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FREE HEEGAARD DIAGRAMS AND EXTENDED NIELSEN TRANSFORMATIONS, II

BY
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Introduction

This is the second of two papers (see also [Cr]) relating a free analogue, called a free splitting homomorphism, of the Stallings–Jaco formalism for Heegaard splittings of 3-manifolds (see [St] and [Jc]) with the theory of extended Nielsen transformations. Free splitting homomorphisms are homorphisms of the form

$$\psi = \psi_1 \times \psi_2 \colon G^m \to X^n \times Y^n$$

where G^m, X^n, and Y^n are free groups of ranks m, n, and n respectively and each of the factor homomorphisms $\psi_1\colon G^m \to X^n$ and $\psi_2\colon G^m \to Y^n$ is surjective. A theory of equivalence and stable equivalence for free splitting homomorphisms is developed in [Cr] modelled on the corresponding theory for Heegaard diagrams. Extended Nielsen transformations may be regarded as certain special Tietze transformations that preserve the deficiency of a group presentation. They will be reviewed in §1. Sometimes extended Nielsen transformations include operations that allow one to add new generators and relations to a presentation—i.e. to stabilize the presentation—and sometimes these operations are excluded.

A connection between the free splitting homomorphism theory and the theory of extended Nielsen transformations is made in [Cr] by normalizing a free splitting homomorphism $\psi = \psi_1 \times \psi_2\colon G^m \to X^n \times Y^n$ so that for free bases $\{g_i : i \leq m\}$ for G^m and $\{x_i : i \leq n\}$ for X^n we have $\psi_1(g_i) = x_i$ $(i \leq n)$ and $\psi_1(g_{i+n}) = 1$ $(i \leq m-n)$. In a manner analogous to the reading of a group presentation for the fundamental group of a 3-manifold from a Heegaard diagram for the 3-manifold, we associate a group presentation $\mathcal{P}(\psi) = \langle Y^n : (r_i)\rangle$ with the normalized ψ where (r_i) is the $(m-n)$-tuple $(\psi_2(g_{i+n}))$. Theorem 4.1 of [Cr] says that two free splitting homomorphisms ψ and ϕ in normal form and with associated group presentations $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are stably equivalent if and only if the presentations $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are equivalent under extended Nielsen transformations including stabilization.

Our interest in this paper will be in establishing a more detailed connection between the free splitting homomorphism theory and the theory of extended Nielsen transformations and in using the information thus obtained to gain further insight into the difficult Andrews–Curtis conjecture

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[AC, FH] and [AC, EN] which says that balanced presentations for the trivial group can be converted to the standard ones through the use of extended Nielsen transformations. Two questions that we will be moving towards are these: When does an equivalence between normalized free splitting homomorphisms ψ and ϕ convert to an extended Nielsen equivalence between $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ that avoids stabilization of $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$? If for equivalent ψ and ϕ as above, it is necessary to stabilize $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ in order to make them equivalent under extended Nielsen transformations, is there a reasonable bound on the amount of stabilization needed? We will obtain answers to these two questions through a detailed analysis of methods due to Rapaport [Rp] (see also [Pf]). This analysis will enable us to give a purely algebraic proof of the stable classification theorem mentioned above, Theorem 4.1 of [Cr], as opposed to the geometric proof given in [Cr]. The Rapaport techniques will lead us also to some new criteria for simplifying presentations for the trivial group by means of extended Nielsen transformations. Finally our investigations will provide us with a possible method for producing examples of balanced presentations for the trivial group of the form $\langle Y^n : (r_i)\rangle$ that cannot be converted to the standard presentation

$$\langle Y^n : (y_1, \ldots, y_n)\rangle$$

unless stabilization is used.

Some familiarity with [Cr], especially §§2 and 3, would be helpful to the reader, although we will review some things in §1 here. The logical dependence on [Cr] here is limited to the two normalization theorems in §3 of [Cr] and the first half of the proof of Theorem 4.1 in §4 of [Cr].

The paper is divided into five sections. In §1 we will review extended Nielsen transformations and the free splitting homomorphism theory, and we will show how considerations of equivalence for free splitting homomorphisms lead us directly to Rapaport's work on invertible transformations [Rp]. In §2 we will consider Rapaport's methods in detail. In §3 we will give a criterion for simplifying presentations for the trivial group through the use of extended Nielsen transformations. This criterion mostly involves normal closure properties of auxilliary presentations connected with the original group presentation. In §4 we will give a partial classification theorem relating the two theories, and we will give the bounds on stabilization promised earlier. These bounds will lead us to the promised algebraic proof of the stable classification theorem in [Cr.]. In §5 we will examine the failure of the partial classification in §4 to completely resolve matters without stabilization. We will describe an example and a conjecture, Conjecture 5.3, designed to clarify the reasons for this failure. The conjecture is aimed at determining whether there are examples of balanced presentations for the trivial group that cannot be made standard by extended Nielsen transformations unless stabilization is allowed. The conjecture concerns subgroups of the automorphism group of a free group; the falsity of the conjecture will be seen to guarantee that such examples exist.

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1. Review of extended Nielsen transformations and free splitting homomorphisms

We denote by G^m, X^n, and Y^n $(m=1,2,\ldots, n=1,2,\ldots)$ the free groups with canonical free bases $\{g_i: i\leq m\}$, $\{x_i: i\leq n\}$, and $\{y_i: i\leq n\}$. In general, for a set of elements $\{r_i\}$ in a free group W, we denote by $\mathrm{Gp}(\{r_i\})$ or $\mathrm{Gp}_W(\{r_i\})$ the subgroup generated by $\{r_i\}$ and we denote by $\mathrm{Cl}(\{r_i\})$ or $\mathrm{Cl}_W(\{r_i\})$ the smallest normal subgroup of W containing $\{r_i\}$.

Extended Nielsen transformations are regarded as acting on p-tuples (r_i) of elements of Y^n $(p=0,1,\ldots, n=1,2,\ldots)$ or as acting on the corresponding group presentations $\langle Y^n: (r_i)\rangle$. Because we will sometimes be changing the rank of Y^n we will, on occasion, write $(r_i)_n$ to indicate that we are considering (r_i) as a tuple of elements in Y^n. We consider the following elementary transformations acting on a p-tuple $(r_i)_n$:

Type 1. Replace r_k by r_k^{-1} for some k.
Type 2. Replace r_k by r_kr_j for some k and some $j\neq k$.
Type 3. Replace r_k by $y_j^{\mp 1}r_ky_j^{\pm 1}$ for some k and some $j\leq n$.
Type 4. Replace $(r_i)_n$ by $(s_i)_n=(\lambda(r_i))_n$ where λ is an automorphism of Y^n.
Type 5. Replace $(r_i)_n$ by the $(p+1)$-tuple of elements in Y^{n+1}, $((r_i), y_{n+1})_{n+1}$. Notice that $n+1$ now replaces n and $p+1$ replaces p.
Type 6. If $r_p=y_n$ and each r_i $(i<p)$ is an element of Y^{n-1}, replace $(r_i)_n$ by the $(p-1)$-tuple of elements in Y^{n-1}, $(r_1,\ldots, r_{p-1})_{n-1}$. Notice that $n-1$ now replaces n and $p-1$ replaces p.

A Q, Q^*, resp. Q^{**}-*transformation* $(r_i)_{n_1}\to(s_i)_{n_2}$ is any composition of transformations of Types 1–3, 1–4, resp. 1–6, and two tuples $(r_i)_{n_1}$ and $(s_i)_{n_2}$ or the corresponding group presentations $\langle Y^{n_1}: (r_i)\rangle$ and $\langle Y^{n_2}: (s_i)\rangle$ are Q, Q^*, resp. Q^{**}-*equivalent* if one is a Q, Q^*, resp. Q^{**}-transform of the other. See [AC, FH], [Rp], and [Mz, HT] for other descriptions of extended Nielsen transformations. The Q^{**}-equivalence relation will be used sparingly here.

Example 1. The transformation $(y_1, y_2)\to(y_2^{-1}y_1y_2y_1^{-2}, y_1^{-1}y_2y_1y_2^{-2})$ of elements in Y^2 is not a Nielsen transformation since the two elements in the second tuple do not generate Y^2. The transformation above is, however, a Q-transformation as is seen by the following sequence of steps (some elementary steps have been combined):

$$(y_1, y_2) \to (y_2^{-1}, y_1) \to (y_2^{-1}y_1^{-1}, y_1) \to (y_2^{-1}y_1^{-1}, y_2y_1y_2^{-1})$$
$$\to (y_2^{-1}y_1^{-1}, y_2^{-1}y_1^{-1}y_2y_1y_2^{-1})$$
$$\to (y_2^{-1}y_1^{-1}, y_1^{-1}y_2y_1y_2^{-2}) \to (y_2y_1^{-1}y_2^{-2}, y_1^{-1}y_2y_1y_2^{-2})$$
$$= (y_2y_1^{-2}y_2^{-1}y_1y_1^{-1}y_2y_1y_2^{-2}, y_1^{-1}y_2y_1y_2^{-2}) \to (y_2y_1^{-2}y_2^{-1}y_1, y_1^{-1}y_2y_1y_2^{-2})$$
$$\to (y_2^{-1}y_1y_2y_1^{-2}, y_1^{-1}y_2y_1y_2^{-2}).$$

This example will be used as a model to illustrate several constructions to be introduced later.

A *free splitting homomorphism* is a homomorphism $\psi = \psi_1 \times \psi_2 \colon G^m \to X^n \times Y^n$ where each of the factor homomorphisms $\psi_1 \colon G^m \to X^n$ and $\psi_2 \colon G^m \to Y^n$ is surjective. Two free splitting homomorphisms ψ and ϕ are defined to be *equivalent* if there are automorphisms $\eta \colon G^m \to G^m$, $\eta_1 \colon X^n \to X^n$, and $\eta_2 \colon Y^n \to Y^n$ such that the diagram below is commutative:

$$\begin{array}{ccc} G^m & \xrightarrow{\phi} & X^n \times Y^n \\ \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle (\eta_1,\eta_2)} \\ G^m & \xrightarrow{\psi} & X^n \times Y^n \end{array} \tag{1}$$

Let $\psi \colon G^{m_1} \to X^{n_1} \times Y^{n_1}$ and $\phi \colon G^{m_2} \to X^{n_2} \times Y^{n_2}$ be two free splitting homomorphisms. Let

$$\mu \colon X^{n_2} \times Y^{n_2} \to X^{n_1+n_2} \times Y^{n_1+n_2}$$

be defined by $\mu(x_i, 1) = (x_{i+n_1}, 1)$ and $\mu(1, y_i) = (1, y_{i+n_1})$. We define the *sum* $\psi \# \phi$ of ψ and ϕ to be the free splitting homomorphism from $G^{m_1+m_2}$ to $X^{n_1+n_2} \times Y^{n_1+n_2}$ given by

$$g_i \to \psi(g_i)\ (i \le n_1), \qquad g_{i+n_1} \to \mu\phi(g_i)\ (i \le n_2),$$
$$g_{i+n_1+n_2} \to \psi(g_i)\ (i \le m_1 - n_1), \qquad g_{i+m_1+n_2} \to \mu\phi(g_{i+n_2}).$$

We define a standard free splitting homomorphism $\chi_n \colon G^{2n} \to X^n \times Y^n$ by $\chi_n(g_i) = (x_i, y_i)\ (i \le n)$ and $\chi_n(g_{i+n}) = (1, y_i)$. Finally, we define two free splitting homomorphisms ψ and ϕ to be *stably equivalent* provided that for some p and q, the homomorphisms $\psi \# \chi_p$ and $\phi \# \chi_q$ are equivalent.

A free splitting homomorphism $\psi \colon G^m \to X^n \times Y^n$ is said to be in *normal form* if $\psi(g_i) = (x_i, w_i)\ (i \le n)$ and $\psi(g_{i+n}) = (1, r_i)$ where $w_i\ (i \le n)$ and $r_i\ (i \le m - n)$ are elements in Y^n. For ψ in the normal form as above, we refer to the group presentation $\mathcal{P}(\psi) = \langle Y^n : (r_i) \rangle$ as the *presentation associated with* ψ. If ψ is in the normal form above, and if $w_i = y_i\ (i \le n)$, then ψ is said to be a *Mihailova map* (compare [Mh] and [Ml, pp. 35–42]). Notice that every finite group presentation $\mathcal{P} = \langle Y^n : (r_i) \rangle$ is the associated group presentation $\mathcal{P}(\psi)$ for some Mihailova map ψ. If (r_i) has p elements, set

$m=p+n$ and just define $\psi: G^m \to X^n \times Y^n$ so that it is a Mihailova map with $\mathcal{P}(\psi)=\mathcal{P}$. In [Cr, Section 3] it is shown that every free splitting homomorphism ψ is equivalent to one in normal form and is stably equivalent to a Mihailova map. Theorem 4.1 of [Cr] says that two free splitting homomorphisms ψ and ϕ in normal form and with associated group presentations $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are stably equivalent if and only if $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are Q^{**}-equivalent.

