ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 1, March 1979

REMARKS ON STRONGLY M-PROJECTIVE MODULES

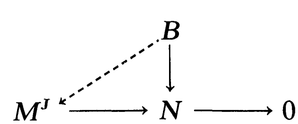
BY
PAUL E. BLAND

In [11], Varadarajan introduced the notion of strongly M-projective modules. He showed that every $B \in \operatorname{Mod} R$ satisfying $B\mathscr{A}(M) = 0$ possesses a strong M-projective cover if and only if $R/\mathscr{A}(M)$ is a right perfect ring where $\mathscr{A}(M)$ denotes the right annihilator of M in R. We show that if a certain class of modules in $\operatorname{Mod} R$ is closed under factors, then every $B \in \operatorname{Mod} R$ possesses a strong M-projective cover if and only if $R/\mathscr{A}(M)$ is right perfect, thereby conditionally extending Varadarajan's result to $\operatorname{Mod} R$. We also show via a pullback diagram that $B \in \operatorname{Mod} R$ is strongly M-projective if and only if $B/B\mathscr{A}(M)$ is a projective $R/\mathscr{A}(M)$-module. Varadarajan has shown this for the special case when $\mathscr{A}(M) = 0$.

If M is injective and $(\mathscr{T}, \mathscr{F})$ is the hereditary torsion theory on $\operatorname{Mod} R$ cogenerated by M, then it is shown that $B \in \operatorname{Mod} R$ is codivisible with respect to $(\mathscr{T}, \mathscr{F})$ if and only if B is strongly M-projective. From this it follows that if B has a projective cover, then B is codivisible with respect to $(\mathscr{T}, \mathscr{F})$ if and only if B is M-projective in the sense of G. Azumaya [1].

Throughout this paper R will denote an associative ring with identity and our attention will be confined to the category $\operatorname{Mod} R$ of unital right R-modules. We will often abuse notation and write $B \in \operatorname{Mod} R$ for an object of $\operatorname{Mod} R$. Furthermore all maps will be morphisms in $\operatorname{Mod} R$ while $\mathscr{A}(M)$ and M^J will denote the right annihilator of M in R and the direct product of the family $\{M_a = M\}$ $(a \in J)$ respectively. In addition, M will denote a fixed right R-module which is not necessarily injective.

Following Varadarajan [11], we call $B \in \operatorname{Mod} R$ strongly M-projective if every row exact diagram of the form



where J is any indexing set can be completed commutatively. This is a natural generalization of M-projective modules first studied by G. Azumaya [1]. Azumaya called B M-projective if the diagram above can be completed commutatively when J is a singleton.

If K is a submodule of $B \in \operatorname{Mod} R$, then K is said to be M-independent in B if for each $0 \neq x \in K$ there is an $f \in \operatorname{Hom}_R(B, M)$ such that $f(x) \neq 0$.

Received September 12, 1977.

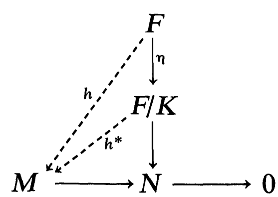
$f \in \mathrm{Hom}_R(B, M)$ is said to be M-independent if $\ker f$ is M-independent in B while B is called M-independent if B is M-independent in itself.

A module $B \in \mathrm{Mod}\, R$ is said to have a strong M-projective cover if there exists a strongly M-projective module A and an M-independent epimorphism $\varphi: A \to B$ with small kernel. Recall that if K is a submodule of A, then K is a small submodule of A if whenever B is a submodule of A such that $K + B = A$, $B = A$.

The first of the following two lemmas shows that $\mathrm{Mod}\, R$ has enough strongly M-projective modules.

LEMMA 1. *For any $B \in \mathrm{Mod}\, R$, there is a strongly M-projective module A and an M-independent epimorphism $\varphi: A \to B$.*

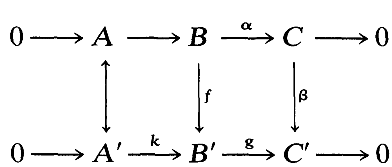
Proof. Let $g: F \to B$ be a free module on B and set
$K = \{x \in \ker g \mid f(x) = 0$ for all $f \in \mathrm{Hom}_R(F, M^J)$ and every indexing set $J\}$. If given a row exact diagram



