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ON THE "STABLE" HOMOTOPY TYPE OF KNOT COMPLEMENTS

BY

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1. Introduction

This paper is concerned with knots of codimension two, that is, embeddings of the (q-2)-sphere S^{q-2} in the q-sphere S^q. By Alexander duality, the *complement* C of the knot (see Paragraph 2) has the same homology groups as the circle S^1, and Levine [7], [8] has proved that if $q \neq 4$, then C is homotopy equivalent to S^1 if and only if the knot is trivial. We will consider those knots for which there is a positive integer n such that $\pi_i C \cong \pi_i S^1$ for $i \leq n$. Thus, all fundamental groups will be infinite cyclic, and all higher homotopy groups will be modules over $\Lambda = \mathbf{Z}[\mathbf{Z}]$, the group ring of the integers. Then, in Theorem 1 (see Paragraph 2), we prove that *$\pi_i C$ is a finitely generated acyclic Λ-module* (see Paragraph 2) *for $n+1 \leq i \leq 2n$* (the "stable range").

Conversely, let X be any space for which $\pi_i X \cong \pi_i S^1$ for $i \leq n$, and $\pi_i X$ is a finitely generated acyclic Λ-module for $n+1 \leq i \leq 2n$. Then, in Theorem 2 (see Paragraph 2), we prove that *there is a knot complement C with the same homotopy type as X through dimension $2n$.*

The first work in this direction was done by Kervaire. In [6] he proved, under the assumptions above, that $\pi_{n+1} C$ is a finitely generated acyclic Λ-module and that any finitely generated acyclic Λ-module can be so realized. In [1], Brown and Dror showed that for $n \geq 2$, the module $\pi_{n+2} C$ has the same characterization as $\pi_{n+1} C$, and that these two modules are independent of one another. In [3], Dror and Dwyer obtained results on homology localizations in the stable range, which imply most of our Theorem 1.

Our Theorems 1 and 2 have analogues for arbitrary *homology circles*, i.e., spaces with the same homology groups as the circle. In Theorems 1′ and 2′ (see Paragraph 2) we show that in the "stable range", the homotopy type of a homology circle has the same characterization as that of a knot complement, except that the acyclic Λ-modules involved are not required to be finitely generated.

Organization of the paper. Paragraph 2 contains the definitions and the statement of our results. Paragraph 3 begins with a review of perfect and acyclic modules, and then proves Theorem 1. Paragraph 4 proves Theorem 2, and the proofs of Theorems 1′ and 2′ are sketched in Paragraph 5.

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Added in proof. We have been able to obtain similar results for link complements; see *Link complements and coherent group rings*, to appear in the Illinois Journal of Mathematics.

2. Statement of Results

By a *knot* we mean an embedding $f: S^{q-2} \to S^q$ which is either *smooth* or *piecewise linear*. Given such an embedding, there is always a tubular neighborhood (in the smooth case) or a regular neighborhood (in the piecewise linear case) U of $f(S^{q-2})$ in S^q, and this neighborhood is essentially unique (see [11, Appendix 2] for the smooth case and [5, Chapter 2] for the piecewise linear case). The *complement* C of the knot is the closure of $S^q - U$. It has the following properties:

2.1. The knot complement C is a *finite CW-complex* homotopy equivalent to $S^q - f(S^{q-2})$, and by Alexander duality, C is a *homology circle*, i.e., C has the homology groups of the circle S^1.

This paper will only consider those knots for which there is a positive integer n such that $\pi_i C \cong \pi_i S^1$ for $i \leq n$. Such knots have an infinite cyclic fundamental group, and so all higher homotopy groups will be modules over $\Lambda = \mathbf{Z}[\mathbf{Z}]$, the group ring of the integers. A fundamental property of Λ is that it is *Noetherian* [9, p. 136], and so:

2.2. *The class of finitely generated Λ-modules is closed under the operations of taking submodules, quotient modules, and module extensions.*

If G is a group and M is a $\mathbf{Z}[G]$-module, the *homology groups* $H_i(G; M)$ are, by definition, the homology groups of a $K(G, 1)$ with (twisted) coefficients in M, and a $\mathbf{Z}[G]$-module M is called *acyclic* if $H_i(G; M) \cong 0$ for all $i \geq 0$.