We describe in the next two paragraphs a lifting technique that will be used to convert equivalences between free splitting homomorphisms ψ and ϕ in normal form to Q^*-equivalences between $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$. There appear to be, in some cases, obstructions to making this conversion, and these obstructions will be considered in §5.

The notation in this paragraph will be assumed to hold throughout the remainder of the paper. Consider the free product $X^n * Y^n$. It is freely generated by $\{x_i\}\cup\{y_i\}$, but it is also freely generated by $\{y_i\}\cup\{x_iy_i\}$. Set $z_i=x_iy_i$ and set $Z^n=\mathrm{Gp}(\{z_i\})$. Then $X^n * Y^n$ has the internal free product decomposition $X^n * Y^n = Y^n * Z^n$ corresponding to the free basis $\{y_i\}\cup\{z_i\}$. We define as follows three projection homomorphisms for $X^n * Y^n = Y^n * Z^n$ to subgroups: $\xi(x_i)=x_i$, $\xi(y_i)=1$, $(\xi(z_i)=x_i)$, $\upsilon(x_i)=1$, $\upsilon(y_i)=y_i$, $(\upsilon(z_i)=y_i)$, and $\zeta(y_i)=1$, $\zeta(z_i)=z_i$, $(\zeta(x_i)=z_i)$. Let π denote the natural factor map $\pi=\xi\times\upsilon: X^n * Y^n \to X^n\times Y^n$.

Consider now (1), the model for equivalence. Assume that ψ and ϕ are in normal form, and assume for simplicity that $\eta_1=\mathrm{id}$ and $\eta_2=\mathrm{id}$. Let $\psi(g_{i+n})=(1, r_i)$ and $\phi(g_{i+n})=(1, s_i)$. We define lifts $\hat{\psi}$ and $\hat{\phi}$ of ψ and ϕ to homomorphisms from G^m to $X^n * Y^n$. First choose elements $v_i \in X^n * Y^n$ $(i\leq n)$ such that $\pi(v_i)=\psi(g_i)$. Set $\hat{\psi}(g_i)=v_i$ $(i\leq n)$ and $\hat{\psi}(g_{i+n})=r_i$. Then set $\hat{\phi}=\hat{\psi}\eta$. We have then, $\psi=\pi\hat{\psi}$ and $\phi=\pi\hat{\phi}$, and what we have done, in effect, is to respell each element s_i as $\hat{s}_i=\hat{\phi}(g_{i+n})$ using the symbols v_j and r_j and using the homomorphism υ to translate back to the old spelling. We will be able to use this respelling to convert the equivalence in (1) to a Q^*-equivalence between $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ provided that η takes $\mathrm{Gp}(\{g_i: i\leq n\})$ onto itself; i.e. $\hat{\psi}\eta\hat{\phi}^{-1}\mid V$ defines an automorphism of V where $V=\mathrm{Gp}(\{v_i\})$.

Example 2. Let $\psi: G^4\to X^2\times Y^2$ and $\phi: G^4\to X^2\times Y^2$ be defined by $\psi=\chi_2$,

$$\phi(g_i)=(x_i, y_i)\ (i\leq 2),\qquad \phi(g_3)=(1, y_2^{-1}y_1y_2y_1^{-2})$$

and

$$\phi(g_4)=(1, y_1^{-1}y_2y_1y_2^{-2}).$$

Define an automorphism $\eta: G^4\to G^4$ by

$$\eta(g_i)=g_i\ (i\leq 2),\qquad \eta(g_3)=g_2^{-1}g_1g_2^2g_4^{-1}g_3^{-2}g_2^{-1}g_3g_4g_2^{-1}g_1^{-1}g_2$$

and

$$\eta(g_4)=g_2g_4^{-1}g_3^{-1}g_2g_3g_2^{-2}.$$

Then η, $\eta_1=\mathrm{id}$, and $\eta_2=\mathrm{id}$ define an equivalence in (1) between ψ and ϕ. If we define $\hat{\psi}$ and $\hat{\phi}$ as in the preceding paragraph, then corresponding to the relator tuple $(\phi_2(g_3), \phi_2(g_4))$ we have the 2-tuple of elements in $X^2 * Y^2$,

$$(\hat{\phi}(g_3), \hat{\phi}(g_4)) = (z_2^{-1}z_1z_2^2y_2^{-1}y_1^{-2}z_2^{-1}y_1y_2z_2^{-1}z_1^{-1}z_2,\ z_2y_2^{-1}y_1^{-1}z_2y_1z_2^{-2}).$$

This particular 2-tuple of elements in $X^2 * Y^2$ is an example of what will be called an invertible respelling (of $(\phi_2(g_3), \phi_2(g_4))$) in §2. The Q-equivalence of $(y_1, y_2)_2$ and $(\phi_2(g_3), \phi_2(g_4))_2=(y_2^{-1}y_1y_2y_1^{-2}, y_1^{-1}y_2y_1y_2^{-2})$, already shown in Example 1, will be seen in §2 to follow from the fact that $\{z_1, z_2, \hat{\phi}(g_3), \hat{\phi}(g_4)\}$ freely generates $X^2 * Y^2$. Also the equivalence between ψ and ϕ is typical of the kind of equivalence which converts to a Q^*-equivalence between $\mathscr{P}(\psi)$ and $\mathscr{P}(\phi)$.

Respelling techniques are essential features in papers by Rapaport [Rp] and Peiffer [Pf] on group presentations. The Rapaport respelling technique corresponds in the situation above to the case where

$$v_i = z_i \quad\text{and}\quad \eta\,|\,\mathrm{Gp}(\{g_i\colon i\le n\}) = \mathrm{id}.$$

Instead of using the second alphabet $\{z_i\}$, Rapaport uses an exponent language. Thus $\hat{s}_i$ would be written as $\prod r_{i_j}^{\pm\alpha_j}$ where $\alpha_j \in Y^n$. If $\{\beta_j\}$ were a set of elements in Z^n with $\upsilon(\beta_j)=\alpha_j$, then $\prod r_{i_j}^{\pm\alpha_j}$ would correspond to the element $\prod \beta_j^{-1}r_{i_j}^{\pm 1}\beta_j$ in $X^n * Y^n$.

2. Respellings, the Rapaport invertibility criterion

In this section we develop a variant form of Rapaport's invertibility criterion [Rp, Th. 1] characterizing Q-equivalent tuples. Some of the material here can be extracted from [Rp, §§3 and 5], but in some cases it is only by an analysis of the proof of Rapaport's invertibility criterion that we are able to obtain tools needed in later sections of the paper. For example, there is a lovely technique exhibited in Rapaport's proof of her Theorem 1 that we call "conservative Nielsen transformations"; Metzler and Browning use the term "relative Nielsen transformations" (see [Mz, ÄG, §4] and [Br]). This technique is the key to understanding (a) the invertibility criterion, (b) the criterion in §3 for simplifying presentations for the trivial group by Q-transformations, (c) the revised, algebraic proof of Theorem 4.1 of [Cr] in §4, and (d) the possible obstructions described in §5 to converting an equivalence of normalized free splitting homomorphisms to a Q^*-equivalence of the associated group presentations. Because of this importance, the technique of conservative Nielsen transformations will be isolated and discussed in detail.

Let V be a finitely generated subgroup of $X^n * Y^n = Y^n * Z^n$ such that the natural homomorphism from $Y^n * V$ to $\mathrm{Gp}\,(Y^n \cup V)$ induced by inclusion on each of the factors is an isomorphism. Note that V is free by the

Nielsen–Schreier subgroup theorem (see [MKS, Cor. 2.9]). Because $\mathrm{Gp}(Y^n \cup V)$ is naturally isomorphic to $Y^n * V$, there is a well-defined projection homomorphism

$$\theta_V : \mathrm{Gp}(Y^n \cup V) \to V$$

given by $\theta_V(y) = 1$ for $y \in Y^n$ and $\theta_V(v) = v$ for $v \in V$. Let (r_i) be a p-tuple of elements in Y^n, and let $s \in Y^n$. By a *respelling of s relative to* $((r_i), V)$ we mean an element $\hat{s} \in X^n * Y^n$ such that

(2) (i) $\upsilon(\hat{s}) = s$,
(ii) $\theta_V(\hat{s}) = 1$, and
(iii) $\hat{s} = \prod u_i$ where each syllable u_i either belongs to V or has the form $r_j^{\pm 1}$ for some r_j in (r_i). If the relator tuple (r_i) contains 1's these may appear in the syllabification.

By a *respelling* of a q-tuple (s_i) of elements in Y^n relative to $((r_i), V)$ we mean a q-tuple $(\hat{s}_i)$ such that each $\hat{s}_i$ is a respelling of s_i relative to $((r_i), V)$. Condition (ii) shows that any respelling $\hat{s}$ of s relative to $((r_i)\ V)$ can also be regarded as a respelling relative to $((y_i), Z^n)$ by a possible revision of the syllables u_i, for $\zeta(\hat{s}) = \zeta\theta_V(\hat{s}) = 1$. Sometimes we will refer to elements or tuples as respellings by which we will mean respellings relative to $((y_i), Z^n)$.

The discussion at the end of §1 suggests that we should have $\xi \mid V$ an isomorphism of V onto X^n, but this is not necessary and would be in fact misleading. The property of V that makes it work in respellings is the existence of the projection map $\theta_V : \mathrm{Gp}(Y^n \cup V) \to V$ rather than the character of $\xi \mid V$. We remark also that condition (ii) in (2) enables us to avoid heavy reliance on conjugations found in [Pf] and [Rp]. We will, however, find it necessary in a couple of places to express certain elements of $X^n * Y^n$ as products of conjugates of powers of the elements r_j. The following lemma is designed for such needs:

LEMMA 2.1. *If $\hat{s} = \prod u_i$ is a respelling of s relative to $((r_i), V)$, then $s \in \mathrm{Cl}_Y(\{r_i\})$. In fact, $\hat{s} \in \mathrm{Cl}_{X*Y}(\{r_i\})$, and $\hat{s}$ can be written as a product $\prod c_j^{-1} r_{i_j}^{\varepsilon_j} c_j$ where each $\varepsilon_j = \pm 1$ and each $c_j \in V$.*

Proof. The last assertion in the conclusion of the lemma implies the others so we establish it. By redefining the syllables of $\hat{s}$, if necessary, we may suppose that $\theta_V(\hat{s}) = \prod \theta_V(u_i)$ cancels to 1 by removal of successive pairs of syllables of the form $v^{-\varepsilon} v^{\varepsilon}$ where $\varepsilon = \pm 1$ and $v \in V$. Check off pairs $(v^{-\varepsilon}, v^{\varepsilon})$ in $\hat{s}$ as their projections cancel in $\theta_V(\hat{s})$. Consider the last pair to be cancelled. We have $\hat{s} = \hat{s}_1 v^{-\varepsilon} \hat{s}_2 v^{\varepsilon} \hat{s}_3$ where $\hat{s}$, $\hat{s}_2$, and $\hat{s}_3$ are respellings of elements s_1, s_2, and s_3 relative to $((r_i), V)$. By an induction argument, we may suppose that $\hat{s}_1$, $\hat{s}_2$, and $\hat{s}_3$ can be expressed in the desired form as products of V-conjugates of powers of the r_i's. But then $\hat{s}$ can also be expressed in this form and so the lemma is proved. ■

Let (r_i) and (s_i) be p-tuples of elements in Y^n, and let $(\hat{s}_i)$ be respelling of (s_i) relative to $((r_i), V)$. We say that $(\hat{s}_i)$ is *invertible* if there is a respelling $(\hat{r}_i)$ of (r_i) relative to $((s_i), V)$ with syllabification $\hat{r}_i = \prod w_{ij}$ (as required by (iii) in (2)) so that for each $\hat{r}_i$, when every syllable w_{ij} of the form $s_k^{\pm 1}$ $(1 \le k \le p)$ is replaced by $(\hat{s}_k)^{\pm 1}$, the resulting word is equal to r_i in the free group $X^n * Y^n$. The respelling $(\hat{r}_i)$ will be called an *inverse respelling* to $(\hat{s}_i)$.