where η is the natural projection, then the projectivity of F yields a completing map $h: F \to M^J$ which makes the diagram commutative. But $h(K) = 0$ and so there is an induced map $h^*: F/K \to M^J$ which makes the inner diagram commute. Thus F/K is strongly M-Projective. Next let $A = F/K$ and suppose that φ is the map induced by g. If $0 \neq x + K \in \ker \varphi$, then for some indexing set J there is an $f \in \mathrm{Hom}_R(F, M^J)$ such that $f(x) \neq 0$. Now $f(K) = 0$ and so there is an $f^* \in \mathrm{Hom}_R(A, M^J)$ such that $f^*(x + K) = f(x) \neq 0$. But since $0 \neq f(x) \in M^J$, one can certainly find a map $p: M^J \to M$ such that $p(f(x)) \neq 0$. Consequently, $p \circ f^* \in \mathrm{Hom}_R(A, M)$ is such that $p \circ f^*(x + K) \neq 0$. Thus φ is M-independent.

The following lemma seems to be known. Since we have been unable to find a proof in the literature, we include a proof for the sake of completeness.

LEMMA 2. *Let*



be a row exact commutative diagram such that the right hand square is a pullback diagram. Then the splitting of the top row follows from the splitting of the bottom row.

Proof. Suppose that the bottom row splits and let

$$0 \longrightarrow C' \xrightarrow{g'} B' \xrightarrow{k'} A' \longrightarrow 0$$

be the splitting maps. Since A and A' are isomorphic we can assume, without loss of generality, that $A = A'$. Let

$$p_1\colon A \oplus C \to A \quad \text{and} \quad p_2\colon A \oplus C \to C$$

be the canonical projections and define $\varphi\colon A \oplus C \to B'$ by

$$\varphi(a, c) = k(a) + g'(\beta(c)).$$

Then $g \circ \varphi = \beta \circ p_2$ and so since the right hand square is a pullback diagram there is a unique mapping $\phi\colon A \oplus C \to B$ such that $f \circ \phi = \varphi$ and $\alpha \circ \phi = p_2$. Notice next that $k' \circ \varphi = p_1$ and so since $A \oplus C$ is a product there is a unique mapping $\phi^*\colon B \to A \oplus C$ such that $p_1 \circ \phi^* = k' \circ f$ and $p_2 \circ \phi^* = \alpha$. Hence it follows that $\alpha \circ \phi \circ \phi^* = f \circ 1_B$. Thus by the uniqueness of factorization through products we see that $\phi \circ \phi^* = 1_B$. Similarly by the uniqueness of factorization through pullbacks $\phi^* \circ \phi = 1_{A \oplus C}$. Thus φ is an isomorphism and if $i_2\colon C \to A \oplus C$ is the canonical injection, then $\phi \circ i_2$ is a splitting map for the top row of the diagram.

PROPOSITION 3. *$B \in \operatorname{Mod} R$ is strongly M-projective if and only if $B/B\mathscr{A}(M)$ is a projective $R/\mathscr{A}(M)$-module.*

Proof. Let B be a strongly M-projective and consider the row exact diagram

$$\begin{array}{ccccc} & & B/B\mathscr{A}(M) & & \\ & & \downarrow & & \\ M^J & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

of $R/\mathscr{A}(M)$-modules and $R/\mathscr{A}(M)$-maps. (Note M^J is an $R/\mathscr{A}(M)$-module since $M^J\mathscr{A}(M) = 0$ for any indexing set J.) If we view these as R-modules and R-maps in the natural fashion, then we have a commutative diagram

$$\begin{array}{ccccc} & & B & & \\ & \swarrow^{h} & \downarrow & & \\ & & B/B\mathscr{A}(M) & & \\ & & \downarrow & & \\ M^J & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

where h is the completing map given by the strong M-projectivity of B. But $h(B\mathcal{A}(M)) \subseteq M^J\mathcal{A}(M) = 0$ and so there is an induced map $h^*: B/B\mathcal{A}(M) \to M^J$ which makes the original diagram commute. Hence $B/B\mathcal{A}(M)$ is a strongly M-projective $R/\mathcal{A}(M)$-module. Now Varadarajan has shown [11, Proposition 3.6] that when M is faithful, any strongly M-projective module is projective. Thus $B/B\mathcal{A}(M)$ is a projective $R/\mathcal{A}(M)$-module since M is a faithful $R/\mathcal{A}(M)$-module.

