ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 1, March 1979

UNIFORM FILTER CONVERGENCE AND POINTWISE FILTER CONVERGENCE FOR FUNCTION SPACES

BY
J. W. BRACE AND J. D. THOMISON

Let G be a vector space of scalar valued functions on a set S and let X denote the formal linear span of S. It was shown in [4], [5] that every pseudonorm topology on G is equivalent to the topology of convergence on a filter $\mathscr{F}$ of subsets of X.

In the present paper we introduce 4 notions of convergence on a filter, viz. uniform, subuniform, pointwise and subpointwise convergence and investigate spaces $(G, \mathscr{F})$ where convergence on $\mathscr{F}$ implies convergence of any of these four types. The notion of completion of $(G, \mathscr{F})$ with preservation of the latter property is also studied.

Section I contains the basic definitions and discussion of relations between various notions of convergence, Section II gives characterizations of spaces with various convergence properties, Section III is devoted to examples, and Section IV examines completions.

We use the notation of [5] and [8].

I. Several notions of convergence

In the following definitions $\mathscr{F}$ is a filter in a set S. The functions are scalar valued and have S as their common domain; G is a linear space composed of such functions. A filter in a linear space X will always be assumed to possess a basis consisting of balanced and convex sets.

I.1. DEFINITION. A sequence $\{f_n\}$ converges to f_0 uniformly (pointwise) on $\mathscr{F}$ when there is a set F in $\mathscr{F}$ such that $\{f_n\}$ converges uniformly (pointwise) to f_0 on F. A sequence $\{f_n\}$ converges to f_0 subuniformly (subpointwise) on $\mathscr{F}$ when every subsequence of $\{f_n\}$ has in turn a subsequence converging to f_0 uniformly (pointwise) on $\mathscr{F}$.

I.2. DEFINITION [3]. The topology on G with subbasis at the zero function given by the sets $U(\varepsilon, \mathscr{F}) = \{g \in G$: there exists $F_g \in \mathscr{F}$ such that $|g(x)| < \varepsilon$ if $x \in F_g\}$ will be referred to as the topology of convergence on $\mathscr{F}$. A sequence $\{f_n\}$ is said to converge to f_0 on the filter $\mathscr{F}$ if $\{f_n\}$ converges to f_0 in the $\mathscr{F}$-topology. The space G equipped with the $\mathscr{F}$-topology will be denoted by $(G, \mathscr{F})$.

Received June 23, 1977.

I.3. *Remarks.* (i) The $\mathscr{F}$-topology on G is linear if and only if for any g in G there exists F in $\mathscr{F}$ such that g is bounded on F [3].

(ii) Every seminormed topology on G can be obtained via convergence on a filter in X, the formal linear span of S [4].

(iii) If the sequence $\{f_n\}$ converges to f_0 uniformly or subuniformly on $\mathscr{F}$ then it converges to f_0 on $\mathscr{F}$.

(iv) The $\mathscr{F}$-topology on G coincides with the topology of uniform convergence on the set $A=\overline{\bigcap\{e(F): F\in\mathscr{F}\}}$. Here $e: X\to G^*$ denotes the natural evaluation and the closures are taken with respect to the $\sigma(G^*, G)$-topology, G^* denoting the algebraic dual of G [4].

I.4 DEFINITION [5]. Consider filters $\mathscr{F}_1$ and $\mathscr{F}_2$ in a linear space X. We write $\mathscr{F}_1>\mathscr{F}_2$ when there exists a number $r\geq 1$ such that $\mathscr{F}_1$ is a refinement of $r\mathscr{F}_2=\{rF: F\in\mathscr{F}_2\}$. We say $\mathscr{F}_1$ is equivalent to $\mathscr{F}_2$, when $\mathscr{F}_1>\mathscr{F}_2$ and $\mathscr{F}_2>\mathscr{F}_1$.