The 2-tuple $(\phi(g_3), \phi(g_4))$ in Example 2 is an invertible respelling of $(\phi_2(g_3), \phi_2(g_4))$ relative to $((y_1, y_2), Z^2)$. It is easy to check this by using the lemma below which offers a more convenient description of invertibility.

LEMMA 2.2. *Let (r_i) and (s_i) be p-tuples of elements in Y^n, and let $(\hat{s}_i)$ be a respelling of (s_i) relative to $((r_i), V)$. Let $\{v_i\}$ be a free basis for V. Then $(\hat{s}_i)$ is invertible if and only if $\{\hat{s}_i\} \cup \{v_i\}$ generates* $\mathrm{Gp}(\{r_i\} \cup V)$.

Proof. First suppose that $(\hat{s}_i)$ is invertible. The substitution condition insures that each r_i can be expressed as a product of powers on the elements $\hat{s}_1, \ldots$ and $v_1, \ldots$. Thus $\mathrm{Gp}(\{r_i\} \cup V) = \mathrm{Gp}(\{\hat{s}_i\} \cup V)$ and so $\{\hat{s}_i\} \cup \{v_i\}$ generates $\mathrm{Gp}(\{r_i\} \cup V)$.

Suppose that $\{\hat{s}_i\} \cup \{v_i\}$ generates $\mathrm{Gp}(\{r_i\} \cup V)$. For each r_i set $r_i = \prod w_{ij}^{\varepsilon j}$ where each w_{ij} belongs to the set $\{\hat{s}_i\} \cup \{v_i\}$. Now for $w_{ij} \in \{v_i\}$ set $u_{ij} = w_{ij}^{\varepsilon j}$, and for $w_{ij} \in \{\hat{s}_i\}$, set $u_{ij} = s_k^{\varepsilon j}$ where $w_{ij} = \hat{s}_k^{\varepsilon j}$. Set $\hat{r}_i = \prod u_{ij}$. By (ii) in (2), each $\theta_V(u_{ij}) = \theta_V(w_{ij}^{\varepsilon j})$ so $\theta_V(\hat{r}_i) = \theta_V(r_i) = 1$. By the same argument we have $v(\hat{r}_i) = v(r_i) = r_i$. Thus $(\hat{r}_i)$ is a respelling of (r_i) relative to $((s_i), V)$. By resubstituting the terms $\hat{s}_k$ for s_k $(1 \le k \le p)$ in $(\hat{r}_i)$ we convert $(\hat{r}_i)$ back to (r_i); thus $(\hat{r}_i)$ is an inverse respelling to $(\hat{s}_i)$ and $(\hat{s}_i)$ is invertible. ■

Rapaport's invertibility criterion characterizes Q-equivalent p-tuples in terms of the existence of invertible respellings. Before we describe this result, we wish to examine a technique from her proof. Consider a free group W and a Nielsen transformation $(s_i) \to (t_i)$ of p-tuples of elements of W such that for some number q, $s_i = t_i$ $(i \le q)$. We would like, if possible, to diminish the participation of the terms indexed by $i \le q$ in the individual steps of the Nielsen transformation $(s_i) \to (t_i)$ so that they may be inverted or used to conjugate the terms indexed by $i > q$ but are not otherwise to be involved in the individual steps. An example with $p = 3$ and $q = 1$ is illustrated below with the old steps on the left and a revised sequence of steps on the right:

Example 3.

$s_1 = w_1,\ s_2 = w_3 w_1 w_2 w_1^{-1},\ s_3 = w_3$	$t_1 = w_1,\ t_2 = w_2,\ t_3 = w_3$
$(w_1, w_3 w_1 w_2 w_1^{-1}, w_3)$	$(w_1, w_3 w_1 w_2 w_1^{-1}, w_3)$
$(w_1, w_3 w_1 w_2, w_3)$	$(w_1, w_1^{-1} w_3 w_1 w_2, w_3)$
$(w_1, w_2^{-1} w_1^{-1} w_3^{-1}, w_3)$	$(w_1, w_2^{-1} w_1^{-1} w_3^{-1} w_1, w_3)$
$(w_1, w_2^{-1} w_1^{-1}, w_3)$	$(w_1, w_1 w_2^{-1} w_1^{-1} w_3^{-1}, w_3)$
(w_1, w_2^{-1}, w_3)	$(w_1, w_1 w_2^{-1} w_1^{-1}, w_3)$
(w_1, w_2, w_3)	(w_1, w_2^{-1}, w_3)
	(w_1, w_2, w_3)

We are led to the following definition: Let W be a free group and $(s_i)\to(t_i)$ a Nielsen transformation of a p-tuple (s_i) of elements in W. Let $q\leq p$ be a number and suppose that $s_i=t_i$ $(i\leq q)$. We say that the Nielsen transformation $(s_i)\to(t_i)$ is *conservative relative to* $(s_1,\ldots,s_q)$ provided that there is a sequence of Nielsen transformations

$$(s_i)=(s_i(0))\cdots(s_i(c))\to(s_i(c+1))\cdots(s_i(e))=(t_i)$$

where each of the transformations $(s_i(c))\to(s_i(c+1))$ changes exactly one term, $s_k(c)$ for some $k=k(c)$, and is of one of the following three types:

(3) *Type* 1. $s_k(c+1)=s_k(c)^{-1}$,
Type 2. $s_k(c+1)=s_k(c)s_j(c)$ where $k>q$, $j>q$, and $k\neq j$, or
Type 3$'$. $s_k(c+1)=s_j(c)^{\mp 1}s_k(c)s_j(c)^{\pm 1}$ where $k>q$ and $j\leq q$.

Notice that the transformation $(s_i)\to(t_i)$ in Example 3 is conservative relative to (s_1); although the original sequence of elementary steps defining $(s_i)\to(t_i)$ has to be revised to the right hand sequence to establish this.

The lemma and three corollaries below are designed to cover a variety of situations where conservative Nielsen transformations will be needed later. For the lemma and corollaries we assume that there is given a free group W freely generated by $w_1,\ldots,w_n$ and a number $q\leq n$. We denote by W_L the subgroup $\mathrm{Gp}(\{w_i\colon i\leq q\})$ of W, and we denote by μ the projection map $\mu\colon W\to W_L$ defined by $\mu(w_i)=w_i$ $(i\leq q)$ and $\mu(w_i)=1$ $(i>q)$. The lemma and corollaries are similar to results in [Mz, ÄG, Th. 7] and [Br].

LEMMA 2.3 (Prefix lemma). *Let $(s_i)\to(t_i)$ be a Nielsen transformation of a p-tuple (s_i) of elements of W where $p\geq q$ and $s_i=t_i=w_i$ $(i\leq q)$. Let $u_1,\ldots,u_p$ be elements of W_L such that $u_i=1$ $(i\leq q)$. Then there are elements $u_1',\ldots,u_p'$ in W_L where $u_1'=1$ $(i\leq q)$, and there is a Nielsen transformation $(u_is_i)\to(u_i't_i)$ that is conservative relative to*

$$(u_1s_1,\ldots,u_qs_q)=(w_1,\ldots,w_q).$$

Proof. By a theorem of Nielsen's (see [MKS, Section 3.2]) there are sequences of length preserving and length reducing elementary Nielsen transformation (with respect to the alphabet $\{w_i\}$) converting the p-tuples (s_i) and (t_1) to a Nielsen reduced p-tuple (r_i). By a length argument we may assume that the terms $s_i=t_i=w_i$ $(i\leq q)$ do not change in the sequence except possibly by inversion. By combining these two sequences of transformations we find that there is a sequence of elementary Nielsen transformations

$$(s_i)=(s_i(0))\cdots(s_i(c))\to(s_i(c+1))\cdots(s_i(e))=(t_i)$$

such that $s_i(c+1)=s_i(c)^{\pm 1}$ $(i\leq q)$ and such that a transformation of Type 2 defined by

$$s_k(c+1)=s_k(c)s_j(c)\quad\text{with}\quad k\leq q$$

never occurs in the sequence.

We will define recursively prefixes $u_i(c) \in W_L$ $(c \leq e, i \leq p)$ so that up to certain inversions, each of the transformations

$$(u_i(c)s_i(c)) \to (u_i(c+1)s_i(c+1))$$

is conservative relative to $(u_1(c)s_1(c), \ldots, u_q(c)s_q(c))$. We begin by setting $u_i(0) = u_i$, and we will require that $u_i(c) = 1$ $(c \leq e, i \leq q)$. In the end we will set $u_i' = u_i(e)$ $(i \leq p)$ to get the desired conservative transformation $(u_i s_i) \to (u_i' t_i)$.

Suppose, by way of induction, that the prefixes $u_i(0), \ldots, u_i(c)$ $(i \leq p)$ have been defined so that, up to inversions of the terms $u_i(c)s_i(c)$ $(i \leq q)$, the transformation

$$(u_i(0)s_i(0)) \to (u_i(c)s_i(c))$$

is conservative relative to $(u_1(0)s_1(0), \ldots, u_q(0)s_q(0))$. The inversion hedge is necessary because the terms $u_i(0)s_i(0)$ $(i \leq q)$ may be inverted in the intervening stages of the conservative transformation but must eventually return to $u_i(0)s_i(0)$. Consider the old step $(s_i(c)) \to (s_i(c+1))$. We describe new prefixes $u_i(c+1)$ so that, up to the inversions mentioned before, the transformation

$$(u_i(0)s_i(0)) \to (u_i(c+1)s_i(c+1))$$

is conservative relative to $(u_1(0)s_1(0), \ldots, u_q(0)s_q(0))$. We consider the possible types of elementary Nielsen transformations separately:

Type 1. Let $s_k(c+1) = s_k(c)^{-1}$ and $s_i(c+1) = s_i(c)$ otherwise. If $k > q$, set $u_k(c+1) = u_k(c)^{-1}$ and $u_i(c+1) = u_i(c)$ otherwise. Because $u_k(c) \in W_L$, we can convert

$$(u_k(c)s_k(c))^{-1} = s_k(c)^{-1}u_k(c)^{-1}$$

to $u_k(c)^{-1}s_k(c)^{-1}$ by a sequence of conservative transformations effecting the conjugation

$$u_k(c)^{-1}s_k(c)^{-1}u_k(c)^{-1}u_k(c) = u_k(c)^{-1}s_k(c)^{-1}.$$

Thus, up to the inversions mentioned before,

$$(u_i(0)s_i(0)) \to (u_i(c+1)s_i(c+1))$$

is conservative relative to $(u_1(0)s_1(0), \ldots, u_q(0)s_q(0))$. If $k \leq q$, then set $u_i(c+1) = u_i(c)$ for all i. Remember that $u_i(c) = 1$ $(i \leq q)$. Then, as before, the transformation

$$(u_i(0)s_i(0)) \to (u_i(c+1)s_i(c+1))$$

is, up to inversions, conservative relative to $(u_1(0)s_1(0), \ldots, u_q(c+1)s_q(c+1))$.

Type 2. Let $s_k(c+1)=s_k(c)s_j(c)$ and $s_i(c+1)=s_i(c)$ otherwise where $k\neq j$. We divide considerations into two cases; recall that by assumption, $k>q$.

Case I ($j\leq q$). Set $u_k(c+1)=s_j(c)^{-1}u_k(c)$ and $u_i(c+1)=u_i(c)$ otherwise. Then we have

$$u_k(c+1)s_k(c+1)=s_j(c)^{-1}u_k(c)s_k(c)s_j(c)$$

and $u_i(c+1)s_i(c+1)=u_i(c)s_i(c)$ otherwise. This gives us, up to inversions, the desired conservative transformation.

Case II ($j>q$). Set $u_k(c+1)=u_j(c)u_k(c)$ and $u_i(c+1)=u_i(c)$ otherwise. Then we have the following sequence of conservative transformations (the unmodified terms are ignored):

(a) $u_j(c)s_j(c)\rightarrow s_j(c)u_j(c)$
(b) $u_k(c)s_k(c)\rightarrow u_k(c)s_k(c)s_j(c)u_j(c)\rightarrow u_j(c)u_k(c)s_k(c)s_j(c)$
$=u_k(c+1)s_k(c+1)$
(c) $s_j(c)u_j(c)\rightarrow u_j(c)s_j(c)=u_j(c+1)s_j(c+1)$.

Thus again $(u_i(0)s_i(0))\rightarrow(u_i(c+1)s_i(c+1))$ is, up to inversions, conservative relative to $(u_1(0)s_1(0),\ldots,u_q(0)s_q(0))$, and the induction step is complete.

