contrary to the fact that $\mathscr{L}$ holds in R. Hence $\mathscr{L}$ holds in W. Q.E.D.

Theorem 3.23 can be used to impart negative information to W; for example,

$$(\exists x_1)(\forall x_2)(x_1 x_2 x_1 \neq x_1)$$

is a special Horn sentence. We return to the universal Horn sentences. Of course, identities are Horn sentences. A generalization of the identity is the sentence,

$$(\forall x_1, \ldots, x_m)((f_1(x_1, \ldots, x_m) = 0 \wedge \cdots \wedge f_k(x_1, \ldots, x_m) = 0) \rightarrow f(x_1, \ldots, x_m) = 0),$$

called a *quasiidentity* by Mal'cev and studied in depth in [6, Chapter V]. Clearly, Theorem 2.1 implies that all quasiidentities of R are quasiidentities of W.

With minor modifications, all of the above results could be done in a general theory of varieties of arbitrary algebraic structures, not necessarily for rings only. On the other hand, some very important sentences cannot be analyzed in such general ways; for example, $(\forall x, y)(x = 0 \vee y = 0 \vee xy \neq 0)$ is *not* Horn (since the direct product of domains need not be a domain), but the crucial step in much of the theory of (associative) universal PI-algebras is that the universal PI-algebra of a **Q**-division algebra is a domain. Recall that a *universal sentence* (in logic) is a sentence of the form

$$(\forall x_1)\cdots(\forall x_m)(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_v),$$

where each $\mathscr{A}_i$ has some type n_i.

THEOREM 3.25. *Suppose every identity of R is R-stable. If $\mathscr{L}$ is a universal sentence holding in R, then $\mathscr{L}$ holds in W under either one of the additional hypotheses:* (1) *$\mathscr{L}$ holds in the polynomial ring $R[\lambda_1, \ldots, \lambda_n]$, for every n;* (2) *R and $R[\lambda]$ are strongly equivalent (as defined in Section* 1).

Proof. We start as in the proof of Theorem 3.22. Suppose

$$\mathscr{L} = (\forall x_1, \ldots, x_m)(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_v)$$

is a sentence holding in R, and assume that $\sim \mathscr{L}$ holds in W. Then, for some i, $\sim\mathscr{A}_i'$ holds in W, where $\mathscr{A}_i'$ is quantifier-free, of type n_i. Let $n = n_i$. We can write $\sim\mathscr{A}_i'$ as follows: "For $1 \le j \le n$, $f_j(X_1, \ldots, X_u)$ are not identities of W; for $n < j < t, f_j(X_1, \ldots, X_u)$ are identities of W." Since R is equivalent to W, we can find x_{jk}, $1 \le j \le n$, $1 \le k \le u$, such that $f_j(x_{j1}, \ldots, x_{ju}) \neq 0$ for all j, $1 \le j \le n$; note that $f_j(x_{j1}, \ldots, x_{ju}) = 0$ for all $j > n$. Let $\lambda_1, \ldots, \lambda_n$ be associative indeterminates over R, and let $x_k = \sum_{j=1}^{n} x_{jk}\lambda_j$, elements of $R[\lambda_1, \ldots, \lambda_n]$. Clearly, $f_j(x_1, \ldots, x_u) \neq 0$ for $1 \le j \le n$. Thus, letting

$$\mathscr{L}_1 = (\forall x_1, \ldots, x_u)(f_1(x_1, \ldots, x_u) = 0 \vee \cdots \vee f_n(x_1, \ldots, x_u) = 0),$$

we see that $\sim \mathscr{L}_1$ holds in $R[\lambda_1, \ldots, \lambda_n]$.

At this point, under the additional hypothesis (1) we reach an immediate contradiction since $f_{n+1}, \ldots, f_t$ are also identities of $R[\lambda_1, \ldots, \lambda_n]$ (since R and $R[\lambda_1, \ldots, \lambda_n]$ are equivalent). On the other hand, assume hypothesis (2). Then, by Proposition 1.6, $\sim \mathscr{L}_1$ holds in R, and, of course, $f_{n+1}, \ldots, f_t$ are identities of R, so we would conclude $\sim \mathscr{L}$ holds in R, a contradiction. Thus $\mathscr{L}$ must hold in W, after all. Q.E.D.

COROLLARY 3.26. *If Z contains an infinite domain over which R is torsion free, then every universal sentence of R also holds for $\mathscr{U}(R)$.*

Proof. Apply Theorem 1.7 to Theorem 3.25. Q.E.D.