I.5. PROPOSITION [5]. *Let (G, p) be a seminormed space consisting of linear forms on a linear space X, and let $\mathscr{F}$ be a filter in X such that $(G,\mathscr{F})=(G,p)$. The collection of all F belonging to $\mathscr{F}$ of the form $F=\{x: |f(x)|<a\}$ for some f in G and $a>0$, is a subbasis for a filter $\mathscr{M}$ in X inducing the p-topology on G. Moreover, if $\mathscr{G}$ is any filter in X satisfying $(G,\mathscr{G})=(G,p)$, then $\mathscr{G}>\mathscr{M}$. We will refer to $\mathscr{M}$ as a minimal filter.*

Defining $\mathscr{M}$ via a given filter $\mathscr{F}$ is for convenience only: All filters inducing the p-topology on G will give rise to the same minimal filter (up to equivalence).

II. Uniform and pointwise filter convergence spaces

Throughout this section $(G,\mathscr{F})$ is a pseudonormed space consisting of linear forms on a linear space X. The pseudonorm is p; $\mathscr{M}$ is the minimal filter determined by p; p-convergence is equivalent to convergence on $\mathscr{F}$; all filters are composed of subsets of X. Proofs appear at the end of the section.

II.1. DEFINITION. $(G,\mathscr{F})$ is a uniform (subuniform, pointwise, subpointwise) filter convergence space when convergence on $\mathscr{F}$ implies uniform (subuniform, pointwise, subpointwise) convergence on $\mathscr{F}$. These are abbreviated as u.f.c., s.u.f.c., p.f.c., and s.p.f.c. spaces respectively.

II.2. THEOREM. (a) *$(G,\mathscr{F})$ is a u.f.c. (s.u.f.c.) space if and only if there is a filter $\mathscr{H}<\mathscr{F}$ such that $(G,\mathscr{H})=(G,\mathscr{F})$ and for every sequence $\{H_n\}$ from $\mathscr{H}$ and sequence $\{b_n\}$ of reals, $b_n\geq 1$ and $\lim_{n\to\infty} b_n=\infty$, it follows that $\bigcap_{n=1}^{\infty} b_nH_n$ belongs to $\mathscr{F}$ (there exist subsequences $\{H_k\}$ and $\{b_k\}$ with the property that $\bigcap_{k=1}^{\infty} b_kH_k$ belongs to $\mathscr{F}$).*

(b) *The following are equivalent:*

(i) *$(G,\mathscr{F})$ is a p.f.c. (s.p.f.c.) space.*

(ii) *There is a filter $\mathscr{H} < \mathscr{F}$ such that $(G, \mathscr{H}) = (G, \mathscr{F})$ and for every sequence $\{H_n\}$ from $\mathscr{H}$ and sequence $\{b_n\}$ of real numbers, $\lim_{n\to\infty} b_n = \infty$, it follows that $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k H_k$ belongs to $\mathscr{F}$ (there are subsequences $\{H_k\}$ and $\{b_k\}$ with $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k H_k$ belonging to $\mathscr{F}$).*

(iii) *For each sequence $\{f_n\}$ in G converging on $\mathscr{F}$ to a function f_0 in G there is a nested sequence $\{D_m\}$, $D_m \subset D_{m+1}$, of subsets of X (and a subsequence $\{f_k\}$) such that the sequence (the subsequence $\{f_k\}$) converges uniformly to f_0 on each D_m and $\bigcup_{m=1}^{\infty} D_m$ belongs to $\mathscr{F}$.*

II.3. Corollary. *If $\mathscr{F}$ is closed under countable intersections then $(G, \mathscr{F})$ is a u.f.c. space.*

II.4. Definition. Let $\mathscr{H}$ be a filter in a linear space X. By $u(\mathscr{H})$ we denote the filter with subbasis consisting of all sets of the form $\bigcap_{n=1}^{\infty} b_n H_n$ with H_n in $\mathscr{H}$, $b_n \geq 1$, and $\lim b_n = \infty$.