We may now suppose that the prefixes $u_i(c)$ have been defined for $c\leq e$. But $(s_i(e))=(t_i)$ so we take $u_i'=u_i(e)\,(i\leq p)$ to complete the proof of the lemma. ■

COROLLARY 2.4. *In Lemma* 2.3, *let H be a normal subgroup of W_L containing each $\mu(u_is_i)\,(i>q)$. Then H also contains each $\mu(u_i't_i)\,(i>q)$.*

Proof. This follows directly from a case by case analysis of the changes in the projected sets $\{\mu(u_i(c)s_i(c)): i>q\}\rightarrow\{\mu(u_i(c+1)s_i(c+1)): i>q\}$. ■

COROLLARY 2.5. *In Lemma* 2.3, *suppose that $\mu(s_i)=1$ and $\mu(t_i)=1\,(i>q)$. Then $(s_i)\rightarrow(t_i)$ is conservative relative to $(s_1,\ldots,s_q)$.*

Proof. Set $u_i=1\,(i\leq p)$ and note that by Corollary 2.4 we must also have $u_i'=1\,(i\leq p)$ in the conservative transformation $(u_is_i)\rightarrow(u_i't_i)$. ■

COROLLARY 2.6. *In Lemma* 2.3, *suppose that the Nielsen transformation $(s_i)\rightarrow(t_i)$ restricts to a Nielsen transformation*

$$(s_{q+1},\ldots,s_p)\rightarrow(t_{q+1},\ldots,t_p).$$

Then the elements u_i' and the conservative transformation $(u_is_i)\rightarrow(u_i't_i)$ in the conclusion of Lemma 2.3 *may be chosen so that*

$$(u_{q+1},\ldots,u_p)\rightarrow(u_{q+1}',\ldots,u_p')$$

is also a Nielsen transformation.

Proof. Under the hypothesis of the corollary we may ignore the original length analysis and suppose instead that there is a sequence of elementary Nielsen transformations $(s_i(c)) \to (s_i(c+1))$ with $(s_i) = (s_i(0))$ and $(t_i) = (s_i(e))$ where, for $i \leq q$, $s_i(c) = s_i(c+1)$ and $s_i(c)$ is not involved in the Nielsen transformation $(s_i(c)) \to (s_i(c+1))$. But then the Case I analysis for a Type 2 transformation never comes up, and a case by case analysis of the remaining possibilities reveals that each prefix transformation

$$(u_{q+1}(c), \ldots, u_p(c)) \to (u_{q+1}(c+1), \ldots, u_p(c+1))$$

is a Nielsen transformation. Thus $(u_{q+1}, \ldots, u_p) \to (u'_{q+1}, \ldots, u'_p)$ is a Nielsen transformation as desired. ■

LEMMA 2.7. *Let (r_i) and (s_i) be p-tuples in Y^n, and let $(\hat{r}_i)$ and $(\hat{s}_i)$ be respellings of (r_i) and (s_i). Suppose that for some finite tuple of elements (v_i) in $X^n * Y^n$ there is a Nielsen transformation $((v_i), (\hat{s}_i)) \to ((v_i), (\hat{r}_i))$ that is conservative relative to (v_i). Then (r_i) and (s_i) are Q-equivalent p-tuples.*

Proof. Apply υ to the reduced tuples in the elementary steps in the conservative transformation, $(\hat{s}_i(c)) \to (\hat{s}_i(c+1))$, to get a sequence of Q-transformations, $(\upsilon(\hat{s}_i(c)) \to (\upsilon(\hat{s}_i(c+1))$, beginning at (s_i) and terminating at (t_i). ■

The following theorem gives the Rapaport invertibility criterion in alternate form [Rp, Th. 1] (compare [Pf, Section 2, Th. 4]):

THEOREM 2.8 (Rapaport). *Let (r_i) and (s_i) be p-tuples of elements of Y^n. Then (r_i)* and *(s_i) are Q-equivalent if there is a subgroup V of $X^n * Y^n$ and an invertible respelling $(\hat{s}_i)$ of (s_i) relative to $((r_i), V)$. Moreover, if (r_i) and (s_i) are Q-equivalent, then there is an invertible respelling $(\hat{s}_i)$ of (s_i) relative to $((r_i), Z^n)$.*

Proof. Suppose that $(\hat{s}_i)$ is an invertible respelling of (s_i) relative to $((r_i), V)$. Let $v_1, \ldots, v_q$ be a free basis for V. By Lemma 2.2, $\{v_i\} \cup \{\hat{s}_i\}$ generates $\mathrm{Gp}(\{r_i\} \cup V)$. By the assumption on V that $\mathrm{Gp}(Y^n \cup V)$ is canonically equivalent to $Y^n * V$, it follows that there is a free basis $w_1, \ldots, w_m$ for $\mathrm{Gp}(Y^n \cup V)$ such that $w_i = v_i$ $(i \leq q)$. By a theorem of Nielsen's (see [MKS, Section 3.2]) there is a Nielsen transformation

$$((v_i), (\hat{s}_i)) \to ((v_i), (r_i)).$$

Apply Lemma 2.3 and Corollary 2.5 with $\mathrm{Gp}(Y^n \cup V)$ in place of W and θ_V in place of μ. Since each $\theta_V(\hat{s}_i) = 1$ and each $\theta_V(r_i) = 1$, it follows from Corollary 2.5 that the transformation $((v_i), (\hat{s}_i)) \to ((v_i), (r_i))$ is conservative relative to (v_i). Now (r_i) is a respelling of (r_i); thus by Lemma 2.7, (r_i) and (s_i) are Q-equivalent.

Where (r_i) and (s_i) are Q-equivalent, the invertible respelling $(\hat{s}_i)$ of (s_i) relative to $((r_i), Z^n)$ can be constructed from Rapaport's proof by using

Lemma 2.1 together with the change in notation described at the end of §1. ■

It is helpful to look at the invertibility criterion in terms of isomorphisms between subgroups of $X^n * Y^n$. In general, a respelling $(\hat{s}_i)$ of (s_i) relative to $((r_i), V)$ does not induce a homomorphism from $\mathrm{Gp}(\{s_i\} \cup V)$ to $\mathrm{Gp}(\{r_i\} \cup V)$ by $s_i \to \hat{s}_i$, for there may be relations among the s_i's that are not reflected in relations among the $\hat{s}_i$'s. In fact, this excess redundancy in the s_i's is sometimes just what is needed to establish Q-equivalence of (r_i) and (s_i). But if $\{r_i\}$ and $\{s_i\}$ freely generate subgroups of Y^n this situation does not arise. The following corollary restates the invertibility criterion for this special case:

COROLLARY 2.9. *Let (r_i) and (s_i) be p-tuples of elements in Y^n that freely generate rank p subgroups R and S of Y^n. Then (r_i) and (s_i) are Q-equivalent if for some V as in the definition of respelling there is an isomorphism $\lambda: \mathrm{Gp}(S \cup V) \to \mathrm{Gp}(R \cup V)$ such that* (i) $\lambda \mid V = \mathrm{id}$, (ii) $\upsilon \mid \mathrm{Gp}(S \cup V) = \upsilon\lambda$, *and* (iii) $\theta_V \lambda \mid S$ *is the trivial homomorphism. Furthermore, if (r_i) and (s_i) are Q-equivalent then there is an isomorphism λ: $\mathrm{Gp}(S \cup Z^n) \to \mathrm{Gp}(R \cup Z^n)$ with the properties described above.*

Proof. Any isomorphism λ with the three properties listed above gives rise to a respelling $(\hat{s}_i) = ((\lambda(s_i))$ relative to $((r_i), V)$ that is invertible with an inverse respelling $(\hat{r}_i) = (\lambda^{-1}(r_i))$; thus the first part of the corollary follows from the first part of Theorem 2.8. On the other hand, in the special case dealt with here, respellings $(\hat{s}_i)$ and $(\hat{r}_i)$ of (s_i) and (r_i) relative to $((r_i), Z^n)$ and $((s_i), Z^n)$ give rise to homomorphisms from $\mathrm{Gp}(S \cup V)$ to $\mathrm{Gp}(R \cup V)$ and vice versa satisfying (i)–(iii). Invertible respellings correspond to the case where the second homomorphism is the inverse of the first. Thus the homomorphisms are isomorphisms in this case and the second half of the corollary follows from the second half of Theorem 2.8. ■

3. Matched respellings, simplifying presentations for the trivial group

In this section we develop a criterion for simplifying p-tuples. It will apply both to the case of simplifying p-tuples by Q-transformations and to the case of simplifying p-tuples by Q^{**}-transformations that ultimately result in destabilization via Type 6 transformations.

Consider a p-tuple (r_i) of elements in Y^n such that $Y^n \subseteq \mathrm{Cl}(\{r_i\})$. The set $\{r_i\}$ can be regarded as a generating set for the commutator quotient group $Y^n/([Y^n, Y^n])$. Thus by performing Nielsen transformations on the p-tuple (r_i) we can always arrange things so that the exponent sum of r_1 on the generator y_1 is $+1$ (see [MKS, p. 76] for the definition of exponent sum). Assume that this modification has already been made. Then, when regarded as a word on the alphabet $\{y_i\}$, r_1 has the form uy_1v where the exponent sums on y_1 in u and v add up to zero. Replace each syllable y_1^ε in u and v

by z_1^ε to get an element $\hat{r}_1 \in X^n * Y^n$ respelling r_1. Let $\hat{r}_2, \ldots, \hat{r}_p$ be any respellings of $r_2, \ldots, r_p$. There is a partial matching of $(\hat{r}_1)$ with an invertible respelling $(\hat{t}_i)$ of an n-tuple (t_i) relative to $((y_i), Z^n)$. To see this set $t_1 = r_1$, $t_i = y_i$ $(i > 1)$, $\hat{t}_1 = \hat{r}_1$, and $\hat{t}_i = y_i$ $(i > 1)$. Then $\{z_i\} \cup \{\hat{t}_i\}$ is a free basis for $X^n * Y^n$ so by Lemma 2.2, $(\hat{t}_i)$ is invertible. By the invertibility criterion, or by direct observation, (t_i) is Q-equivalent to $(y_1, \ldots, y_n)$. It seems natural to ask whether this partial matching of the respellings might be useful in simplifying the p-tuple (r_i). The matching would be useful if the answer to the following question were "yes": Given a p-tuple (r_i) and an n-tuple (t_i) of elements in Y^n such that $Y^n \subseteq \mathrm{Cl}(\{r_i\})$, $Y^n \subseteq \mathrm{Cl}(\{t_i\})$, and $r_i = t_i$ $(i \leq d)$ where $d \geq 1$, and given a Q-transformation $(t_i) \to (u_i)$, is there a Q-transformation $(r_i) \to (s_i)$ such that $s_i = u_i$ $(i \leq d)$? The matching property below will provide us with conditions under which the answer to the preceding question is "yes".

Matching property. Let (r_i) and (t_i) be a p-tuple and an n-tuple of elements in Y^n such that $Y^n \subseteq \mathrm{Cl}(\{r_i\})$ and $Y^n \subseteq \mathrm{Cl}(\{t_i\})$. Suppose, moreover, that for some integer $d \geq 1$, $r_i = t_i$ $(i \leq d)$. Let $(\hat{r}_i)$ and $(\hat{t}_i)$ be respellings of (r_i) and (t_i) relative to $((y_i), V)$ for some V. The pair $((\hat{r}_i), (\hat{t}_i))$ is defined to be a *(d, V)-matched pair of respellings* provided that the following conditions are satisfied:

(i) $(\hat{t}_i)$ is invertible,
(ii) $\hat{r}_i = \hat{t}_i$ $(i \leq d)$, and
(iii) $\upsilon(V) \subseteq \mathrm{Cl}_H(\{\hat{r}_i\})$ where $H = \mathrm{Gp}(Y^n \cup V)$.

In the case where $V = Z^n$ condition (iii) says that $\mathrm{Cl}_{X*Y}(\{\hat{r}_i\})$ must contain Y^n. We will show in Theorem 3.3 that when a (d, Z^n)-matched pair of respellings exists for (r_i) and (t_i) the answer to the question above is "yes", and we will show in Theorem 3.2 that regardless of what V is, a (d, V)-matched pair of respellings for (r_i) and (t_i) causes (r_i) to be Q^{**}-equivalent to a $(p-d)$-tuple of elements in Y^{n-d}.

The proof of the following lemma is left as an exercise.