Conversely, suppose that $B/B\mathcal{A}(M)$ is a projective $R/\mathcal{A}(M)$-module. Now by Lemma 1 there is an exact sequence

$$0 \longrightarrow K \xrightarrow{k} A \xrightarrow{\varphi} B \longrightarrow 0$$

such that A is strongly M-projective and K is M-independent in A. This yields a row exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{\quad k \quad} & A & \xrightarrow{\quad \varphi \quad} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \\
0 & \longrightarrow & \dfrac{K + A\mathcal{A}(M)}{A\mathcal{A}(M)} & \xrightarrow{k^*} & A/A\mathcal{A}(M) & \xrightarrow{\varphi^*} & B/B\mathcal{A}(M) & \longrightarrow & 0
\end{array}$$

where k^* and φ^* are the maps induced by k and φ respectively and η_1, η_2 and η_3 are the natural projections. Since $K \cap A\mathcal{A}(M) = 0$,

$$\frac{K + A\mathcal{A}(M)}{A\mathcal{A}(M)} \cong K$$

and so Lemma 2 will apply if we can show that the right hand square is a pullback. Toward this end let $P = \{(x + A\mathcal{A}(M), y) \in A/A\mathcal{A}(M) \oplus B \mid \varphi^*(x + A\mathcal{A}(M)) = \eta_3(y)\}$. Then

$$\begin{array}{ccc}
P & \xrightarrow{p_2} & B \\
\downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \eta_3} \\
A/A\mathcal{A}(M) & \xrightarrow{\varphi^*} & B/B\mathcal{A}(M)
\end{array}$$

where p_1 and p_2 are the obvious maps is well known to be a pullback diagram. Hence there is a unique map $\phi: A \to P$ such that $p_1 \circ \phi = \eta_2$ and $p_2 \circ \phi = \varphi$ and so it must be the case that

$$\phi(a) = (a + A\mathcal{A}(M), \varphi(a)).$$

We claim that ϕ is an isomorphism. If $\phi(a) = 0$, then $a \in A\mathcal{A}(M)$ and $a \in \ker \varphi = K$. Hence $a \in K \cap A\mathcal{A}(M) = 0$. Also if

$$(x + A\mathcal{A}(M), y) \in P,$$

then there is an $a \in A$ such that $\varphi(a) = y$. But then

$$\phi(a) = (a + A\mathscr{A}(M), y) \in P$$

and so $\varphi^*(a + A\mathscr{A}(M)) = \varphi^*(x + A\mathscr{A}(M))$. Therefore

$$(x - a) + A\mathscr{A}(M) \in \ker \varphi^*.$$

Let $z \in K$ be such that $(x - a) + A\mathscr{A}(M) = z + A\mathscr{A}(M)$ and set $a' = a + z$. Then $\varphi(a') = y$ and $x + A\mathscr{A}(M) = (a + z) + A\mathscr{A}(M)$. Therefore $\phi(a') = (x + A\mathscr{A}(M), y)$ and so ϕ is an isomorphism as was asserted. That B is strongly M-projective now follows from the assumption that $B/B\mathscr{A}(M)$ is a projective $R/\mathscr{A}(M)$-module, Lemma 2 and the fact that a direct summand of a strongly M-projective module is strongly M-projective.

Corollary 4. *If $B\mathscr{A}(M) = B$, then B is strongly M-projective.*

Now let $C(M)$ denote the class of all modules in Mod R which are M-independent in some over-module. We will say that $C(M)$ is closed under factors if whenever K is M-independent in B, K/K' is M-independent in B/K' for each submodule K' of K.

Proposition 5. *If $B \in \operatorname{Mod} R$ has a strong M-projective cover, then $B/B\mathscr{A}(M)$ has a projective cover as an $R/\mathscr{A}(M)$-module. Conversely, if $C(M)$ is closed under factors and $B/B\mathscr{A}(M)$ has a projective cover as an $R/\mathscr{A}(M)$-module, then B has a strong M-projective cover.*

Proof. Our proof follows closely that given for Theorem 10 in [8]. First suppose that $B \in \operatorname{Mod} R$ has a strong projective cover, then we have an exact sequence $0 \to K \to A \to B \to 0$ where A is strongly M-projective and K is small and M-independent in A. But this yields an exact sequence

$$0 \to \frac{K + A\mathscr{A}(M)}{A\mathscr{A}(M)} \to A/A\mathscr{A}(M) \to B/B\mathscr{A}(M) \to 0$$

where by Proposition 3, $A/A\mathscr{A}(M)$ is a projective $R/\mathscr{A}(M)$-module. Now it is known that if $f: X \to Y$ is R-linear and K is small in X, then $f(K)$ is small in Y [7, Hilfssatz 3.1]. Hence $(K + A\mathscr{A}(M))/A\mathscr{A}(M)$ is small in $A/A\mathscr{A}(M)$ and so $B/B\mathscr{A}(M)$ has a projective cover as an $R/\mathscr{A}(M)$-module.