Theorem 3.25 raises the following very interesting question: "What types of sentences lift from R to $R[\lambda]$?" Another issue worth raising is that much of the proof of Theorem 3.25 is applicable to $\mathscr{U}^{(m)}(R)$, the universal Ω-ring built only from the identities of R in $X_1, \ldots, X_m$.

THEOREM 3.27. *Suppose every identity of R is R-stable. If*

$$\mathscr{L} = (\forall x_1, \ldots, x_m)(\mathscr{A}_1 \wedge \cdots \wedge \mathscr{A}_v)$$

is a sentence of R and $v \leq t$, *then* $\mathscr{L}$ *holds in* $\mathscr{U}^{(t)}(R)$, *under either hypothesis* (1) *or hypothesis* (2) *of Theorem* 3.25.

In particular, if R is a domain then $\mathscr{U}^{(t)}(R)$ *is a domain for each* $t \geq 2$.

4. Ω-rings without 1, with applications to the radicals

There are a number of interesting relationships among the various radicals of R, and of Z, which we recall again now. Nil $(R) \subseteq J(R) \subseteq$ Jac $(R) \subseteq BM(R)$; Jac $(R) = BM(R)$ when R is in a Kaplansky class, by [10, Proposition 3.22]; $Z \cap BM(R) \subseteq$ Jac (Z), so $BM(R) = 0$ whenever Jac $(Z) = 0$; in a prime Ω-ring with regular central polynomial Jac $(Z) = 0$ whenever $BM(R) = 0$ (cf. [10, Corollary 3.15]). Thus the question arises in general: When does Jac $(R) = 0$ imply Jac $(Z) = 0$? The answer turns out to be "in almost every Kaplansky class," but, to see this fact, we need to prove some facts about Ω-rings without 1. Henceforth, R_0 denotes a Ω-ring without 1, defined in the obvious way.

Let $\Omega\{X\}_0$ be the Ω-subring without 1 of $\Omega\{X\}$, consisting of polynomials with constant term 0; identities of R_0 are defined to be those polynomials in the kernel of all homomorphisms from $\Omega\{X\}_0$ to R_0. In case $1 \in R_0$, this gives all the identities of R_0 (as ring with 1) having constant term 0, which, as we saw in [10, Section 1B] is sufficient to yield the entire PI-theory.

Ring theoretic terms are now given in the category of Ω-rings without 1. For a multiplicative set S of $Z(R_0)$, we can define $(R_0)_S$, which, by the proof of [10, Corollary 2.1], satisfies all identities of R_0. But clearly $1 \in (R_0)_S$ iff $S \neq \emptyset$, so we have a very useful way of passing to Ω-rings with 1; the major goal of this section is to exploit this passage between categories.

A more formal passage from the category of Ω-rings without 1 to the category of Ω-rings (with 1) is the *formal adjunction of* 1, which can be done as follows if we assume that Ω is a ring and R_0 is a Ω-algie (cf. [1, Section 1B]). Define $R' = \Omega \oplus R_0$ as an additive group, and view R' as an Ω-ring with the following operations:

$$(w_1, r_1)(w_2, r_2) = (w_1 w_2, w_1 r_2 + r_1 w_2 + r_1 r_2);$$
$$w(w_1, r_1) = (ww_1, wr_1) \quad \text{and} \quad (w_1, r_1)w = (w_1 w, r_1 w).$$

Clearly R' is a Ω-algie with multiplicative unit (1, 0). There is a canonical injection $r \mapsto (0, r)$, under which every ideal of R_0 is identified with an ideal of R', and we shall use this identification implicitly. Also, there is a Ω-ring homomorphism of R' to Ω, given by $(w, r) \mapsto w$.

For convenience, assume Ω *is a commutative, associative ring* ϕ *and* R_0 *is a* ϕ*-algebra* (*without* 1). Write Z_0 for $Z(R_0)$ and Z' for $Z(R')$; clearly Z' is isomorphic to the ϕ-algebra obtained (by adjoining) formally 1 to Z_0. Let $^-$ denote the canonical homomorphism from R' to $R'/\mathrm{Ann}_{R'} R_0$, and let $R = \overline{R'}$, $Z = Z(R)$. Call R the *reduced* ϕ*-algebra with* 1 *of* R_0. We give two straightforward observations of the flavor of [9], without proof.