II.5. Theorem. (a) *The set of all filters $\mathscr{H}$ such that $(G, \mathscr{H}) = (G, p)$ is a u.f.c. (p.f.c.) space has, when nonempty, a unique (within equivalence) smallest element. In the u.f.c. case this smallest element is $u(\mathscr{M})$.*

(b) *If $(G, \mathscr{F}_i) = (G, p)$ is a s.p.f.c. space, $i = 1, 2, 3, \ldots$, then there is a filter $\mathscr{F}$, $\mathscr{F} < \mathscr{F}_i$, $i = 1, 2, 3, \ldots$, such that $(G, \mathscr{F}) = (G, p)$ is a s.p.f.c. space.*

II.6. *Remark.* We do not know if II.5 (b) can be extended to an arbitrary family of filters. A related open question is found in the theory of functional spaces (see [1], pages 140–141).

II.7. *Proof of Theorem II.2.* Only the u.f.c. and p.f.c. cases will be examined. First we will establish the necessity of the condition in (a) and the implication (i)$\Rightarrow$(ii) of (b) by showing that we can take $\mathscr{H} = \mathscr{M}$. It suffices to consider a sequence $\{M_n\}$ from $\mathscr{M}$ where each M_n is of the form $M_n = \{x \in X: |g_n(x)| < a_n\}$, with g_n in G and a_n positive. Let $f_n = (b_n a_n)^{-1} g_n$, and note that the sequence $\{f_n = (b_n a_n)^{-1} g_n\}$ converges to the zero function on $\mathscr{F}$. In the u.f.c. case this implies that there is a set F in $\mathscr{F}$ and a positive integer k_0 such that $f_k(x)| < 1$ for each x in F and $k \geq k_0$. Thus $F \subset b_k M_k$ for all $k \geq k_0$ and it follows that $\bigcap_{n=1}^{\infty} b_n M_n$ belongs to $\mathscr{F}$.

For the p.f.c. case observe that $\lim_k \sup |f_k(x)| \geq 1$ for all x which do not belong to $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} b_k M_k$. Next we prove the sufficiency of the condition in (a) and the implication (ii)$\Rightarrow$(iii) of (b). Let $f_n \to 0$ on $\mathscr{F}$ as $n \to \infty$, f_n in G. Let

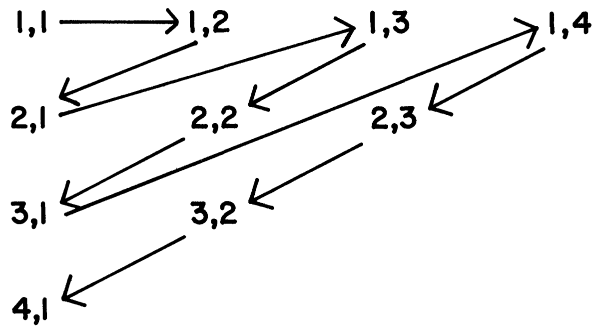
$$M_n = \{x \in X: |f_n(x)| < p(f_n)\}.$$

Then there exists $r \geq 1$ such that rM_n belongs to $\mathscr{H}$ for each positive integer n (see Section 5 of [5]). Let $b_n = (rp(f_n))^{-1/2}$ and choose n_0 such that $b_n \geq 1$ whenever $n \geq n_0$. By assumption the set $\bigcap_{n=n_0}^{\infty} b_n r M_n$ belongs to $\mathscr{F}$. Observing that $\{f_n\}$ converges uniformly to zero on this set establishes the u.f.c. result. The sets $\bigcap_{n=m}^{\infty} b_n r M_n$, $m = 1, 2, 3, \ldots$ provide the nested sequence required for the p.f.c. case. The implication (iii)$\Rightarrow$(i) is immediate.