Conversely, let

$$P \xrightarrow{u} B/B\mathscr{A}(M)$$

be a projective cover of $B/B\mathscr{A}(M)$ as an $R/\mathscr{A}(M)$-module and suppose that $C(M)$ is closed under factors. By Lemma 1 there is an exact sequence

$$0 \longrightarrow K \longrightarrow A \xrightarrow{\varphi} B \longrightarrow 0$$

where A is strongly M-projective and φ is M-independent. Hence we have a row exact diagram

$$\begin{array}{ccccccccc} & & & & & & P & & \\ & & & & & & \downarrow & & \\ 0 \longrightarrow & \dfrac{K+A\mathscr{A}(M)}{A\mathscr{A}(M)} & \longrightarrow & A/A\mathscr{A}(M) & \xrightarrow{\varphi^*} & & B/B\mathscr{A}(M) & \longrightarrow & 0 \end{array}$$

with φ^* being the map induced by φ. Now by Proposition 3, $A/A\mathscr{A}(M)$ is a projective $R/\mathscr{A}(M)$-module and so there is a map $f\colon A/A\mathscr{A}(M)\to P$ such that $\mu\circ f=\varphi^*$. But φ^* is an epimorphism and so it follows that $P=\operatorname{Im} f+\ker\mu$. Therefore f is an epimorphism since $\ker\mu$ is small in P. Now P is projective and so f splits. Hence we have submodules X and Y of A such that

$$A/A\mathscr{A}(M)=X/A\mathscr{A}(M)\oplus Y/A\mathscr{A}(M)$$

with $X/A\mathscr{A}(M)=\ker f$ and $Y/A\mathscr{A}(M)\cong P$. Also since $\ker f\subseteq\ker\varphi^*$, it follows that

$$\frac{K+A\mathscr{A}(M)}{A\mathscr{A}(M)}=X/A\mathscr{A}(M)\oplus Z/A\mathscr{A}(M)$$

where $Z/A\mathscr{A}(M)\subseteq Y/A\mathscr{A}(M)$ is small in $Y/A\mathscr{A}(M)$ and consequently in $A/A\mathscr{A}(M)$. Notice next that since $K\cap A\mathscr{A}(M)=0$, $K+A\mathscr{A}(M)=K\oplus A\mathscr{A}(M)$ and so

$$X=X'\oplus A\mathscr{A}(M)\quad\text{and}\quad Z=Z'\oplus A\mathscr{A}(M)$$

where $X'=X\cap K$ and $Z'=Z\cap K$. Also $K\oplus A\mathscr{A}(M)=X+Z$ yields $K=X'\oplus Z'$. Now let $A^*=A/X'$ and $K^*=K/X'$; then

$$A^*\mathscr{A}(M)=\frac{X'+A\mathscr{A}(M)}{X'}=X/X'$$

and so

$$A^*/A^*\mathscr{A}(M)\cong A/X\cong(A/A\mathscr{A}(M))/X/A\mathscr{A}(M))\cong Y/A\mathscr{A}(M)\cong P.$$

Hence $A^*/A^*\mathscr{A}(M)$ is a projective $R/\mathscr{A}(M)$-module and so, by Proposition 3, A^* is a strongly M-projective R-module. Note also that

$$A^*/K^*=(A/X')/(K/X')\cong A/K\cong B.$$

Next we claim that K^* is small in A^*. Suppose $A^*=K^*+W^*$ where $W^*=W/X'$ for some $W\subseteq A$. Since $K^*=K/X'\cong Z'$ and Z' is M-independent in A, it follows that $Z'\mathscr{A}(M)=0$ and consequently that $K^*\mathscr{A}(M)=0$. Hence $A^*\mathscr{A}(M)=K^*\mathscr{A}(M)+W^*\mathscr{A}(M)=W^*\mathscr{A}(M)\subseteq W^*$. But $A^*\mathscr{A}(M)=X/X'$ and so

$$\begin{aligned} A/A\mathscr{A}(M)&=\frac{K+A\mathscr{A}(M)}{A\mathscr{A}(M)}+W/A\mathscr{A}(M)\\ &=Z/A\mathscr{A}(M)+X/A\mathscr{A}(M)+W/A\mathscr{A}(M)\\ &=Z/A\mathscr{A}(M)+W/A\mathscr{A}(M)=W/A\mathscr{A}(M)\end{aligned}$$

because $Z/A\mathscr{A}(M)$ is small in $A/A\mathscr{A}(M)$. Therefore $A = W$ and so $A^* = W^*$.

Since it follows easily from the fact that $C(M)$ is closed under factors that K^* is M-independent in A^*, our proof is complete.