LEMMA 3.1. *Let (r_i) be a p-tuple of elements in Y^n.*

(i) *Let $r_k = uvw$ where $v \in \mathrm{Cl}(\{r_i\colon i \neq k\})$. Set $s_k = uw$ and $s_i = r_i$ $(i \neq k)$. Then (r_i) is Q-equivalent to (s_i).*

(ii) *Let $r_k = w_a u w_b$ and for some $j \neq k$ let $r_j = uv^{-1}$. Set $s_k = w_a v w_b$ and $s_i = r_i$ $(i \neq k)$. Then (r_i) is Q-equivalent to (s_i).* ∎

THEOREM 3.2. *Let (r_i) and (t_i) be a p-tuple and an n-tuple of elements in Y^n such that a (d, V)-matched pair of respellings $((\hat{r}_i), (\hat{t}_i))$ exists for (r_i) and (t_i). Then (r_i) is Q^{**}-equivalent to a $(p-d)$-tuple (s_i) of elements in Y^{n-d}.*

Proof. Let $\{v_i\colon i \leq q\}$ be a free basis for V. By Lemma 2.2, $\{\hat{t}_i\} \cup \{v_i\}$ generates the group $H = \mathrm{Gp}(Y^n \cup V)$. Below we describe a sequence of presentation transformations. They are not described explicitly as Q^{**}-transformations, but by associating each v_i with y_{i+n}, and using Nielsen transformations that permute the relators, these transformations can be converted to Q^{**}-transformations.

1. Stabilize by adding each v_i as a generator and adding relators $v_i\upsilon(v_i^{-1})$ (see [Cr, Le. 1.1]):

$$\mathcal{P}(0)=\langle Y^n: (r_i)\rangle \to \mathcal{P}(1)=\langle H: ((r_i), (v_i\upsilon(v_i^{-1})))\rangle.$$

2. Respell the relators r_i as $\hat{r}_i$ using the new relations $v_i=\upsilon(v_i)$ and Lemma 3.1_{ii}:

$$\mathcal{P}(1)\to \mathcal{P}(2)=\langle H: ((\hat{r}_i), (v_i\upsilon(v_i^{-1})))\rangle.$$

3. Remove the subwords $\upsilon(v_i^{-1})$ from the relators $v_i\upsilon(v_i^{-1})$ using Lemma 3.1_{i} and the normal closure property in the matching condition:

$$\mathcal{P}(2)\to \mathcal{P}(3)=\langle H: ((\hat{r}_i), (v_i))\}.$$

4. By rank considerations on the abelianization of H (see [MKS, §§2.2–2.4]), $\{\hat{t}_i\}\cup\{v_i\}$ freely generates H. Define an isomorphism $\lambda: H\to H$ by $\lambda(v_i)=v_i$ and $\lambda(\hat{t}_i)=y_{n-i}$:

$$\mathcal{P}(3)\to \mathcal{P}(4)=(H: ((y_n, \ldots, y_{n-d+1}, \lambda(\hat{r}_{d+1}), \ldots, \lambda(\hat{r}_p)), (v_i)).$$

5. Permute the relators:

$$\mathcal{P}(4)\to \mathcal{P}(5)=\langle H: ((\lambda(\hat{r}_{d+1}), \ldots, \lambda(\hat{r}_p), y_{n-d+1}, \ldots, y_n), (v_i)).$$

6. Using Lemma 3.1_{i}, remove the syllables $y_{n-i+1}^{\pm 1}$ $(i\le d)$ and $v_i^{\pm 1}$ from the terms in $(\lambda(\hat{r}_{d+1}), \ldots, \lambda(\hat{r}_p))$ to convert this tuple to $(s_1, \ldots, s_{p-d})$, a tuple of elements in Y^{n-d}:

$$\mathcal{P}(5)\to \mathcal{P}(6)=\langle H: ((s_1, \ldots, s_{p-d}, y_{n-d+1}, \ldots, y_n), (v_i))\rangle.$$

7. Destabilize:

$$\mathcal{P}(6)\to \mathcal{P}(7)=\langle Y^{n-d}: (s_1, \ldots, s_{p-d})\rangle. \qquad \blacksquare$$

THEOREM 3.3. *Let (r_i) and (t_i) be a p-tuple and an n-tuple of elements in Y^n such that a (d, Z^n)-matched pair of respellings $((\hat{r}_i), (\hat{t}_i))$ exists for (r_i) and (t_i). Let $(t_i)\to(u_i)$ be a Q-transformation. Then there is an invertible respelling $(\hat{u}_i)$ of (u_i) relative to $((y_i), Z^n)$, and for any such invertible respelling there is a Q-transformation $(r_i)\to(s_i)$ such that $s_i=u_i (i\le d)$ and there is a respelling $(\hat{s}_i)$ of (s_i) so that $((\hat{s}_i), (\hat{u}_i))$ is a (d, Z^n)-matched pair of respellings for (s_i) and (u_i).*

Proof. By the Rapaport invertibility criterion, Theorem 2.8, there is an invertible respelling $(\hat{u}_i)$ of (u_i) relative to $((y_i), Z^n)$. Let $(\hat{u}_i)$ be given now. By Lemma 2.2 both $\{z_i\}\cup\{\hat{t}_i\}$ and $\{z_i\}\cup\{\hat{u}_i\}$ generate $X^n * Y^n$. Thus by the proof of the invertibility criterion, there is a Nielsen transformation

$$((z_i), (\hat{t}_i))\to((z_i), (\hat{u}_i))$$

that is conservative relative to (z_i). We wish to find a Q-transformation $(r_i)\to(s_i)$ so that $s_i=u_i$ $(i\le d)$ and we wish to find a respelling $(\hat{s}_i)$ so that $((\hat{s}_i),(\hat{u}_i))$ is a (d, Z^n)-matched pair of respellings.

We begin by considering the following special case:

Special case. The transformation $((z_i),(\hat{t}_i))\to((z_i),(\hat{u}_i))$ *is an elementary conservative Nielsen transformation as defined in* (3). To define (s_i) and $(\hat{s}_i)$ we will consider the types of transformations separately, and after the definitions are made in all cases we will verify that (s_i) and $(\hat{s}_i)$ have the desired properties.

Type 1. If some $\hat{t}_k$ $(k>d)$ is inverted, set $(s_i)=(r_i)$ and $(\hat{s}_i)=(\hat{r}_i)$. If some $\hat{t}_k$ $(k\le d)$ is inverted, set $s_k=r_k^{-1}$ and $\hat{s}_k=\hat{r}_k^{-1}$, and $s_i=r_i$ and $\hat{s}_i=\hat{r}_i$ otherwise.

Type 2. Suppose that $\hat{u}_k=\hat{t}_k\hat{t}_j$. We consider three subcases:

Case I. $k>d$. Set $(s_i)=(r_i)$ and $(\hat{s}_i)=(\hat{r}_i)$.

Case II. $k\le d$, $j\le d$. Set $s_k=r_kr_j$ and $\hat{s}_k=\hat{r}_k\hat{r}_j$, and set $s_i=r_i$ and $\hat{s}_i=\hat{r}_i$ otherwise.

Case III. $k\le d$, $j>d$. This is the difficult case. We pass to the new tuples in two stages each involving several transformations. By Lemma 2.2 and rank considerations as in the proof of Theorem 3.2, $\{z_i\}\cup\{\hat{t}_i\}$ is a free basis for $X^n * Y^n$. Define first a projection homomorphism $\mu\colon X^n * Y^n\to X^n * Y^n$ by $\mu(z_i)=z_i$, $\mu(\hat{t}_i)=\hat{t}_i$ $(i>d)$, and $\mu(\hat{t}_i)=1$ $(i\le d)$. Define a p-tuple $(\hat{r}_i(0))$ by $\hat{r}_i(0)=\hat{r}_i$ $(i\le d)$ and $\hat{r}_i(0)=\mu(\hat{r}_i)$ $(i>d)$. Set $(r_i(0))=(\upsilon(\hat{r}_i(0))$. By expressing each $\hat{r}_i$ $(i>d)$ in terms of the alphabet $\{z_i\}\cup\{\hat{t}_i\}$, we see that each $\hat{r}_i(0)$ $(i>d)$ is obtained from $\hat{r}_i$ by deleting the syllables $\hat{t}_l^{\varepsilon}=\hat{r}_l^{\varepsilon}$ $(l\le d)$. Thus by Lemma 3.1_{ii}, $(r_i(0))$ is Q-equivalent to (r_i). By definition, $\mathrm{Cl}_{X*Y}(\{\hat{r}_i(0)\})=\mathrm{Cl}_{X*Y}(\{\hat{r}_i\})$ so this group contains $Y^n=\upsilon(Z^n)$. Thus $((\hat{r}_i(0),(\hat{t}_i))$ is a (d, Z^n)-matched pair of respellings.

We claim next that each $\hat{t}_i$ $(i>d)$ is a product of conjugates of powers of the elements $\hat{r}_{d+1}(0),\dots,\hat{r}_p(0))$. To see this, observe that by Lemma 2.1 and the fact that $Y^n\subseteq \mathrm{Cl}(\{\hat{r}_i(0)\})$ we have $\hat{t}_i=\prod c_l^{-1}\hat{r}_{i_l}(0)^{\varepsilon l}c_l$ $(i>d)$. But for $i>d$, we have $\mu(\hat{t}_i)=\hat{t}_i=\prod \mu(c_l)^{-1}\mu(\hat{r}_{i_l}(0))^{\varepsilon l}\mu(c_l)$. Now $\mu(\hat{r}_{i_l}(0)=1$ for $i_l\le d$ and $\mu(\hat{r}_{i_l}(0))=\hat{r}_{i_l}(0)$ for $i_l>d$. Thus $\hat{t}_i$ belongs to $\mathrm{Cl}(\{\hat{r}_{d+i}(0)\})$ as desired.

We define now $s_k=r_k(0)t_j$ and $\hat{s}_k=\hat{r}_k(0)\hat{t}_j$, and for $i\ne k$ we set $s_i=r_i(0)$ and $\hat{s}_i=\hat{r}_i(0)$.

Type 3′. Suppose that $\hat{u}_k=z_j^{\mp1}\hat{t}_kz_j^{\pm1}$. If $k>d$, set $(s_i)=(r_i)$ and $(\hat{s}_i)=(\hat{r}_i)$. If $k\le d$, set $s_k=y_j^{\mp1}r_ky_j^{\pm1}$ and $\hat{s}_k=z_j^{\mp1}\hat{r}_kz_j^{\pm1}$ and set $s_i=r_i$ and $\hat{s}_i=\hat{r}_i$ otherwise.

We claim that (s_i) is Q-equivalent to (r_i) and that $((\hat{s}_i),(\hat{u}_i))$ is a (d, Z^n)-matched pair of respellings. Except for Case III in the Type 2 consideration, this verification is straightforward. Consider Case III. We showed that each $\hat{t}_i$ $(i>d)$ belongs to $\mathrm{Cl}_{X*Y}(\{\hat{r}_i(0)\colon i>d\})$; thus each $t_i=\upsilon(\hat{t}_i)$ $(i>d)$ belongs to

$$\mathrm{Cl}(\{r_i(0)\colon i>d\}).$$

Now s_k results from $r_k(0)$ by insertion of conjugates of the $r(0)$-terms $(i>d)$. From Lemma 3.1, $(r_i(0))$ and (s_i) are Q-equivalent, and by transitivity of equivalence, (r_i) and (s_i) are equivalent. Finally, since each $\hat{t}_i$ $(i>d)$ belongs to Cl_{X*Y} $(\{\hat{r}_i(0): i>d\})$, it follows that

$$\mathrm{Cl}_{X*Y}(\{\hat{r}_i(0)\}) = \mathrm{Cl}_{X*Y}(\{\hat{s}_i\})$$

and so Y^n is contained in this latter subgroup as required.

It follows now that $((\hat{s}_i), (\hat{u}_i))$ is a (d, Z^n)-matched pair of respellings and the verification of the theorem for the special case is complete.

General Case. Divide the conservative transformation $((z_i), (\hat{t}_i)) \to ((z_i), (\hat{u}_i))$ into a sequence of elementary conservative transformations

$$((z_i), (\hat{t}_i)) = ((z_i(0)), (\hat{t}_i(0))) \ldots ((z_i(c)), (\hat{t}_i(c)))$$
$$\to ((z_i(c+1)), (\hat{t}_i(c+1))) \ldots ((z_i(e)), (\hat{t}_i(e))) = ((z_i), (\hat{u}_i)).$$

Use the special case to define, inductively, respellings $(\hat{r}_i(c))$ $(0 \le c \le e)$ of p-tuples $(r_i(c))$ so that

(i) $(r_i(0)) = (r_i)$ and $(\hat{r}_i(0)) = (\hat{r}_i)$,

(ii) each $(r_i(c+1))$ is Q-equivalent to $(r_i(c))$, and

(iii) each pair $((\hat{r}_i(c)), (\hat{t}_i(c)))$ is a matched pair of respellings for $(\upsilon(r_i(c))$ and $(\upsilon(t_i(c)))$.

