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CHEVALLEY GROUPS AS STANDARD SUBGROUPS, I

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1. Introduction

Let G be a finite group and A a quasisimple subgroup of G. Then A is called a *standard subgroup* if $K = C_G(A)$ is tightly embedded (i.e. $|K|$ is even, but $|K \cap K^g|$ is odd for $g \notin N_G(K)$), $N_G(A) = N_G(K)$, and $[A, A^g] \neq 1$ for all $g \in G$. The importance of such subgroups is evident from the work of Aschbacher (see Theorem 1 of [1]).

The recent approach to the classification of all finite simple groups requires the determination of those groups, G, having a standard subgroup, A, such that $A/Z(A) = \tilde{A}$ is one of the currently known simple groups. This paper and its sequels are concerned with the case of $\tilde{A}$ a group of Lie type defined over a field of characteristic 2. Our results aim at finding the possibilities for G when $\tilde{A}$ has Lie rank at least 3, although we will not treat the cases $\tilde{A} \cong Sp(6, 2)$, $U_6(2)$, or $O^{\pm}(8, 2)'$. Our proofs will be inductive so we require information about the rank 1 and rank 2 configurations as well as information about the four cases above. The necessary results, not covered to date, are assembled in the following hypothesis.

Hypothesis(*). Let P be quasisimple with $|Z(P)|$ odd and $P/Z(P) \cong Sp(6, 2)$, $U_6(2)$, or $O^{\pm}(8, 2)'$. If P is a standard subgroup of a group X with $O(X) = 1$ and $C_X(P)$ having cyclic Sylow 2-subgroups, then one of the following occurs:

(a) $P \trianglelefteq X$.
(b) $E(X) \cong P \times P$.
(c) $E(X)$ is a group of Lie type defined over a field of characteristic 2.
(d) $P \cong O^+(8, 2)'$ and $E(X) \cong M(22)$.

For a group, X, we set $\tilde{X} = X/Z(X)$. Our main result is as follows.

MAIN THEOREM. *Assume that Hypothesis* (*) *holds and that the $B(G)$-conjecture holds. Let A be a quasisimple group with $|Z(A)|$ odd and $\tilde{A}$ a finite group of Lie type defined over a field of characteristic* 2 *and having Lie rank at least* 3. *Suppose that A is a standard subgroup of G and that $C_G(A)$ has*

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cyclic Sylow 2-subgroups. Let t be an involution in $C_G(A)$. Then $G_0 = E(G) \geq A$ and one of the following holds:

(a) $t \in Z^*(G)$.
(b) $\tilde{G}_0 \cong \tilde{A} \times \tilde{A}$ *and t interchanges the factors.*
(c) $\tilde{G}_0$ *is a group of Lie type defined over a field of characteristic 2, and t induces an outer automorphism of G_0.*
(d) $A \cong GL(4, 2)$ *and* $G/O(G) \cong \mathrm{Aut}\,(HS)$.
(e) $A \cong O^+(8, 2)'$ *and* $G/O(G) \cong \mathrm{Aut}\,(M(22))$.

Listed below are the possible pairs $(\tilde{A}, \tilde{G}_0)$ that occur in (c) of the theorem above.

$\tilde{A}$	$\tilde{G}$
$PSL(n, q)$	$PSL(n, q^2)$
$O^+(n, q)'$, n even	$O^+(n, q^2)'$
$O^-(n, q)'$, n even	$O^+(n, q^2)'$
$PSU(n, q)$	$PSL(n, q^2)$
$PSp(n, q)$	$PSL(n, q)$, $PSL(n+1, q)$, $O^+(n+2, q)'$, $O^-(n+2, q)'$ $PSU(n, q)$, $PSU(n+1, q)$, $PSp(n, q^2)$
$E_n(q)$, $n = 6, 7, 8$	$E_n(q^2)$
${}^2E_6(q)$	$E_6(q^2)$
$F_4(q)$	$E_6(q)$, ${}^2E_6(q)$, $F_4(q^2)$

The assumption that $C_G(A)$ has cyclic Sylow 2-subgroups is justified by Corollary 2 of [2] together with the theorem in [6]. We remark that if $\tilde{A}$ is defined over $\mathbf{F}_q$ for $q \geq 4$, then we do not need Hypothesis (*) for the cases $PSp(6, 2)$, $PSU(6, 2)$, or $O^{\pm}(8, 2)'$.