II.8. *Proof of Theorem II.5: u.f.c. case.* It suffices to prove that $u(u(\mathcal{M})) = u(\mathcal{M})$. This identity is, in fact, valid for any filter $\mathcal{H}$ in X. Let $H_k = \bigcap_{n=1}^{\infty} b_{kn}H_{kn}$ where the b_{kn}'s and H_{kn}'s are as stipulated in II.4. The set $\bigcap_{k=1}^{\infty} b_k H_k$ can be shown to belong to the subbasis for $u(\mathcal{H})$ by enumerating the coefficients

$$\{b_k b_{kn} \colon k = 1, 2, \ldots ; n = 1, 2, \ldots\}$$

as follows:



II.9. Lemma. *Let (G, p) be a seminormed space and $\mathcal{B}$ a filter with a basis consisting of linear subspaces of X. If every function which vanishes on a member of $\mathcal{B}$ has zero seminorm, then $(G, p) = (G, \mathcal{M} \vee \mathcal{B})$.*

Proof. By I.3(iv) it suffices to show that

$$A \subset \bigcap\{\overline{e(M \cap B)} \colon M \in \mathcal{M}, B \in \mathcal{B}\} \quad \text{where} \quad A = \bigcap\{\overline{e(M)} \colon M \in \mathcal{M}\}.$$

Consider arbitrary $M \in \mathcal{M}$, $B \in \mathcal{B}$, $x_0 \in A$, and $\sigma(G^*, G)$-neighborhood V of x_0. Without loss of generality we may assume $M = e^{-1}(e(M))$, $\overline{e(M)} \bigcap e(X) = e(M)$, and $\overline{e(M)}$ a $\sigma(G^*, G)$-neighborhood of A. It follows from the bipolar theorem that $\overline{e(B)} \supset A$. Thus

$$e(M \bigcap B) \bigcap V = e(M) \bigcap e(B) \bigcap V = \overline{e(M)} \bigcap V \bigcap e(B) \neq \emptyset.$$

II.10. *Proof of Theorem II.5: p.f.c. and s.p.f.c. cases.* Let $\mathcal{B}$ denote the glb of the collection of all filters having a basis composed of the linear spans of members of a filter $\mathcal{G}$ where $(G, \mathcal{G}) = (G, p)$ is a p.f.c. space. It follows from the lemma that $\mathcal{M} \vee \mathcal{B}$ is the required smallest filter for the p.f.c. case. Now let $\mathcal{B}_i$ denote the filter with basis consisting of the linear spans of members of $\mathcal{F}_i$, $i = 1, 2, 3, \ldots$. Note that $(G, \mathcal{M} \vee \mathcal{B}) = (G, p)$ where $\mathcal{B}$ is the glb of the $\mathcal{B}_i$. A standard diagonal argument shows that $(G, \mathcal{M} \vee \mathcal{B})$ is a s.p.f.c. space.

III. Examples

The following inclusions are immediate from II.1.

$$\begin{array}{ccc} \text{u.f.c.} & \Rightarrow & \text{s.u.f.c.} \\ \Downarrow & & \Downarrow \\ \text{p.f.c.} & \Rightarrow & \text{s.p.f.c.} \end{array}$$

It may be inferred from the examples in this section that the four classes are distinct. Our only explicit effort in this direction, III.3, is to exhibit a s.u.f.c. space which is not a u.f.c. space.

III.1. *A seminormed function space not belonging to any of the four classes.*

Let $S=(0,1]$ and let G be the space consisting of all real valued bounded continuous functions defined on S. Equip G with the topology τ of uniform convergence on $\beta(S)-S$, i.e., the Stone-Čech compactification of S with the points of S removed, and extend the functions in G to X, the formal linear span of S. For each $n\in\mathbf{N}$ let

$$\begin{aligned} f_n(s) &\equiv 1 && \text{if} \quad s<1/n, \\ &\equiv -ns+2 && \text{if} \quad 1/n\leq s<2/n, \\ &\equiv 0 && \text{if} \quad s\geq 2/n, \end{aligned}$$

and let f denote the function which is identically 1 on S. Let $\mathcal{F}$ denote an arbitrary filter in X inducing the τ-topology on G. Since $\{f_n\}$ converges to f in this topology it is clear that (G,τ) cannot be a s.p.f.c. space with respect to $\mathcal{F}$.