Our main results are then the following two theorems.

THEOREM 1. *Let n be a positive integer, and let C be a knot complement such that $\pi_i C \cong \pi_i S^1$ for $i \leq n$. Then $\pi_i C$ is a finitely generated acyclic Λ-module for $n+1 \leq i \leq 2n$.*

The only properties of C that we use here (see Paragraph 3) are that it is a finite CW-complex and a homology circle. Thus, Theorem 1 still holds if we only assume that C is the complement of a compact homology (q-2)-sphere embedded in S^q.

THEOREM 2. *Let n be a positive integer, and let X be a space for which $\pi_i X \cong \pi_i S^1$ for $i \leq n$, $\pi_i X$ is a finitely generated acyclic Λ-module for $n+1 \leq i \leq 2n$, and $\pi_i X \cong 0$ for $i \geq 2n+1$. Then for any $q \geq 4n+3$, there is an* *embedding* $f: S^{q-2} \to S^q$ *with complement* C, *and a map* $C \to X$ *inducing isomorphisms* $\pi_i C \cong \pi_i X$ *for* $i \le 2n$.

We have analogous results for arbitrary homology circles.

THEOREM 1′. *Let* n *be a positive integer, and let* C *be a homology circle such that* $\pi_i C \cong \pi_i S^1$ *for* $i \le n$. *Then* $\pi_i C$ *is an acyclic* Λ*-module for* $n+1 \le i \le 2n$.

THEOREM 2′. *Let* n *be a positive integer, and let* X *be a space for which* $\pi_i X \cong \pi_i S^1$ *for* $i \le n$, $\pi_i X$ *is an acyclic* Λ*-module for* $n+1 \le i \le 2n$, *and* $\pi_i X \cong 0$ *for* $i \ge 2n+1$. *Then there is a homology circle* C *and a map* $C \to X$ *inducing isomorphisms* $\pi_i C \cong \pi_i X$ *for* $i \le 2n$.

3. Proof of Theorem 1

If G is a group and M is a $\mathbf{Z}[G]$-module, then M is called *perfect* if $H_0(G; M) \cong 0$, and *acyclic* if $H_i(G; M) \cong 0$ for all $i \ge 0$. (Since $H_0(G; M) \cong M/$(action of G), this is equivalent to the usual definition of perfect.)

3.1 LEMMA. *If* $G \cong \mathbf{Z}$, *then* M *is acyclic if and only if* $H_i(G; M) \cong 0$ *for* $i = 0, 1$.

Proof. This follows easily from the fact that S^1 is a $K(\mathbf{Z}, 1)$.

3.2 LEMMA. *If* M *is a finitely generated* Λ*-module, then* M *is acyclic if and only if* $H_0(G; M) \cong 0$.

Proof. See [4].

3.3 PROPOSITION. *The class of finitely generated acyclic* Λ*-modules is closed under the operations of taking submodules, quotient modules, and module extensions.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence of Λ-modules. If A and C are both finitely generated and acyclic, then 2.2 and the exact homology sequence imply that B is also. If B is finitely generated and acyclic, then (3.2) so is C, and then 2.2 and the exact homology sequence imply that A is also.

We will also need the following lemma.

3.4 LEMMA. *If* D *is a finitely generated acyclic* Λ*-module, then so is* $H_i K(D, m)$ *for* $i \le 2m-1$.

Proof. This follows from 3.3 together with the computation of these groups given in [2, p. 11–11].

Proof of Theorem 1. Let $\tilde{C}$ be the universal cover of C. Then $\mathbf{Z} \cong \pi_1 C$ acts on $\tilde{C}$, and (2.1 and 2.2) $H_i \tilde{C}$ is a finitely generated Λ-module for all $i \ge 0$, and we have the following lemma (which we prove at the end of this section).