The proof of the main theorem is in three parts, after assuming $\tilde{A} \ntrianglelefteq \tilde{G}$. The first part, the subject of this paper, is fusion-theoretic. For t an involution in K, we first study $t^G \cap N(A)$ and then find a subgroup $X \leq A$ such that $N_G(X)$ contains a standard subgroup Y of Lie rank less than that of A and having $\langle t \rangle$ as a Sylow 2-subgroup of $N(X) \cap C(Y)$. At this point induction can be applied. In the second paper we use this information to construct a subgroup $G_0 \leq G$ with $\tilde{G}_0$ either a group of Lie type on which t acts as an outer automorphism, or $\tilde{G}_0$ isomorphic to the direct product of two copies of $\tilde{A}$, interchanged by t. In the third paper we will show that $G_0 \trianglelefteq G$ (hence $G_0 = E(G)$).

As mentioned above this paper concerns the fusion-theoretic information needed for the proof of the main theorem. These results are in §3. The proofs use a theorem about transitive extensions, which is proved in §2. In §4 we apply the results of §3 to show that certain proper sections of G have standard subgroups.

Notation is as in [5]. Throughout the paper we make use of standard isomorphisms and only consider orthogonal groups of dimension at least 8.

Furthermore, the groups $O(2n+1, 2^a)'$ will be considered as the symplectic group $Sp(2n, 2^a)$.

2. A result on transitive extensions

In this section let X be a perfect central extension of $L_n(q_0)$, for q_0 even, and let σ act on X, inducing a graph field or graph-field automorphism on $\tilde{X}$. Setting $Y=O^{2'}(C_X(\sigma))$, we have Y a central extension of $PSp(n-1, q)$, $PSp(n, q)$, $PSU(n, q)$, or $PSL(n, q)$, for some $q \mid q_0$.

(2.1). *Let G be a 2-transitive permutation group on a finite set Ω.* Choose *$\alpha \neq \beta$ in Ω and assume that $X \trianglelefteq G_\alpha$, $Y \trianglelefteq G_{\alpha\beta}$, and $C_{G_\alpha}(X)$ is cyclic. Then $X \cong L_2(4)$, $Y \cong L_2(2)$, and $G' \cong L_2(11)$.*

Let $\hat{X}$ denote the subgroup of $\mathrm{Aut}(\tilde{X})$ generated by $\tilde{X}$ together with all field and diagonal automorphisms of $\tilde{X}$. Let M be the usual module for $SL(n, q_0)$. Even though $\tilde{X} \cong L_n(q_0)$ does not necessarily act on M, there will be times when we consider subgroups of $\tilde{X}$ acting on M. For example, if T is any 2-subgroup of $\tilde{X}$, then there is no ambiguity in discussing the action of T on M.

If we view $\tilde{X}$ as a Chevalley group, the root subgroups of $\tilde{X}$ are groups of transvections in a given direction and fixing a given hyperplane of M. From the action of σ on the root subgroups of $\tilde{X}$, and from the fact that q_0 is even, it follows that there is a root subgroup V_0, of Y, such that $V=\Omega_1(V_0)$ is contained in a root subgroup of X. In fact, $V_0=V$ unless n is odd and $\tilde{Y} \cong PSU(n, q)$.

The following is well known.

(2.2) *Assume $Z(X)=1$, so that $\tilde{X}=X$.*

(a) *There is a unique root subgroup, D, of X such that $V \leq D$. In fact, $D=Z(O_2(C_X(V)))$.*

(b) *For $y \in Y$ either*

$$[V, V^y]=1, \quad \langle V, V^y\rangle \cong L_2(q), \quad \textit{or} \quad \langle V, V^y\rangle \in Syl_2(L_3(q)).$$

In the latter case $Z(\langle V, V^y\rangle) \in V^Y$.

(c) *For $x \in X$,*

$$[D, D^x]=1, \quad \langle D, D^x\rangle \cong L_2(q_0) \quad \textit{or} \quad \langle D, D^x\rangle \in Syl_2(L_3(q_0)).$$

In the latter case $Z(\langle D, D^x\rangle) \in D^X$.

(d) *$N_Y(V)$ is transitive on $\{V^y : y \in Y, \langle V, V^y\rangle \cong L_2(q)\}$.*

Fix a subgroup $V_- \in V^Y$ such that $H=\langle V, V_-\rangle \cong L_2(q)$.