III.2. *The u.f.c. space F^∞ and its subspaces.* Let μ be a complete positive measure defined on a σ-ring of subsets of a set S, and let F^∞ denote the collection of all scalar valued functions f for which the following seminorm is finite:

$$p(f)=\inf\{M\colon \mu\{x\colon |f(x)|>M\}=0\}.$$

(See [10], page 468.) As usual we regard F^∞ as a space of linear forms on X, the formal linear span of S. Let $\mathcal{H}$ denote the filter with basis formed from the convex balanced hulls of subsets whose complements have measure zero. Since $p(f)=\lim_{\mathcal{H}}\inf|f|$ and $\mathcal{H}$ is closed under countable intersections, it follows from II.3 that $(G,\mathcal{H})=(G,p)$ is a u.f.c. space. The subspaces L^∞ and, when S is locally compact and μ is a Borel measure, $C_c(S)$ are also u.f.c. spaces with respect to $\mathcal{H}$. Of course, when S is compact the topology on $C(S)$ can be obtained from the coarser filter composed of all subsets of X containing the convex balanced hull of S.

III.3. *A s.u.f.c. space $(G, \mathcal{F})$ which is not a u.f.c. space.* Let $(G, \|\ \|)$ be an infinite dimensional Hilbert space, $X = G'$, $\mathcal{M}$ the filter of $\sigma(G', G)$-neighborhoods of the unit ball in G'. The filter $\mathcal{F}$ is obtained using the construction of II.4 with the added condition $b_n \geq n^4$. By II.2(b), $(G, \mathcal{F}) = (G, \|\ \|)$ is a s.p.f.c. space.

Let $\{f_n\}$ be an orthonormal sequence in G and let $h > 1$ be arbitrary. Define $D_n = \{x \in X : |(x \cdot f_n)| < h\}$. We will invoke II.2(a) by showing that $\bigcap_{n=1}^{\infty} nD_n$ is not contained in $\mathcal{F}$. Let

$$\bigcap_{i=1} \bigcap_{n=1}^{\infty} b_{ni} M_{ni}$$

be an arbitrary set from the basis for $\mathcal{F}$. This set may be expressed as $\bigcap_{j=1}^{\infty} r_j E_j$ where $j = i + (n-1)m$, $r_j = b_{ni}$, $E_j = M_{ni}$ and each E_j is of the form $\{x \in X : |(x \cdot g_j)| < a_j\}$ where $\|g_j\| = 1$ and $a_j > 1$. Choose j_0 so that $j^2 > hm^4$ when $j \geq j_0$, and let u_0 be a unit vector in $\langle f_1, f_2, \ldots, f_{j_0}\rangle$ such that u_0 belongs to the kernel of g_j for $j = 1, 2, \ldots, j_0 - 1$. Suppose that $j \geq j_0$. Then $r_j \geq (j/m)^4 > hj_0^2$ and

$$|(hj_0^2 u_0 \cdot g_j)| \leq hj_0^2 \|u_0\| \cdot \|g_j\| = hj_0^2 < r_j.$$

It follows that $hj_0^2 u_0 \in \bigcap_{j=1}^{\infty} r_j E_j$. Now suppose that

$$hj_0^2 u_0 \in \bigcap_{n=1}^{\infty} nD_n,$$

and let $c_n = (hj_0^2 u_0 \cdot f_n)$. Thus $c_n < hn$ and

$$hj_0^2 u_0 = c_1 f_1 + c_2 f_2 + \cdots + c_{j_0} f_{j_0}.$$

But then

$$hj_0^2 = \|hj_0^2 u_0\| = \left(\sum_{j=1}^{j_0} c_j^2\right)^{1/2} < h(1^2 + 2^2 + 3^2 + \cdots + j_0^2)^{1/2} < hj_0^{3/2} < hj_0^2,$$

a contradiction.

III.4. *Pointwise and subpointwise filter convergence spaces.* The Hardy spaces H^p, $1 \leq p < \infty$, are p.f.c. spaces since $\|f_n - f\| \to 0$ implies pointwise convergence at every point of the open unit disk. (See [7].) Each L^p space may be realized as a s.p.f.c. space as follows. Let $\mathcal{B}$ denote the filter in X having as basis the linear spans of the subsets of S whose complements have measure zero. Since every convergent sequence has a subsequence converging pointwise on a member of $\mathcal{B}$ and any function which is identically zero on a member of $\mathcal{B}$ has seminorm zero, we may infer from II.9 the existence of a suitable filter $\mathcal{F}$.