3.5 LEMMA. *$H_i\tilde{C}$ is an acyclic Λ-module for all $i>0$.*

To complete the proof we will show by induction on k, that for $n+1\leq k\leq 2n$,

(1) $\pi_k C$ is a finitely generated acyclic Λ-module, and
(2) $H_i P^k\tilde{C}$ is a finitely generated acyclic Λ-module for $i\leq 2n+1$ where $P^k\tilde{C}$ is the k-*th* Postnikov approximation of $\tilde{C}$.

The induction is begun by Lemma 3.4 (with $m=n+1$), together with the fact that $\pi_{n+1}C\cong H_{n+1}\tilde{C}$.

Now assume that we have shown (1) and (2) through dimension k-1, and consider the Serre spectral sequence of the fibration

$$K(\pi_k C, k)\to P^k\tilde{C}\to P^{k-1}\tilde{C}.$$

This is a spectral sequence of Λ-modules, and so it follows easily (3.3) that $\pi_k C$ is a finitely generated acyclic Λ-module. Lemma 3.4 (with $m=k$) now implies that $E^2_{0,q}$ is a finitely generated acyclic Λ-module for $q\leq 2k-1$, and so statement (2) is clear, and the induction is complete.

Proof of Lemma 3.5. Consider the Serre spectral sequence of the fibration $\tilde{C}\to C\to S^1$. Since S^1 is a $K(\mathbf{Z},1)$, we have $E^2_{p,q}\cong H_p(\mathbf{Z};H_q\tilde{C})$, and since S^1 is one-dimensional, $E^2_{p,q}\cong 0$ for $p\geq 2$. Thus, all the differentials vanish, and so $H_p(\mathbf{Z};H_q\tilde{C})\cong E^2_{p,q}\cong 0$ for $q>0$.

4. Proof of Theorem 2

We will use the following theorem of Wall [13, p. 17].

THEOREM. *Let W be a finite CW-complex of dimension $2n+2$ such that $\pi_1 W\cong\mathbf{Z}$ and $H_*W\cong H_*S^1$, and let $q\geq 4n+3$. Then there is a smooth embedding $S^{q-2}\to S^q$ with complement C, and a map $C\to W$ inducing isomorphisms $\pi_i C\cong\pi_i W$ for $i\leq 2n$.*

Thus, it is sufficient to construct W as above, together with a map $W\to X$ inducing isomorphisms $\pi_i W\cong\pi_i X$ for $i\leq 2n$. To do this we will need the following theorem, also due to Wall [12, Theorems A and B].

THEOREM. *Let X be a CW-complex for which $\pi_1 X\cong\mathbf{Z}$, and let $\tilde{X}$ be the universal cover of X. If there is an integer k such that $H_i\tilde{X}$ is a finitely generated Λ-module for $i\leq k$, then X is homotopy equivalent to a CW-complex with a finite k-skeleton.*

To make use of this theorem, we prove the following proposition.

PROPOSITION. *$H_i\tilde{X}$ is a finitely generated Λ-module for $i\leq 2n+1$.*

Proof. One shows by induction on k that for $k\leq 2n+1$ and $i\leq 2n+1$, the groups $H_iP^k\tilde{X}$ are finitely generated Λ-modules (where $P^k\tilde{X}$ is the k-*th*

Postnikov approximation of $\tilde{X}$). The induction follows easily from the Serre spectral sequence, using Lemma 3.4.

Thus, we may assume that X^{2n+1}, the $(2n+1)$-skeleton of X, is finite, and we have the following proposition (which we prove at the end of this section).

4.1 PROPOSITION. *$H_iP^{2n}X^{2n+1} \cong H_iS^1$ for $i \leq 2n+1$.*

In particular, we have isomorphisms $H_iX^{2n+1} \cong H_iS^1$ for $i \leq 2n$. Now $H_{2n+1}X^{2n+1}$ is a (finitely generated) free abelian group, so we can choose a (finite) free basis for this group and, because we have the exact sequence

$$\pi_{2n+1}X^{2n+1} \to H_{2n+1}X^{2n+1} \to H_{2n+1}P^{2n}X^{2n+1},$$

we can lift each element of this basis to $\pi_{2n+1}X^{2n+1}$. If we now use these elements of $\pi_{2n+1}X^{2n+1}$ to attach $(2n+2)$-cells, we obtain our complex W with $H_*W \cong H_*S^1$. The inclusion $X^{2n+1} \to X$ can now be extended over all of W (because $\pi_{2n+1}X \cong 0$), and so the proof is complete (except for Proposition 4.1).