(2.3) *Let $Z(X)=1$ and regard $Y \leq X \leq \mathrm{Aut}(X)=K$.*

(a) *$D^K=D^X$.*

(b) *If $\tau \in \mathrm{Aut}(Y)$, then either $V^\tau \in V^Y$ or $Y \cong Sp(4, q)$ and V, V^τ are root subgroups of Y for roots of different lengths.*

(c) $V^K \cap Y = V^Y$.
(d) $J^K \cap Y = J^Y$.

Proof. To prove (a) one shows that $K = XN_K(D)$. This is proved by simply checking that, under the assumption of q_0 even, D is normalized by suitable graph, field, and diagonal automorphisms of X. Similarly we prove (b), taking into account the one exceptional situation. To see (c), first use (a) to observe that if $k \in K$ with $V^k \leq Y$, then $V^k \leq D^g$ for some $g \in X$. Consequently, V^k is a group of transvections with a given direction and fixed hyperplane of M. We conclude $V^k \in V^Y$.

Finally we prove (d). Assume $k \in K$ and $J^k \leq Y$. Conjugating by an element of Y we may assume $V^k = V$ (here we use (c)). So $V_-^k \in V^Y$ and satisfies $\langle V, V_-^k \rangle \cong SL(2, q)$. So by (2.2)(d), $\langle V, V_-^k \rangle$ is conjugate to J by an element of $N_Y(V)$.

(2.4) *If* $Z(X) = 1$, *then* $Z(Y) = 1$ *and* $Y = O^{2'}(N_X(Y))$.

Proof. Assume $Z(X) = 1$. Let $Y_1 = N_X(Y)$, $C = C_X(Y)$, and for $A \leq X$ let A^* denote the preimage of A in $X^* = SL(n, q_0)$.

Write $M = M \oplus \cdots \oplus M_k \oplus M_0$, where $\dim(M_i) = 2$ for $i = 1, \ldots, k$ and $\dim(M_0) \leq 1$. We may choose this decomposition so that there are Y-conjugates $J_1, \ldots, J_k$ of J satisfying $[O^{2'}(J_i^*), M_j] = 0$ for each $i \neq j$. Letting $I = J_1 \cdots J_k$ we certainly have $[C, I] = 1$, and so $[C^*, I_0] = 1$, where $I_0 = O^{2'}(I^*)$. From the action of I_0 on M we conclude that for each $g \in C^*$ and each $i = 0, 1, \ldots, k$, g induces a scalar matrix on M_i. As $O^{2'}(J_i^*)$ and $O^{2'}(J_j^*)$ are conjugate in $O^{2'}(Y^*)$, for each $j \in \{1, \ldots, k\}$, and since $[O^{2'}(Y^*), C^*] = 1$, we conclude that g induces scalar matrices on $M_1 \oplus \cdots \oplus M_k$. In particular, $|g|$ is odd and $Z(Y) = 1$.

It will suffice to show $O^{2'}(Y_1) \leq C_X(\sigma)$. Let $S \in Syl_2(Y_1)$. From

$$[Y, Y_1, \langle\sigma\rangle] = [Y, \langle\sigma\rangle, Y_1] = 1,$$

we conclude $[Y_1, \langle\sigma\rangle, Y] = 1$. Setting $C_0 = [Y_1, \sigma]$, we then have $C_0 \leq C$. From the above paragraph we have $[S_0, C_0^*] = 1$, where $S_0 = O^{2'}(S^*)$. Also, $S_0 = O^{2'}(SC_0)^*$ is σ-invariant. So

$$S_0^\sigma = S_0 \quad \text{and} \quad [S_0, \sigma] \leq C_0^* \cap S_0 = 1.$$

This proves $[S, \sigma] = 1$, and the result follows.