IV. Cauchy sequences and completions

Proofs appear at the end of the section.

IV.1. THEOREM. *A sequence $\{f_n\}$ in a u.f.c. space $(G, \mathcal{F})$ is Cauchy (i.e., with respect to the seminorm p determined by $\mathcal{F}$) if and only if there is an F in $\mathcal{F}$ such that $\{f_n\}$ is Cauchy for uniform convergence on F.*

IV.2. DEFINITION [5]. $C(X, G, \mathcal{F}) = \{f \in X^*: \lim_{\mathcal{H}} f$ *exists for each refinement* $\mathcal{H}$ *of* $\mathcal{F}$ *such that* $\lim_{\mathcal{H}} g$ *exists for all* g *in* $G\}$.

IV.3. THEOREM. *If* $(G, \mathcal{F})$ *is a u.f.c. space then* $C(X, G, \mathcal{F})$ *is a completion of* $(G, \mathcal{F})$ *and every Cauchy sequence converges uniformly to one of its limits on a member of* $\mathcal{F}$.

Theorem IV.3 may be regarded as a generalization of Grothendieck's completion theorem [8], [9].

IV.4. THEOREM. *If* $\{f_n\}$ *is a Cauchy sequence in a p.f.c. space then there is a nested sequence* $\{D_i\}$, $D_i \subset D_{i+1}$, *of subsets of* X *such that*

$$\bigcup\{D_i: i = 1, 2, 3, \ldots\}$$

belongs to $\mathcal{F}$ *and the sequence is Cauchy for uniform convergence on each* D_i, $i = 1, 2, 3, \ldots$.

IV.5. THEOREM. *Let* $(G, \mathcal{F})$ *be a p.f.c. space. The following are equivalent.*

(i) $C(X, G, \mathcal{F})$ *is a completion of* G *and every Cauchy sequence in* G *converges pointwise to one of its limits on a set* $F \in \mathcal{F}$.

(ii) *For each Cauchy sequence from* $(G, \mathcal{F})$ *and every refinement* $\mathcal{H}$ *of* $\mathcal{F}$ *for which* $\lim_{\mathcal{H}} g$ *exists for each* $g \in G$, *the filter* $\mathcal{H}$ *converges on the (filter of sections of) the Cauchy sequence.*

IV.6. *Proofs of IV.1 and IV.4.* A sequence which is Cauchy for uniform convergence on a set F in $\mathcal{F}$ is clearly Cauchy in $(G, \mathcal{F})$. Suppose $\{f_n\}$ is Cauchy with respect to p. Choose a subsequence $\{n_k\}$ of the positive integers such that $p(f_{n_k} - f_m) < 4^{-k}$ whenever $m > n_k$, and define $\phi(n) = k$, $n_k < n \leq n_{k+1}$. If $(G, \mathcal{F})$ is a u.f.c. space, the sequence $\{2^{\phi(n)}(f_{n_{\phi(n)}} - f_n)\}$ converges uniformly to the zero function on a set F in $\mathcal{F}$. If $(G, \mathcal{F})$ is a p.f.c. space the convergence is uniform on each member of an increasing nested sequence $\{D_i\}$ of sets such that $\bigcup_{i=1}^{\infty} D_i$ belongs to $\mathcal{F}$. Let $\varepsilon > 0$ and choose N, a positive integer, such that

$$\sum_{k=\phi(n)}^{\infty} 2^{-k} < \varepsilon/2 \quad \text{and} \quad |2^{\phi(n)}(f_{n_{\phi(n)}} - f_n)| < 1$$

whenever $n > N$ and x is in F (D_i, in the p.f.c. case). The following inequalities are valid whenever $m > n > N$ and x is in F (D_i):

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq |f_n(x) - f_{n_{\phi(n)}}(x)| + (|f_{n_{\phi(n)}}(x) - f_{n_{\phi(n)+1}}(x)| \\ &+ \cdots + |f_{n_{\phi(m)}}(x) - f_m(x)|) \leq \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

IV.7. *Remark.* The completion theorems can now be obtained by appealing to the following results, reformulated for more convenient application.