Proof of Proposition 4.1. One shows by induction on k that

$$H_iP^kX^{2n+1} \cong H_iS^1 \quad \text{for} \quad k \leq 2n \quad \text{and} \quad i \leq 2n+1.$$

The induction follows easily from the Serre spectral sequence, using Lemma 3.4 together with the fact that $H_p(P^kX^{2n+1}; M) \cong 0$ for $2 \leq p \leq n$ and any local coefficient system M on P^kX^{2n+1}.

5. Homology circles

Sketch of proof of Theorem 1′. We will need the following lemma.

5.1 LEMMA. *If G is an infinite cyclic group generated by t, then for any $\mathbf{Z}[G]$-module M there is a natural exact sequence*

$$0 \to H_1(G; M) \to M \xrightarrow{\varphi} M \to H_0(G; M) \to 0$$

where $\varphi = t - 1_M$. Thus, M is acyclic if and only if $\varphi: M \to M$ is an isomorphism.

Proof. This follows readily from the fact that S^1 is a $K(\mathbf{Z}, 1)$.

Now if we let $\tilde{C}$ be the universal cover of C, then $\tilde{C}$ is n-connected, and we want to show that $\pi_i\tilde{C}$ is an acyclic Λ-module for $i \leq 2n$. As in Lemma 3.5, we know that $H_i\tilde{C}$ is an acyclic Λ-module for all $i > 0$.

Now if $S\tilde{C}$ is the *suspension* of $\tilde{C}$, then we have natural isomorphisms $\pi_i\tilde{C} \cong \pi_{i+1}S\tilde{C}$ for $i \leq 2n$ [10, p. 458] and $H_i\tilde{C} \cong H_{i+1}S\tilde{C}$ for all $i > 0$. Thus, we know that the $H_i S\tilde{C}$ are acyclic Λ-modules for all $i > 0$, and we want to show that the $\pi_i S\tilde{C}$ are acyclic Λ-modules for $i \leq 2n+1$.

If we now let t denote both a generator of $\pi_1 C$ and the automorphism it induces on $S\tilde{C}$, then we can define a map $\psi: S\tilde{C} \to S\tilde{C}$ by $\psi = t - 1_{S\tilde{C}}$ [10, p. 41], and we have the following lemma (which is easily verified).

LEMMA. *The map ψ induces φ: $H_i S\tilde{C} \to H_i S\tilde{C}$ for all $i > 0$, where φ is as in Lemma* 5.1.

Thus, by Lemma 5.1, ψ_*: $H_i S\tilde{C} \to H_i S\tilde{C}$ is an *isomorphism* for all i. Since $S\tilde{C}$ is simply connected, this implies that ψ_*: $\pi_i S\tilde{C} \to \pi_i S\tilde{C}$ is also an isomorphism for all i, and we have the following lemma (which is also easily verified).

LEMMA. *The map ψ induces φ: $\pi_i S\tilde{C} \to \pi_i S\tilde{C}$ for $i \leq 2n+2$.*

Lemma 5.1 now implies that $\pi_i S\tilde{C}$ is an acyclic Λ-module for $i \leq 2n+1$, and the proof is complete.

Sketch of Proof of Theorem 2′. Letting X^{2n+1} denote the $(2n+1)$-skeleton of X one shows, as in Proposition 4.1, that $H_i P^{2n} X^{2n+1} \cong H_i S^1$ for $i \leq 2n+1$ (this is proved using Lemma 5.2 (below) in place of Lemma 3.4). Then, as in the proof of Theorem 2, we can attach (perhaps infinitely many) $(n+2)$-cells to X^{2n+1} to obtain a homology circle.

5.2 LEMMA. *If D is an acyclic Λ-module, then so is $H_i K(D, m)$ for $i \leq 2m-1$.*

Proof. This is similar to the proof of Theorem 1′. We know that $\pi_i SK(D, m)$ is acyclic for $i \leq 2m$, and so the same must be true of $H_i SK(D, m)$.

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