Take $(s_i) = (r_i(e))$ and $(\hat{s}_i) = (\hat{r}_i(e))$. By transitivity, (r_i) and (s_i) are Q-equivalent so $((\hat{s}_i), (\hat{u}_1))$ is the desired (d, Z^n)-matched pair of respellings. ■

The theorem below shows how the matching condition may be used simplify presentations for the trivial group by Q-transformations:

THEOREM 3.4. *Let (r_i) and (t_i) be a p-tuple and an n-tuple of elements in Y^n such that a (d, Z^n)-matched pair of respellings exists for (r_i) and (t_i). Then (r_i) is Q-equivalent to a p-tuple*

$$(s_1, \ldots, s_{p-d}, y_{n-d+1}, \ldots, y_n)$$

where each s_i $(i \le p-d)$ is an element of Y^{n-d}.

Proof. By the invertibility criterion, (t_i) is Q-equivalent to $(y_n, \ldots, y_1)$. By Theorem 3.3, (r_i) is Q-equivalent to a p-tuple

$$(y_n, \ldots, y_{n-d+1}, u_1, \ldots, u_{p-d}).$$

Express each u_i as a product of powers of the generators $y_1, \ldots, y_n$. For each i, let s_i result from u_i by deletion of the syllables y_j^{ε} $(n-d+1 \le j \le n)$. By Lemma 3.1,

$$(y_n, \ldots, y_{n-d+1}, u_1, \ldots, u_{p-d})$$

is Q-equivalent to

$$(y_n, \ldots, y_{n-d+1}, s_1, \ldots, s_{p-d}).$$

Now use elementary Nielsen transformations effecting permutations to convert this tuple to the desired $(s_1, \ldots, s_{p-d}, y_{n-d+1}, \ldots, y_n)$. ∎

Remark. If (r_i) and (t_i) are a p-tuple and an n-tuple of elements in Y^n with (t_i) Q-equivalent to (y_i), $Y^n \subseteq \mathrm{Cl}(\{r_i\})$, and $r_i = t_i$ $(i \leq d)$, it would appear that the existence of a (d, V)-matched pair of respellings $((\hat{r}_i), (\hat{t}_i))$ would depend on the choice of the invertible respelling $(\hat{t}_i)$. This is not so however. If $((\hat{r}_i), (\hat{t}_i))$ is a (d, V)-matched pair and $(\hat{t}'_i)$ is another invertible respelling of (t_i) relative to $((y_i), V)$ then there is an automorphism

$$\kappa: \mathrm{Gp}(Y^n \cup V) \to \mathrm{Gp}(Y^n \cup V)$$

given by $\kappa \mid V = \mathrm{id}$ and $k(\hat{t}_i) = \hat{t}'_i$. Set $\hat{r}'_i = \kappa(\hat{r}_i)$. It is then easy to verify that $((\hat{r}'_i), (\hat{t}'_i))$ is also a (d, V)-matched pair of respellings. It helps to observe that in the group $\mathrm{Gp}(Y^n \cup V)$, $\mathrm{Cl}(Y^n)$ is precisely

$$\pi^{-1}(1 \times Y^n) \cap \mathrm{Gp}(Y^n \cup V),$$

and κ must leave this invariant since $\pi\kappa = \pi$.

4. Converting equivalences, classifying free splitting homomorphisms

In this section we apply the invertibility criterion and the technique of conservative Nielsen transformations to answer the following two questions: (I) When do equivalences between free splitting homomorphisms ψ and ϕ in normal form convert to Q^*-equivalences between $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ and vice versa? (II) If the convertability cannot be assured in (I) how much stabilization is needed to guarantee convertibility? In the process of answering these two questions we will give a purely algebraic proof of Theorem 4.1 of [Cr] which says that two free splitting homomorphisms ψ and ϕ in normal form are stably equivalent if and only if $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are Q^{**}-equivalent.

For $n \leq m$, let G^m_L and G^m_H denote the respective subgroups

$$\mathrm{Gp}(\{g_i : i \leq n\}) \quad \text{and} \quad \mathrm{Gp}(\{g_{i+n}\})$$

of G^m. Let $\lambda_L: G^m \to G^m_L$ be the projection defined by $\lambda_L(g_i) = g_i$ $(i \leq n)$ and $\lambda_L(g_{i+n}) = 1$. In the lemma and corollary which follow, we use the decomposition $G^m = G^m_L * G^m_H$ and the projection λ_L to examine the character of the isomorphisms in (1), the model for equivalence:

LEMMA 4.1. *Let ψ and ϕ be free splitting homomorphisms in normal form from G^m to $X^n \times Y^n$, and let η, η_1, and η_2 be isomorphisms defining an equivalence via* (1) *between ψ and ϕ. Then $\lambda_L\eta \mid G^m_L$ is an isomorphism and $\lambda_L\eta \mid G^m_H$ is the trivial map that sends every element to* 1.

Proof. Each of the maps ϕ_1, $\eta_1\phi_1$, and ψ_1 takes G^m_L isomorphically onto X^n. Now $\eta_1\phi_1 \mid G^m_L = \psi_1\eta \mid G^m_L = \psi_1\lambda_L\eta \mid G^m_L$. The latter equality follows from the fact that $\psi_1 \mid G^m_H$ is the trivial map. Thus $\lambda_L\eta \mid G^m_L$ must be surjective and so be an isomorphism onto G_L^m. Suppose that $\lambda_L\eta(g)\neq 1$ for some element $g\in G_H^m$. Then we have $\psi_1\lambda_L\eta(g)=\psi_1\eta(g)\neq 1$ contradicting the normal form properties. ∎

Remark. Let ψ and ϕ be free splitting homomorphisms in normal form from G^m to $X^n\times Y^n$. Let η, η_1 and η_2 be automorphisms of G^m, X^n, and Y^n defining an equivalence via (1) between ψ and ϕ. In attempting to convert this equivalence between ψ and ϕ to a Q^*-equivalence between $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ it turns out to be a nusiance to have $\eta_1\neq\mathrm{id}$ or $\eta_2\neq\mathrm{id}$. We can avoid this as follows: Consider first the homomorphism $(\eta_1,\eta_2)\phi$. Now this map may not be in normal form, but as we observed in the proof of the preceding lemma, ϕ_1 and so $\eta_1\phi_1$ takes G_L isomorphically onto X^n. Thus there is an automorphism η_0 of G^m with $\eta_0\mid G_H=\mathrm{id}$ and $\eta_0\mid G_L$ an automorphism of G_L onto itself so that $\eta_1\phi_1\eta_0(g_i)=x_i$ $(i\leq n)$. Replace ϕ by $\phi'=(\eta_1,\eta_2)\phi\eta_0$. Then the equivalence in (1) between ψ and ϕ' is induced by $\eta\eta_0$ on G^m, id on X^n, and id on Y^n. By transitivity, a Q^*-equivalence of $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi')$ implies a Q^*-equivalence of $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$.

COROLLARY 4.2. *Let ψ, ϕ, η, η_1, and η_2 be as in Lemma* 4.1. *Suppose that $\eta_1=\mathrm{id}$, $\eta_2=\mathrm{id}$, and $\eta\mid G_L^m$ takes G_L^m isomorphically onto itself. Then $\eta\mid G_L^m=\mathrm{id}$.*

Proof. We observed in the proof of Lemma 4.1 that both ϕ_1 and ψ_1 take G_L^m isomorphically onto X^n. Each of these maps sends g_i $(i\leq n)$ to x_i. Thus since $\eta_1=\mathrm{id}$ it follows from commutativity in (1) that $\eta\mid G_L^m$ must also be the identity. ∎

The theorem which follows gives a partial answer to the first question raised in the beginning of this section:

THEOREM 4.3. *Let ψ and ϕ be free splitting homomorphisms from G^m to $X^n\times Y^n$. Let these be in normal form with associated group presentations $\mathcal{P}(\psi)=\langle Y^n:(r_i)\rangle$ and $\mathcal{P}(\phi)=\langle Y^n:(s_i)\rangle$.*

If there are automorphisms η, η_1, and η_2 of G^m, X^n, and Y^n defining an equivalence between ψ and ϕ via (1) *so that $\eta\mid G_L^m$ takes G_L^m isomorphically onto itself, then (r_i) and (s_i) are Q^*-equivalent. Moreover, if ψ and ϕ are Mihailova maps, and if (r_i) and (s_i) are Q^*-equivalent, then there are automorphisms η, η_1, and η_2 defining an equivalence as above so that $\eta\mid G_L=\mathrm{id}$.*

Proof. Suppose that isomorphisms η, η_1, and η_2 exist as in the hypotheses. We will show that (r_i) and (s_i) are Q^*-equivalent. By Corollary 4.2 and the remark preceding it, it is sufficient to consider the case where $\eta\mid G_L^m=\mathrm{id}$, $\eta_1=\mathrm{id}$, and $\eta_2=\mathrm{id}$.

We define homomorphisms $\hat{\psi}$ and $\hat{\phi}$ from G^m to $X^n * Y^n$ as indicated at the end of §1 so that $\hat{\psi}(g_i)=v_i$ $(i\leq n)$, $\hat{\psi}(g_{i+n})=r_i=\psi_2(g_{i+n})$, and $\hat{\phi}=\hat{\psi}\eta$.

Here $\pi(v_i)=\psi(g_i)$. Note again that $\psi=\pi\hat{\psi}$ and $\phi=\pi\hat{\phi}$. We have a commutative diagram:

$$\begin{tikzcd}
& & X^n * Y^n \arrow[dl, "\pi"'] \arrow[ddd, "\mathrm{id}"] \\
G^m \arrow[urr, "\hat{\phi}"] \arrow[r, "\phi"'] \arrow[d, "\eta"] & X^n \times Y^n \arrow[d, "(\mathrm{id},\mathrm{id})"] & \\
G^m \arrow[r, "\psi"] \arrow[drr, "\hat{\psi}"'] & X^n \times Y^n & \\
& & X^n * Y^n \arrow[ul, "\pi"']
\end{tikzcd} \tag{5}$$

Set $V=\mathrm{Gp}\,(\{v_i\})$, and set $(\hat{s}_i)=(\hat{\phi}(g_{i+n}))$. We claim that $(\hat{s}_i)$ is a respelling of (s_i) relative to $((r_i), V)$: By the commutativity in (5) conditions (i) and (ii) are satisfied in the definition of respelling; so all that is necessary is to specify the syllabification $\hat{s}_i=\prod u_{ij}$ for each $\hat{s}_i$. Let $\eta(g_{i+n})=\prod g_{i_j}^{\varepsilon_j}\ (\varepsilon_j=\pm 1)$. For the syllables u_{ij} take $u_{ij}=\hat{\psi}(g_{i_j})^{\varepsilon_j}$. This takes care of condition (iii).

By the definition of $\hat{\psi}$, $\mathrm{Gp}\,(\{v_i\}\cup\{\hat{s}_i\})=\mathrm{Gp}\,(\{v_i\}\cup\{r_i\})$; so by Lemma 2.2, $(\hat{s}_i)$ is invertible, and by the Rapaport criterion (r_i) and (s_i) are Q- and hence Q^*-equivalent.

When ψ and ϕ are Mihailova maps, and (r_i) and (s_i) are Q^*-equivalent, the existence of the desired automorphisms η, η_1, and η_2 follows from the first half of the proof of Theorem 4.1 of [Cr]. ∎

Lemma 4.4 and Theorem 4.5 below are designed to answer the second question at the beginning of this section.

LEMMA 4.4. *Suppose that ψ and ϕ are equivalent Mihailova maps from G^m to $X^n\times Y^n$ with associated group presentations*

$$\mathscr{P}(\psi)=\langle Y^n:(r_i)\rangle \quad \textit{and} \quad \mathscr{P}(\phi)=\langle Y^n:(s_i)\rangle.$$

Then the stabilized presentations

$$\mathscr{P}'(\psi)=\langle Y^{2n}:((r_i),(y_{n+1},\dots,y_{2n}))\rangle$$

and

$$\mathscr{P}'(\phi)=\langle Y^{2n}:((s_i),(y_{n+1},\dots,y_{2n}))\rangle$$

(*obtained by applying Type 5 transformations n times*) *are Q^*-equivalent.*

Proof. By the remark preceding Corollary 4.2, it is sufficient to establish the lemma for the case where there is an equivalence between ψ and ϕ defined in (1) via automorphisms η, $\eta_1=\mathrm{id}$, and $\eta_2=\mathrm{id}$.