We now begin the proof of (2.1). Suppose the result false and let G be a counterexample of least order. Let

$$S \in Syl_2(G_{\alpha\beta}), \quad \bar{S} \in Syl_2(G_\alpha) \quad \text{and} \quad S \leq \bar{S}.$$

Choose $\bar{S}$ such that $\bar{S}$ contains a Sylow 2-subgroup of $N_{G_\alpha}(G_{\alpha\beta})$. We first show that G does not contain a regular normal subgroup, N. Otherwise $|N_{\bar{S}}(S) : S| = 2$ and $N_{\bar{S}}(S)$ inverts $C_N(S)$. This implies $N_{\bar{S}}(S) = SN_{\bar{S} \cap X}(S)$. This contradicts the facts that $|N_{G_\alpha}(G_{\alpha\beta}) : G_{\alpha\beta}|$ is even and $|N_X(Y) : Y|$ is odd. By Theorem 3 of [3] we may assume $Z(X) = 1 = C_{G_\alpha}(X)$. So by (2.3) we now write $X=\tilde{X}$, $Y=\tilde{Y}$, and regard $Y \leq X \trianglelefteq G_\alpha \leq \mathrm{Aut}(X)$. Also, we may assume $(X, Y) \neq (L_2(4), L_2(2))$, as otherwise G has dihedral Sylow 2-subgroups and G is determined.

(2.5) *Assume $Y \not\cong Sp(4, q)$. Let Δ, Σ be the sets of fixed points of V, J, respectively. Then $N_G(V)^\Delta$ and $N_G(J)^\Sigma$ are each 2-transitive.*

Proof. Suppose $g \in G_\alpha$ and $V^g \leq G_{\alpha\beta}$. As $X \trianglelefteq G_\alpha$, $V^g \leq X \cap G_{\alpha\beta}$. Since $Y \trianglelefteq G_{\alpha\beta}$, $V^g \leq N_X(Y)$, so by (2.4) and the definition of Y, we have $V^g \leq Y$. Now, apply (2.3)(c) and conclude $V^g \in V^Y \subseteq V^{G_{\alpha\beta}}$. Therefore, $V^{G_\alpha} \cap G_{\alpha\beta} = V^{G_{\alpha\beta}}$, and Witt's theorem implies that $N_{G_\alpha}(V)$ is transitive on $\Delta - \{\alpha\}$.

Let t be an involution interchanging α and β. If $Y = Y'$, then $t \in N(G_{\alpha\beta}^{(\infty)}) = N(Y)$. So in this case (2.3)(b) implies $V^t \in V^Y \subseteq V^{G_{\alpha\beta}}$. It follows that $N_G(V)$ moves α, and hence $N_G(V)^\Delta$ is 2-transitive. Similarly $N_G(J)^\Sigma$ is 2-transitive.

Suppose then that $Y' < Y$. Then $Y \cong PSL(2, 2)$ or $PSU(3, 2)$. Even here the above arguments work provided $V^t \in V^Y$. So the only difficulty is when $Y \neq O^{2'}(G_{\alpha\beta})$ and $V^t \not\leq Y$. Here $|V| = 2$, so let $V = \langle v \rangle$. Then $v^t \in G_{\alpha\beta} - Y$, so $v^t \in G_\alpha - X$. Also, $C_G(v) = C_{G_\alpha}(v)$, for otherwise $N_G(V)^\Delta$ would be 2-transitive. However, comparing the structure of $C_{G_\alpha}(v)$ with that of $C_X(v^t)$ (see §19 of [5]) this is seen to be impossible. Consequently, we again have $N_G(V)^\Delta$ 2-transitive and we obtain $N_G(J)^\Sigma$ 2-transitive as well. This proves (2.4).

(2.6) *The Lie-rank of Y is at least 2.*

Proof. Suppose Y has Lie-rank 1. Then $(Y, X) = (L_2(q), L_2(q_0))$, $(L_2(q), L_3(q_0))$, or $(U_3(q), L_3(q_0))$. Since $|X: Y|$ is even, $|\Omega|$ is odd. Suppose $X \cong L_2(q_0)$. As noted before, $q_0 > 4$. Using Theorem 4 of Goldschmidt [9] we conclude that $\bar{S} \cap X$ is strongly closed in $\bar{S} \in Syl_2(G)$. Now apply the main theorem of [9] to obtain a contradiction. Therefore $X \cong L_3(q_0)$.

Let $U = \bar{S} \cap X$ and rechoose notation so that $V \leq Z(U)$. Let $N = N_G(V)$. Then, N_α contains $C_X(V)$ as a normal subgroup and $O_2(C_X(V)) = U$. So $U \trianglelefteq N_\alpha$. Since $U \not\leq Y$ we have $U^\Delta \neq 1$. Also, $C_X(V)$ is solvable, and so N_α^Δ is solvable. From the results of O'Nan [14] and Goldschmidt [9] we conclude that either N^Δ contains a regular normal subgroup or $C_G(V)^\Delta$ is 2-transitive and $(C_G(V)^\Delta)' \cong L_2(q_1)$, $U_3(q_1)$, or $Sz(q_1)$ for some q_1 dividing $|U|$. The cases $(C_G(V)^\Delta)' \cong U_3(q_1)$ or $Sz(q_1)$ are each out by observing $U = \Omega_1(U)$.