LEMMA 4.1. *Suppose* $\mathrm{Ann}_{R_0} R_0 = 0$. *Then* R_0 *is canonically embedded in* R (*by* $r \mapsto \overline{(0, r)}$), *and* $\mathrm{Ann}_R R_0 = 0$. *If* $1 \in R_0$ *already then this map is an isomorphism.*

LEMMA 4.2. *Suppose* R_0 *is semiprime. Then* $\mathrm{Ann}_{R'} R_0 \subseteq Z'$, R *is semiprime, and* $Z = \overline{Z'}$. *If* R_0 *is prime* (*resp. strongly semiprime*) *then* R *is prime* (*resp. strongly semiprime*).

PROPOSITION 4.3. *If* R_0 *is semiprime and* S *is a nonempty multiplicative subset of* Z_0 *containing only regular elements of* R_0, *then all the elements of* S *are regular in* R *and* $(R_0)_S = R_S$. *In this case,* R *and* R_0 *are equivalent.*

Proof. $S \subseteq Z$, by Proposition 4.2. Next, suppose $\overline{(w, r)}\overline{(0, s)} = 0$, for (w, r) in R' and s in S. Since $(0, s) \in Z'$, we have $\overline{(w_1, r_1)}\overline{(0, s)} = 0$ for all (w_1, r_1) in $\langle (w, r) \rangle$. But then, for all r' in R_0,

$$\begin{aligned} 0 &= ((w_1, r_1)(0, s))(0, r') \\ &= ((w_1, r_1)(0, r'))(0, s) \\ &= (0, (w_1 r' + r_1 r')s); \end{aligned}$$

since s is regular in R_0, we have $w_1 r' + r_1 r' = 0$. Thus $\langle (w, r) \rangle R_0 = 0$, implying $\overline{(w, r)} = 0$. This proves s is regular in R.

Clearly (R_0) is an ideal of R_S. But $1 \in (R_0)_S$, so $(R_0)_S = R_S$. Finally, R is equivalent to $R_S = (R_0)_S$, which satisfies all the identities of R_0, so R and R_0 are equivalent. Q.E.D.

COROLLARY 4.4. *Suppose* $\mathscr{V}$ *is a class of* ϕ*-algebras with* 1, $L \in \mathscr{V}$, L *is semiprime, and* $c \in Z(L)$. *If* R *is the reduced* ϕ*-algebra with* 1 *of* cL *then* R *is equivalent to* cL. *If* $\mathscr{V}$ *is a variety then* $R \in \mathscr{V}$.

Proof. Since L is semiprime, c is regular in the ϕ-algebra cL. Thus, by Proposition 4.3, with $R_0 = cL$ and $S = \{c^i \mid i \geq 1\}$, we see that R is equivalent to cL. In particular R satisfies all identities of L. Hence, if $\mathscr{V}$ is a variety then $R \in \mathscr{V}$. Q.E.D.

COROLLARY 4.5. *If* $Z(R_0) \neq 0$ *and if* R, *the reduced* ϕ*-algebra with* 1 *of* R_0, *is absolutely prime, then* R_0 *is absolutely prime, having the same* ϕ*-algebra of central quotients as* R; *in this case* R *and* R_0 *are equivalent.*

Proof. Apply Proposition 4.3, with $S = Z_0 - \{0\}$. Q.E.D.

To apply Proposition 4.3 optimally, we need a decomposition result.

PROPOSITION 4.6. *Let* R_0 *be a subdirect product of the class of* ϕ*-algebras* (*without* 1) $\{R_{0\gamma} = R_0/B_\gamma \mid \gamma \in \Gamma\}$, *and, for each* γ, *let* R_γ *be the reduced algebra with* 1 *of* $R_{0\gamma}$. *Then* R *is a subdirect product of* $\{R_\gamma \mid \gamma \in \Gamma\}$.

Proof. Given r in R_0, let r_γ denote the image of r_γ in $R_{0\gamma}$. Then define $\phi_\gamma: R' \to R_\gamma$ by $\phi_\gamma(w, r) = (w, r_\gamma)$;

$$\ker \phi_\gamma = \{(w, r) \in R' \mid \langle (w, r) \rangle R_0 \subseteq B_\gamma\},$$

so $\bigcap \ker \phi_\gamma = \{(w, r) \in R' \mid \langle (w, r) \rangle R_0 \subseteq \bigcap B_\gamma = 0\} = \mathrm{Ann}_{R'} R_0$. Thus each ϕ_γ induces a homomorphism $\bar{\phi}_\gamma: R = R'/\mathrm{Ann}\, R_0 \to R_\gamma$, and $\bigcap \ker \bar{\phi}_\gamma = 0$. Q.E.D.