The following proposition is now obvious. See [2] for several characterizations of right perfect rings.

PROPOSITION 6. *If $C(M)$ is closed under factors, then every $B \in \text{Mod } R$ has a strong M-projective cover if and only if $R/\mathscr{A}(M)$ is a right perfect ring.*

We conclude with the following observations concerning strongly M-projective modules and torsion theories. The reader can consult [4], [6], [9] for the general results and terminology on torsion theories. If $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod R, then it is well known that $(\mathcal{T}, \mathcal{F})$ is cogenerated by an injective module [5, Theorem 1.1] and that uniquely associated with $(\mathcal{T}, \mathcal{F})$ there is a left exact idempotent radical

$$T: \text{Mod } R \to \text{Mod } R$$

such that $\mathcal{T} = \{B \mid T(B) = B\}$ and $\mathcal{F} = \{B \mid T(B) = 0\}$ [9, Corollary 2.7]. In fact, if M is the injective module cogenerating $(\mathcal{T}, \mathcal{F})$, then $T(B) = \bigcap \ker f$ where $f \in \text{Hom}_R(B, M)$. Hence $\mathcal{F}$ coincides with the class of M-independent modules. Also since every map f from R to M is a multiplication map determined by the action of f on the identity of R, $T(R) = \mathscr{A}(M)$.

A module $B \in \text{Mod } R$ is said to be codivisible with respect to a torsion theory $(\mathcal{T}, \mathcal{F})$ on Mod R if every row exact diagram

$$\begin{array}{ccccc} & & B & & \\ & \swarrow & \downarrow & & \\ A & \xrightarrow{f} & A' & \longrightarrow & 0 \end{array}$$

where $\ker f \in \mathcal{F}$ can be completed commutatively. The interested reader can consult [3], [8], [10] for some recent results on codivisible modules.

PROPOSITION 7. *If $(\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on* Mod R *cogenerated by an injective module M, then the following are equivalent for $B \in$* Mod R:

(1) *B is codivisible with respect to $(\mathcal{T}, \mathcal{F})$.*

(2) *B is strongly M-projective.*

Furthermore if B has a projective cover, then (1) *and* (2) *are equivalent to*:

(3) *B is M-projective.*

Proof. Rangaswamy has shown [8, Theorem 8] that if $(\mathcal{T}, \mathcal{F})$ is hereditary (in fact $(\mathcal{T}, \mathcal{F})$ need only be pseudo-hereditary [10]), then $B \in \text{Mod } R$ is codivisible if and only if $B/BT(R)$ is a projective $R/T(R)$-module where T is as described above. But since $T(R) = \mathscr{A}(M)$, the equivalence of (1) and (2) follows from our above observations and Proposition 3. Next suppose that B

has a projective cover, then if B is M-projective, B is strongly M-projective [11, Lemma 2.2]. Therefore, in this case, (3) is equivalent to (1) and (2).

REFERENCES

1. G. AZUMAYA, *M-projective and M-injective modules* (Unpublished).
2. H. BASS, *Finitistic dimension and a homological generalization of semi-primary rings*, Trans. Amer. Math. Soc., vol 95 (1960), pp. 466–488.
3. P. E. BLAND, *Perfect torsion theories*, Proc. Amer. Math. Soc., vol 41 (1973), pp. 348–355.
4. S. DICKSON, *A torsion theory for abelian categories*, Trans. Amer. Math. Soc., vol 121 (1966), pp. 223–235.
5. J. P. JANS, *Some aspects of torsion*, Pacific J. Math., vol 15 (1965), pp. 1249–1259.
6. J. LAMBEK, *Torsion theories, additive semantics, and rings of quotients*, Lecture notes in mathematics, no. 177, Springer-Verlag, Berlin and New York, 1971.
7. B. PAREIGIS, *Radikale und kleine moduln*, Bayer. Akad. Wiss. Math.-Natur. Kl. S.-B. 1966, Abt. 11, pp. 185–199.
8. K. M. RANGASWAMY, *Codivisible modules*, Communications in Algebra, vol 2 (1974), pp. 475–489.
9. B. STENSTROM, *Rings and modules of quotients*, Lecture notes in mathematics, no. 237, Springer-Verlag, Berlin, 1971.
10. M. L. TEPLY, *Codivisible and projective covers*, Communications in Algebra, vol 1 (1974), pp. 23–38.
11. K. VARADARAJAN, *M-projective and strongly M-projective modules*, Illinois J. Math., vol 20 (1976), pp. 507–515.

EASTERN KENTUCKY UNIVERSITY
RICHMOND, KENTUCKY