Suppose $(C_G(V)^\Delta)' \cong L_2(q_1)$. Then U' is trivial on Δ. As U' is a root subgroup of X, $q = q_0$ and $Y \cong L_2(q) = L_2(q_0)$. So $U \cap G_{\alpha\beta} = V$ and U^Δ is elementary of order $q^2 = q_1$. But then, in N^Δ the normalizer of U^Δ is transitive on $(U^\Delta)^{\#}$. Using a Frattini argument we see that this contradicts the fact that $U = \Omega_1(U)$ and $\exp(U) = 4$. Therefore, N^Δ contains a regular normal subgroup. It follows that U^Δ cannot contain a non-cyclic, abelian, normal subgroup, semiregular on $\Delta - \{\alpha\}$ (otherwise write the regular normal subgroup of N^Δ as a product of centralizers).

If $V = Z(U)$, then $Y \cong L_2(q) = L_2(q_0)$, so $U^\Delta \cong U/V$ is semiregular of order q_0^2. This contradicts the above. So $V < Z(U)$ and $Z(U)^\Delta$ is semiregular or order $q_0 q^{-1}$. Therefore, $q_0 = 4$, $q = 2$, and $|\Delta| - 1 = |N_\alpha : N_{\alpha\beta}| = 2^3 k$, where k is a divisor of $12 = |\mathrm{Out}(X)|$. On the other hand, U^Δ is extraspecial of order 2^5, so the representation theory of U^Δ forces $|\Delta| = r^b$, where r is an odd prime and $b \geq 4$. This is a contradiction, proving (2.6).

The proof of the theorem will be complete once we establish

(2.7) *The Lie-rank of Y is at most* 1.

Proof. Suppose Y has Lie-rank at least 2. Recall the subgroups V, V_-, J, and D. Let D^g be the unique root subgroup of X with $V_- \leq D^g$ (use (2.1) (a)). Then $J \leq \langle D, D^g \rangle = \langle D, D^g \rangle^\sigma \cong L_2(q_0)$. Also,

$$N_X(J) = J \times \hat{L} \leq \langle D, D^g \rangle \times \hat{L} = N_X(\langle D, D^g \rangle) \quad \text{where} \quad \hat{L} \cong GL(n-2, q_0).$$

Finally, $C_X(D) = C_X(V) = Q\hat{L}$, for $Q = O_2(C_X(V))$.

Assume $Y \neq Sp(4, q)$ and set $M = N_G(J)$. Then (2.5) implies that M^Σ is 2-transitive. From the minimality of G we conclude that

$$(X, Y) = (L_4(4), L_4(2)) \quad \text{or} \quad (L_4(4), U_4(2)).$$

Now let $N = N_G(V)$ and consider the 2-transitive group N^Δ. Let $Q = O_2(N_\alpha)$. Since we know $Q_{\alpha\beta}$, it is easily seen that $Q_\Delta = V$. Therefore, Q^Δ is extraspecial of order 2^9 and so the central involution in Q^Δ is in $Z^*(N^\Delta)$. By the Z^*-theorem we see that N^Δ contains a regular normal subgroup, say of order r^b. The representation theory of Q^Δ forces $b \geq 2^4$. However $|\Delta| - 1 = |N_\alpha^\Delta : N_{\alpha\beta}^\Delta| < 2^{10} \cdot 3 \cdot 5 < r^b - 1$. This is a contradiction.

The remaining case is $Y \cong Sp(4, q)$. Here $X \cong L_4(q_0)$ or $L_5(Q_0)$. Suppose that $N_G(J)^\Sigma$ is 2-transitive. Then as above $N_X(J)^\Sigma \cong L_2(4)$ and $N_Y(J)^\Sigma \cong L_2(2)$. This implies $Y \cong Sp(4, 2)$ and $X \cong L_4(4)$. But there is no automorphism of $L_4(4)$ with such a centralizer. This is a contradiction and so we now assume $N_G(J)^\Sigma$ is not 2-transitive.