COROLLARY 4.7. *If R_0 is a subdirect product of prime rings having nontrivial centers, then R_0 is equivalent to its reduced algebra with* 1.

DEFINITION 4.8. A Kaplansky class $\mathscr{C}$ is *sufficient* if for every semisimple member L and for each c in L, the reduced algebra with 1 of cL is in $\mathscr{C}$. (Recall "semisimple" means "$BM(\quad) = 0$.")

Remark 4.9. By Corollary 4.4, every Kaplansky variety of algebras is sufficient.

We are now ready for the theorem promised at the beginning of this section.

THEOREM 4.10. *Suppose $\mathscr{C}$ is a sufficient Kaplansky class of ϕ-algebras, for ϕ some commutative, associative ring, and let $\{g_k \mid 1 \le k \le u\}$ be a given finite set of regular polynomials. Say a semisimple algebra L of $\mathscr{C}$ satisfies* (*) *if there is a set of maximal ideals $\{P_\gamma \mid \gamma \in \Gamma\}$ of L with intersection* 0, *such that for every $\Gamma' \subseteq \Gamma$, a suitable g_k is $L/\bigcap \{P_\gamma \mid \gamma \in \Gamma'\}$-central. For every semisimple L in $\mathscr{C}$ satisfying* (*), *we have* $\mathrm{Jac}\, Z(L) = 0$.

Proof. Order the semisimple members of $\mathscr{C}$ satisfying (*) by defining $L_1 \le L_2$ if every g_k which is an identity of L_2 is an identity of L_1, $1 \le k \le u$. If the theorem were false we could select a semisimple counterexample L (satisfying (*)), which was minimal with respect to the ordering. $\mathrm{Jac}\, Z(L)$ has a nonzero element c; let $R_0 = cL$ and $\Gamma_1 = \{\gamma \in \Gamma \mid c \notin P_\gamma\}$. Now

$$0 = \bigcap \{P_\gamma \mid \gamma \in \Gamma\} \supseteq \bigcap \{P_\gamma \mid \gamma \in \Gamma_1\} \cap R_0 = \bigcap \{P_\gamma \cap R_0 \mid \gamma \in \Gamma_1\}.$$

Letting $R_\gamma = L/P_\gamma$ for each $\gamma \in \Gamma_1$, we see that

$$R_\gamma = (R_0 + P_\gamma)/P_\gamma \approx R_0/(R_0 \cap P_\gamma),$$

so R_0 is a subdirect product of $\{R_\gamma \mid \gamma \in \Gamma_1\}$. Let R be the reduced algebra with 1 of R_0. By Lemma 4.1, each R_γ is its own reduced algebra with 1; hence, by Proposition 4.6, R is the subdirect product of $\{R_\gamma \mid \gamma \in \Gamma\}$. Thus R is semisimple and obviously satisfies (*). Since $\mathscr{C}$ is sufficient, $R \in \mathscr{C}$. On the other hand, one sees easily that $Z(R_0)$ is a quasiregular ideal of $Z(R)$. (Indeed, for any element cz in $Z(R_0)$, one can show that $z \in Z(L)$ (since L is semiprime), so cz has a quasiinverse y. Then $y = czy - cz \in cZ(L) \subseteq Z(R_0)$.) Thus R is a counterexample to the theorem. But, by Corollary 4.4, $R \le L$. Moreover, by assumption, some g_k is L-central and, by [10, Corollary 3.14], $L(g_k)c = 0$. Thus $R_0(g_k)c = 0$, implying $R_0(g_k) = 0$, so g_k is an identity of R, by Corollary 4.4. Therefore $R < L$, contrary to the minimality of our counterexample; thus the theorem is true, after all. Q.E.D.

Property (*) arises very naturally; in fact, all the classes studied in part III satisfy (*) and thus Theorem 4.10. Incidentally, this theorem is very interesting in that its statement and proof seem to necessitate passing back and forth between categories (of algebras with 1 and algebras without 1). Note that the same proof would work for "algies."

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Bar-Ilan University
Ramat-Gan, Israel