Following the proof of Theorem 4.3, we define lifts $\hat{\psi}$ and $\hat{\phi}$ of ψ and ϕ to homomorphisms from G^m to $X^n * Y^n$ by $\hat{\psi}(g_i)=z_i\ (i\le n)$, $\hat{\psi}(g_{i+n})=r_i$, and $\hat{\phi}=\hat{\psi}\eta$. Set $v_i=\hat{\phi}(g_i)\ (i\le n)$ and $\hat{s}_i=\hat{\phi}(g_{i+n})$. Note that $\phi=\pi\hat{\phi}$ so that $\phi(v_i)=(x_i,y_i)$ and $(\hat{s}_i)$ is respelling of (s_i). By construction, $\{v_i\}\cup\{\hat{s}_i\}$ and $\{z_i\}\cup\{r_i\}$ generate the same subgroup of $X^n * Y^n$; thus, by the results on

Nielsen transformations used earlier, there is a Nielsen transformation $((v_i),(\hat{s}_i))\to((z_i),(r_i))$. This transformation will be used presently in an application of Lemma 2.3 and Corollary 2.6.

We switch now from $X^n * Y^n$ to $X^{2n} * Y^{2n} = Y^{2n} * Z^{2n}$, and we describe several correspondences intended to set the hypotheses in Lemma 2.3. First set $W = X^{2n} * Y^{2n}$. Set $w_i = y_{i+n}z_i^{-1}$ $(i\le n)$ and set

$$W_L = \mathrm{Gp}\,(\{w_i : i\le n\}).$$

Let $w_{n+1},\dots,w_{4n}$ denote, in some order, the elements in

$$\{z_i : i\le 2n\}\cup\{y_i : i\le n\}.$$

Set $p = 2n+(m-n) = m+n$, and define two p-tuples $(s_i(a))$ and $(t_i(a))$ by $s_i(a) = t_i(a) = y_{i+n}z_i^{-1}$ $(i\le n)$, $s_{i+n}(a) = v_i$ $(i\le n)$, $t_{i+n}(a) = z_i$ $(i\le n)$, $s_{i+2n}(a) = \hat{s}_i$, and $t_{i+2n}(a) = r_i$. Set $q = n$. By the preceding paragraph $(s_i(a))\to(t_i(a))$ is a Nielsen transformation which restricts to a Nielsen transformation

$$(s_{q+1}(a),\dots,s_p(a))\to(t_{q+1}(a),\dots,t_p(a)).$$

Define prefixes $u_i(a)\,(i\le p)$ in W_L by $u_i(a) = 1\,(i\le n)$, $u_{i+n}(a) = w_i = y_{i+n}z_i^{-1}$ $(i\le n)$, and $u_{i+2n}(a) = 1$.

Apply Lemma 4.3 and Corollary 2.6 using the correspondences described in the last paragraph. There are prefixes $u_i'(a)\,(i\le p)$ in W_L such that $u_i'(a) = 1$ $(i\le n)$, and there is a Nielsen transformation

$$(u_i(a)s_i(a))\to(u_i'(a)t_i(a))$$

that is conservative relative to $(u_1(a)s_1(a),\dots,u_q(a)s_q(a))$ so that

$$(u_{q+1}(a),\dots,u_p(a))\to(u_{q+1}'(a),\dots,u_p'(a))$$

is a Nielsen transformation of prefixes in W_L.

Observe now that for $q+1\le i\le 2n$, we have

$$\xi(u_i(a)s_i(a)) = \xi(y_{i+n}z_i^{-1}v_i) = x_i^{-1}\xi(v_i) = 1$$

and that we have $\xi(u_{i+2n}(a)s_{i+2n}(a)) = \xi(1\cdot\hat{s}_i) = \xi\zeta(\hat{s}_i) = 1$. Thus, since the Nielsen transformation $(u_i(a)s_i(a)\to(u_i'(a)t_i(a))$ is conservative, it follows that we have $\xi(u_i'(a)t_i(a)) = 1$ $(i>q)$. Now

$$u_{i+2n}'(a)t_{i+2n}(a) = u_{i+2n}'(a)r_i$$

so $\xi(u_{i+2n}'(a)r_i) = \xi(u_{i+2n}'(a)) = 1$. But ξ maps W_L isomorphically onto X^n since $\xi(w_i) = \xi(y_{i+n}z_i^{-1}) = x_i^{-1}$. Thus we must have $u_{i+2n}'(a) = 1$. Because, by definition, $\{u_{1+n}(a),\dots,u_{2n}(a)\}$ generates W_L, it follows from our application of Corollary 2.6 to get the Nielsen transformation of prefixes that $\{u_{1+n}'(a),\dots,u_{2n}'(a)\}$ also generates W_L. Because W_L is a free group of rank n, it follows that $\{u_{1+n}'(a),\dots,u_{2n}'(a)\}$ freely generates W_L (see [MKS, Section 2.2]).

Now consider the projection under υ of the conservative transformation

$$(u_i(a)s_i(a)) \to (u_i'(a)t_i(a)).$$

By Lemma 2.7,

$$(\upsilon(u_{1+n}(a)s_{1+n}(a)), \ldots, \upsilon(u_p(a)s_p(a))) \\ \to (\upsilon(u_{1+n}'(a)t_{1+n}(a)), \ldots, \upsilon(u_p'(a)t_p(a)))$$

is a Q-transformation of m-tuples in Y^{2n}. By the analysis on the preceding paragraph, what we have is the Q-transformation,

$$((\upsilon(y_{i+n}z_i^{-1}v_i)), (\upsilon(\hat{s}_i))) = ((y_{i+n}), (s_i)) \to ((\upsilon(u_{1+n}'(a)z_i)), (r_i)).$$

The projection υ maps W_L isomorphically onto the subgroup

$$\mathrm{Gp}\,(\{y_{i+n}y_i^{-1}: i \le n\})$$

of Y^{2n}; thus by our analysis of prefixes in the last paragraph,

$$\{\upsilon(u_{i+n}'(a)): i \le n\} \cup \{y_i: i \le n\}$$

freely generates Y^{2n}. Define an automorphism λ: $Y^{2n} \to Y^{2n}$ by

$$\lambda(\upsilon(u_{i+n}'(a))) = y_{i+n}y_i^{-1}\ (i \le n) \quad \text{and} \quad \lambda(y_i) = y_i\ (i \le n).$$

Then λ induces a Type 4 Q^*-transformation,

$$((\upsilon(u_{i+n}'(a)z_i)), (r_i)) \to ((y_{i+n}y_i^{-1}y_i), (r_i)) = ((y_{i+n}), (r_i)).$$

By using Nielsen transformations to permute elements we then obtain, from the transformations already described, the following sequence of Q^*-transformations:

$$((s_i), (y_{i+n})) \to ((y_{i+n}), (s_i)) \to ((y_{i+n}), (r_i)) \to ((r_i), (y_{i+n})).$$

Thus the stabilized presentations $\mathcal{P}'(\psi)$ and $\mathcal{P}'(\phi)$ are Q^*-equivalent as desired. ■

THEOREM 4.5. *Let ψ and ϕ be free splitting homomorphisms in normal form from G^m to $X^n \times Y^n$ and with associated group presentations $\mathcal{P}(\psi) = \langle Y^n: (r_i)\rangle$ and $\mathcal{P}(\phi) = \langle Y^n: (s_i)\rangle$. If ψ and ϕ are equivalent, then the stabilized presentations*

$$\mathcal{P}'(\psi) = \langle Y^{4n}: ((r_i), (y_{1+n}, \ldots, y_{4n}))\rangle$$

and

$$\mathcal{P}'(\phi) = \langle Y^{4n}: ((s_i), (y_{1+n}, \ldots, y_{4n}))\rangle$$

are Q^*-equivalent. On the other hand, if $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are Q^*-equivalent, then $\psi \# \chi_n$ and $\phi \# \chi_n$ are equivalent.

Proof. Suppose first that ψ and ϕ are equivalent. By Lemma 3.2 of [Cr], $\psi \# \chi_n$ and $\phi \# \chi_n$ are equivalent to Mihailova maps ψ' and ϕ' with associated group presentations

$$\mathcal{P}(\psi') = \langle Y^{2n}: ((r_i), (y_{1+n}, \ldots, y_{2n}))\rangle$$

and

$$\mathcal{P}(\phi') = \langle Y^{2n} : ((s_i), (y_{1+n}, \ldots, y_{2n}))\rangle.$$

Because ψ' and ϕ' are also equivalent, Lemma 4.4 shows that the stabilized presentations $\mathcal{P}'(\psi)$ and $\mathcal{P}'(\phi)$ described in the conclusion of Theorem 4.5 are Q^*-equivalent.

If, on the other hand, $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are Q^*-equivalent, then so are the presentations

$$\langle Y^{2n} : ((r_i), (y_{1+n}, \ldots, y_{2n}))\rangle \quad \text{and} \quad \langle Y^{2n} : ((s_i), (y_{1+n}, \ldots, y_{2n}))\rangle.$$

By Theorem 3.2 of [Cr] these latter two presentations are the associated presentations $\mathcal{P}(\psi')$ and $\mathcal{P}(\phi')$ for Milhailova maps ψ' and ϕ' equivalent to $\psi \# \chi_n$ and $\phi \# \chi_n$ respectively. By Theorem 4.3, ψ' and ϕ' are equivalent; hence so are $\psi \# \chi_n$ and $\phi \# \chi_n$ equivalent. ■

The next theorem gives the promised algebraic proof of Theorem 4.1 of [Cr]. Although it uses a small amount of material from [Cr] one can see by tracing back that the proof below avoids the geometric material introduced in [Cr, Sec. 4].

THEOREM 4.6. *Let ψ and ϕ be free splitting homomorphisms in normal form and with associated group presentations* $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$. *Then ψ and ϕ are stably equivalent if and only if $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are Q^{**}-equivalent.*

Proof. Suppose first that ψ and ϕ are stably equivalent. Stabilization does not change the Q^{**}-classes of $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ so we may assume that ψ and ϕ are equivalent. But then, by Theorem 4.5, $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are Q^{**}-equivalent.

Suppose that $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ are Q^{**}-equivalent. Reorder the sequence of elementary Q^{**}-transformations taking $\mathcal{P}(\psi)$ to $\mathcal{P}(\phi)$ so that the Type 5 transformations come first, the Type 6 transformations come last, and all Type 4 transformations are lumped together in a single step. But then, by stabilizing, we can convert ψ and ϕ to stably equivalent free splitting homomorphisms ψ' and ϕ' so that $\mathcal{P}(\psi')$ and $\mathcal{P}(\phi')$ are Q^*-equivalent. By Theorem 4.5, ψ' and ϕ' are stably equivalent; hence ψ and ϕ are also stably equivalent. ■

The appearance of the invertibility criterion in the proof of Theorem 4.3 viewed in conjunction with the stable classification theorem, Theorem 4.6 here, suggests that there might be, in some cases, a relaxation of the invertibility criterion if one desires to test only for Q^{**}-equivalence. Such a relaxation is described by the following theorem:

THEOREM 4.7. *Let (r_i) and (s_i) be p-tuples of elements in Y^n, and let $(\hat{s}_i)$ be a respelling of (s_i) relative to $((r_i), V)$ where V is a subgroup of $X^n * Y^n$ freely generated by a set of n elements $\{v_i : i \leq n\}$. Set $H = \mathrm{Gp}(\{r_i\} \cup V)$. Suppose that there exists a set of n elements $\{u_i : i \leq n\}$ in $X^n * Y^n$ so that H is freely generated by $\{u_i\} \cup \{\hat{s}_i\}$. Finally, suppose that H is sufficiently large so* *that each of the factor maps* $\xi \mid H\colon H \to X^n$ *and* $\upsilon \mid H\colon H \to Y^n$ *is surjective. Then* (r_i) *and* (s_i) *are* Q^{**}*-equivalent.*

Proof. Set $m = p + n$, and define as follows homomorphisms ψ and ϕ from G^m to $X^n \times Y^n$. First define $\hat{\psi}\colon G^m \to X^n * Y^n$ by $\hat{\psi}(g_i) = v_i$ $(i \leq n)$ and $\hat{\psi}(g_{i+n}) = r_i$. Then define $\hat{\phi}\colon G^m \to X^n * Y^n$ by $\hat{\phi}(g_i) = u_i$ $(i \leq n)$ and $\hat{\phi}(g_{i+n}) = \hat{s}_i$. Finally, set $\psi = \pi\hat{\psi}$ and $\phi = \pi\hat{\phi}$. The hypotheses of the theorem imply that

$$\text{image } \hat{\psi} = \text{image } \hat{\phi} = H = \mathrm{Gp}\,(\{r_i\} \cup V).$$

Furthermore, since ξ and υ map H surjectively onto X^n and Y^n, it follows that both ψ and ϕ are free splitting homomorphisms.