(i) (Theorem 4.5, page 292 of [3]). Suppose $\mathscr{E}$ is a filter in G and $\mathscr{H}$ is a filter in G^* such that $\mathscr{E}$ converges to $f_0 \in G$ pointwise on some member of $\mathscr{H}$ and $\mathscr{H}$ converges to $a \in G^*$ on some member of $\mathscr{E}$. Then $\mathscr{H}$ converges to a on $\mathscr{E}$ if and only if $\mathscr{E}$ converges to f_0 on $\mathscr{H}$.

(ii) (Theorem 1.3 of [5]). Let f_n belong to $C(X, G, \mathscr{F})$ and let $\bar{f}_n$ denote its extension to

$$A = \bigcap\{\overline{e(F)}^{\sigma(G^*,G)} : F \in \mathscr{F}\}, \quad n = 0, 1, 2, \ldots.$$

Suppose $\{\bar{f}_n\}$ converges uniformly to $\bar{f}_0$ on A. Then $\{f_n\}$ converges to f_0 on $\mathscr{F}$.

(iii) (Theorem 2.3 of [5]). $C(X, G, \mathscr{F})$ is the closure of G in $(X^*, \mathscr{F})$.

IV.8. *Proofs of IV.3 and IV.5.* Let $\{f_n\}$ be a Cauchy sequence in $(G, \mathscr{F})$ and let F be a set in $\mathscr{F}$ on which $\{f_n\}$ converges, uniformly in the u.f.c. case, pointwise in the p.f.c. case. Since each f_n belongs to X^*, the function $\lim_{n\to\infty} f_n(x)$, x in F, may be extended to a function f_0 in X^*. Theorem IV.3, and the implication (i)$\to$(ii) of IV.5, now follow from the remarks of IV.7. Assume IV.5(ii) is true, and let $\mathscr{H}$ be a refinement of $\mathscr{F}$ such that $\lim_{\mathscr{H}} g$ exists for each g in G. Remark IV.7(i) tells us that f is a member of $C(X, G, \mathscr{H})$, and thus a member of $C(X, G, \mathscr{F})$, since $\mathscr{H}$ was arbitrary. A standard argument shows that $\lim_n \bar{f}_n(a) = \bar{f}_0(a)$ for each a in A. Note that $\{\bar{f}_n\}$ is Cauchy for uniform convergence on A (I.3(iv)). Thus the sequence $\{\bar{f}_n\}$ converges uniformly to $\bar{f}_0$ on A, and IV.7(ii) applies.

Bibliography

1. N. Aronszajn and K. T. Smith, *Functional spaces and functional completion*, Ann. Inst. Fourier, t. VI (1956), pp. 125–185.
2. N. Bourbaki, *General topology*, Chap. IV, Addison-Wesley, Reading, Mass., 1966.
3. J. W. Brace, *Convergence on filters and simple equicontinuity*, Illinois J. Math., vol. 9 (1965), pp. 286–296.
4. J. W. Brace, G. D. Friend, and P. J. Richetta, *Locally convex topologies on function spaces*, Duke Math. J., vol. 36 (1959), pp. 709–714.
5. J. W. Brace and J. D. Thomison, *Function space completions*, Illinois J. Math., vol. 22 (1978), pp. 581–603.
6. P. R. Halmos, *Measure theory*, D. Van Nostrand, New York, 1950.
7. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N.J., 1962.
8. J. Horváth, *Topological vector spaces and distributions*, Vol. I, Addison-Wesley, Reading, Mass., 1966.
9. J. L. Kelley and I. Namioka, *Linear topological spaces*, D. Van Nostrand, Princeton, 1963.
10. F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.

University of Maryland
College Park, Maryland

Ithaca College
Ithaca, New York