Rank considerations on the abelianization of H (see [MKS, Sections 2.2 and 2.4]) show that $\{r_i\} \cup \{v_i\}$ also freely generates H. Each of the homomorphisms $\hat{\psi}$ and $\hat{\phi}$ is thus an isomorphism from G^m to H; therefore $\eta = \hat{\psi}^{-1}\hat{\phi}$ is an automorphism of G^m. But now η, $\eta_1 = \mathrm{id}$, and $\eta_2 = \mathrm{id}$ define an equivalence via (1) between ψ and ϕ. Since ψ and ϕ are in normal form with associated group presentations $\mathcal{P}(\psi) = \langle Y^n : (r_i)\rangle$ and $\mathcal{P}(\phi) = \langle Y^n : (s_i)\rangle$, Theorem 4.6 shows that (r_i) and (s_i) are Q^{**}-equivalent. ■

Remark. Theorem 4.5 gives a bound of $3n$ on the number of Type 6 transformations needed in a sequence of elementary Q^{**}-transformations to convert $\mathcal{P}(\phi)$ to $\mathcal{P}(\psi)$ in Theorem 4.7.

5. Concluding remarks, some candidates for counter-examples

The partial classification theorem for free splitting homomorphisms, Theorem 4.3, fails to provide a complete answer to the first question about convertibility of equivalences posed at the beginning of §4. It fails in both directions; although Lemma 4.4 and Theorem 4.5 give quantitative bounds on this failure. In this section we will examine this failure with an eye to providing reasons for it.

Consider first the problem of converting a Q^*-equivalence between group presentations $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ associated with normalized splitting homomorphisms ψ and ϕ to an equivalence between ψ and ϕ. To simplify matters assume that ψ is a Mihailova map. There are three possibilities.

(1) The homomorphism ϕ is equivalent to a Mihailova map ϕ' whose associated group presentation $\mathcal{P}(\phi')$ is Q^*-equivalent to $\mathcal{P}(\phi)$. Note that in this case, ψ and ϕ are equivalent by Theorem 4.3.

(2) The homomorphism ϕ is equivalent to a Mihailova map, but any Mihailova map ϕ' equivalent to ϕ has its associated presentation, $\mathcal{P}(\phi')$, Q^**-inequivalent* to $\mathcal{P}(\phi)$.

(3) The homomorphism ϕ is not equivalent to any Milhailova map. Note that in this case ψ and ϕ cannot be equivalent.

We do not know if case (2) can occur, but Example 4 below shows that case (3) can occur even when $\mathcal{P}(\psi)=\mathcal{P}(\phi)$.

Example 4. Two splitting homomorphisms from G^2 to $X^1 \times Y^1$.

$$\psi(g_1)=(x_1, y_1), \quad \psi(g_2)=(1, y_1^5),$$
$$\phi(g_1)=(x_1, y_1^2), \quad \phi(g_2)=(1, y_1^5) \quad (\phi(g_1^{-2}g_2)=(x_1^{-2}, y_1)).$$

The homomorphism ϕ is not equivalent to a Mihailova map. To see this note that if it were, either (x_1, y_1) or (x_1, y_1^{-1}) would have to belong to the image of ϕ. But the only images under ϕ with x_1 in the first coordinate are of the form $(x_1, y_1^2)\ (1, y_1^5)^r=(x_1, y_1^{2+5r})$, and $2+5r=\pm 1$ has no integer solutions.

Consider now the problem of converting free splitting homomorphism equivalences to Q^*-equivalences. Suppose that ψ and ϕ are equivalent free splitting homomorphisms in normal form. Are the associated group presentations $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$ Q^*-equivalent? Let η, η_1, and η_2 define an equivalence in (1) between ψ and ϕ. By the remark preceding Corollary 4.2 we may suppose, without loss of generality, that $\eta_1=\mathrm{id}$ and $\eta_2=\mathrm{id}$. At this point the hypotheses of Theorem 4.3 still may not be satisfied because $\eta \mid G_L^m$ may not take G_L^m onto itself. In general it is hard to determine what kind of a condition might be used in place of the condition above on η to guarantee the Q^*-equivalence of $\mathcal{P}(\psi)$ and $\mathcal{P}(\phi)$. There is one special case, however, where we can pinpoint exactly what is needed. This is the case where both ψ and ϕ are Mihailova maps and $\psi=\chi_n$. Below we describe two subgroups of the automorphism group of $X^n * Y^n$ and then we state a conjecture about the relation between these two. We will show that in the special situation above, the convertibility of equivalences of Mihailova maps ϕ with χ_n to Q^*-equivalences of the corresponding presentations $\mathcal{P}(\phi)$ with $\mathcal{P}(\chi_n)$ is completely determined by the conjecture.

Let $\mathcal{A}_n$ denote the subgroup of the group Auto $(X^n * Y^n)$ of automorphisms of $X^n * Y^n$ consisting of automorphisms α satisfying the two conditions:

(i) $\pi\alpha(z_i)=\pi(z_i)\ (i\le n)$, and
(ii) $\xi\alpha(y_i)=1\ (i\le n)$.

Let $\mathcal{K}_n \subseteq \mathcal{A}_n$ denote the subgroup of Auto $(X^n * Y^n)$ consisting of automorphisms κ satisfying $\pi\kappa=\pi$ (i.e. $\mathcal{K}_n$ consists of the lifts of id: $X^n \times Y^n$ to automorphisms of $X^n * Y^n$). The conjecture and proposition below show how the convertibility of equivalences in the special case above is determined by the relation between the subgroups $\mathcal{A}_n$ and $\mathcal{K}_n$.

Conjecture 5.1. For each n, and for each $\alpha \in \mathcal{A}_n$, there is an element $\kappa \in \mathcal{K}_n$ such that $\kappa\alpha \mid Z^n=\mathrm{id}$.

PROPOSITION 5.2. *For each $\alpha \in \mathcal{A}_n$, there is a Mihailova map*

$$\phi_\alpha : G^{2n} \to X^n \times Y^n$$

equivalent to χ_n. For this map, $\mathcal{P}(\phi_\alpha)$ is Q^*-equivalent to

$$\mathcal{P}(\chi_n) = \langle Y^n : y_1, \ldots, y_n \rangle\rangle$$

if and only if the conclusion of Conjecture 5.1 holds for α. Moreover if ϕ is any Mihailova map equivalent to χ_n, then there is an $\alpha \in \mathcal{A}_n$ such that $\mathcal{P}(\phi)$ is Q^*-equivalent to $\mathcal{P}(\phi_\alpha)$ where ϕ_α is as above.

Proof. Define a lift $\hat{\chi}_n$ of χ_n to an isomorphism from G^{2n} to $X^n * Y^n$ by $\hat{\chi}_n(g_i) = z_i$ $(i \le n)$ and $\hat{\chi}_n(g_{i+n}) = y_i$ $(i \le n)$. For $\alpha \in \mathcal{A}_n$, define $\hat{\phi}_\alpha$ by $\hat{\phi}_\alpha = \alpha\hat{\chi}_n$ and define ϕ_α by $\phi_\alpha = \pi\hat{\phi}_\alpha$. Observe that by the definition of $\mathcal{A}_n$, each ϕ_α is a Mihailova map. The substitution $\hat{\chi}_n^{-1}\hat{\phi}_\alpha$ for η, id for η_1, and id for η_2 in (1) defines an equivalence between χ_n and ϕ_α.

Suppose now that there exists a $\kappa \in \mathcal{K}_n$ such that $\kappa\alpha \mid Z^n = \text{id}$. But $\phi_{\kappa\alpha} = \phi_\alpha$. Set $\lambda = \hat{\chi}_n^{-1}\kappa\alpha\hat{\chi}_n$. The substitution λ for η, id for η_1, and id for η_2 defines another equivalence via (1) between $\psi = \chi_n$ and ϕ_α. Now $\hat{\chi}_n$ takes G_L^{2n} isomorphically onto the subgroup Z^n of $X^n * Y^n$ and $\kappa\alpha \mid Z^n = \text{id}$ so $\lambda \mid G_L^{2n} = \text{id}$ and the hypotheses of Lemma 4.3 are satisfied. We conclude from Lemma 4.3 that $\mathcal{P}(\phi_\alpha)$ is Q^*-equivalent to $\mathcal{P}(\chi_n)$.

Suppose that $\mathcal{P}(\phi_\alpha)$ is Q^*-equivalent to $\mathcal{P}(\chi_n) = (Y^n : (y_i))$. By Theorem 3 of [Rp], $\mathcal{P}(\phi_\alpha)$ is Q-equivalent to $\mathcal{P}(\chi_n)$. By the proof of Theorem 4.3 (the relevant portion goes back to the first part of the proof of Theorem 4.1 of [Cr]), there is an equivalence between $\psi = \chi_n$ and ϕ_α defined in (1) corresponding to the substitution of automorphisms λ for η, id for η_1, and id for η_2 where $\lambda \mid G_L^{2n} = \lambda \mid G^n = \text{id}$. The fact that the equivalence just described has $\eta_1 = \text{id}$ and $\eta_2 = \text{id}$ implies that $\hat{\chi}_n\lambda\hat{\phi}_\alpha^{-1} = \hat{\chi}_n\lambda\hat{\chi}_n^{-1}\alpha^{-1}$ is a lift κ of

$$\text{id}: X^n \times Y^n \to X^n \times Y^n;$$

that is, $\kappa \in \mathcal{K}_n$. But $\kappa\alpha \mid Z^n = \text{id}$ as can be verified by diagram chasing. Thus if $\mathcal{P}(\phi_\alpha)$ is Q^*-equivalent to $\mathcal{P}(\chi_n)$ then there exists a $\kappa \in \mathcal{K}_n$ as promised in Conjecture 5.1.

Finally consider an arbitrary Mihailova map ϕ that is equivalent to χ_n. Let η, η_1, and n_2 define an equivalence in (1) between $\psi = \chi_n$ and ϕ. As in the remark before Corollary 4.2 we may, without modifying the Q^*-equivalence class of $\mathcal{P}(\phi)$, change ϕ to an equivalent map so that in the equivalence with χ_n we have $\eta_1 = \text{id}$ and $\eta_2 = \text{id}$. Define a lift $\hat{\phi} = \hat{\chi}_n\eta$ where $\hat{\chi}_n$ has been defined previously. Note that $\phi = \pi\hat{\phi}$. Also define an automorphism α by $\alpha\hat{\chi}_n = \hat{\phi}$. Then $\alpha\hat{\chi}_n = \chi_n\eta$ so $\alpha = \hat{\chi}_n\eta\hat{\chi}_n^{-1}$. By Lemma 4.1 we have

$$\pi\alpha(z_i) = \pi(\hat{\chi}_n\eta\hat{\chi}_n^{-1}(z_i)) = \pi(\hat{\chi}_n\eta(g_i)) = \pi(\phi(g_i)) = \phi(g_i) = \pi(z_i)$$

since ϕ is a Mihailova map. Similarly we have $\pi\alpha(y_i) = \phi(g_{i+n}) =$

$(1, \phi_2(g_{i+n}))$ so $\xi\alpha(y_i)=1$. Thus $\alpha \in \mathcal{A}_n$. By our simplification we have represented ϕ as ϕ_α as desired.

This completes the proof of the proposition. ■

Remark. We see from Proposition 5.2 that if Conjecture 5.1 is true then for any Mihailova map ϕ equivalent to χ_n, the presentation $\mathcal{P}(\phi)$ is Q^*-equivalent to the standard presentation $\langle Y^n: (y_1, \ldots, y_n)\rangle$. On the other hand, if Conjecture 5.1 is false, say for $\alpha \in \mathcal{A}_n$, and if s_i $(i \leq n)$ denotes $\upsilon\alpha\chi_n(y_i)$, then the presentation $\mathcal{P}(\phi_\alpha) = \langle Y^n: (s_i)\rangle$ *is not* Q^*-equivalent to $\mathcal{P}(\chi_n) = \langle Y^n: (y_i)\rangle$. But by Lemma 4.4 the two presentations $\mathcal{P}(\phi_\alpha)$ and $\mathcal{P}(\chi_n)$ are Q^{**}-equivalent. Thus if Conjecture 5.1 is false, then there are non-standard (in the sense of Q^*-transformations) balanced presentations for the trivial group which can be made standard by stabilization.

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