First, we claim that Y is weakly closed in $G_{\alpha\beta}$, with respect to G. If $q > 2$, then $Y = Y'$, and since $G_{\alpha\beta}/Y$ is solvable, the claim is clear. Suppose $q = 2$. Since $Y = X \cap G_{\alpha\beta}$ we certainly have Y weakly closed in $G_{\alpha\beta}$ with respect to G_α. So let t be an involution interchanging α and β. If $Y^t = Y$, then $N_G(Y)$ is 2-transitive on the fixed points of Y, so the converse of Witt's theorem gives the result. Therefore, we may assume $Y^t \neq Y$. Let x_1, x_2 be representatives of the classes of involutions in $Y - Y'$ (recall $Y \cong Sp(4, 2) \cong S_6$). Since $(Y')^t = T'$, $x_i^t \in G_{\alpha\beta} - Y$ for $i = 1$ and 2. So there is an involution $j \in C_{G_{\alpha\beta}}(Y)$ such that x_i^t is Y-conjugate to $x_1 j$ or $x_2 j$. Now, j induces an outer automorphism on X. It is easily seen from the action of j on the Dynkin diagram of X that $j \sim x_1 j$ or $x_2 j$. Consequently, $C_{G_\alpha}(x_1 j)$ or $C_{G_\alpha}(x_2 j)$ is not 2-constrained. Since $C_{G_\alpha}(x_1)$ and $C_{G_\alpha}(x_2)$ are 2-constrained, $C_G(x_i) \nleq G_\alpha$ for $i = 1$ or 2 (see §4 of [5]). On the other hand, $x_i^{G_\alpha} \cap G_{\alpha\beta} = x_i^{G_{\alpha\beta}}$. So Witt's theorem implies that $C_G(x_i)$ is 2-transitive on the fixed points of x_i on Ω, and consequently $x_i^G \cap G_{\alpha\beta} = x_i^{G_{\alpha\beta}} \subseteq Y$. This contradiction proves the claim.

Let $g \in G_\alpha$ and suppose $J^g \leq G_{\alpha\beta}$. The argument used at the beginning of the proof of (2.4) shows that $J^g \leq Y$, and then (2.2)(d) implies that $J^g \in J^Y \subseteq J^{G_{\alpha\beta}}$. So, $N_{G_\alpha}(J)$ is transitive on $\Sigma - \{\alpha\}$. Since $N_G(J)^\Sigma$ is not 2-transitive this means $N_G(J) = N_{G_\alpha}(J)$.

Let t be any involution interchanging α and β. By the claim, $Y^t = Y$ and by the above, $J^t \notin J^Y$. As above, $V^{G_\alpha} \cap G_{\alpha\beta} = V^{G_{\alpha\beta}}$. If $N_G(V)^\Delta$ is 2-transitive, then we may choose t with $V^t = V$. But this forces $J^t \in J^Y$. Therefore $N_G(V)^\Delta$ is not 2-transitive and as above $N_G(V) = N_{G_\alpha}(V)$. But from $Y = Y^t$ we have the embedding of Y in G_β, and it is easily checked that $C_{G_\beta}(V) \nleq G_{\alpha\beta}$. This is a contradiction, proving the result.

We mention, in passing, that Theorem (2.1) can be generalized to cover Chevalley groups other than $L_n(q_0)$. The arguments are a bit more complicated, but similar to the above.

3. Fusion of involutions

In this section, A will denote a finite group of Lie type having Lie rank at least 2 and defined over a field, $\mathbf{F}_q$, of characteristic 2. Suppose A is a standard subgroup of G and that t is an involution in $C_G(A)$. We assume $|Z(A)|$ odd, $C_G(A)$ has a cyclic Sylow 2-subgroup, R, and $t \in R$. In addition, assume $\tilde{A} \ncong L_3(2)$, $G_2(2)'$, $Sp(4,2)'$, $L_4(2)$, $U_4(2)$, or $O^+(8,2)'$.

Let Σ be the root system for A and $U \in Syl_2(A)$. Set $B = N_A(U)$, a Borel subgroup, and choose a Cartan subgroup, $H \leq B$. In the Lie notation, $N/H = W$, the Weyl group, and $W = \langle s_1, \ldots, s_k \rangle$, where $s_1, \ldots, s_k$ are the fundamental reflections. For each $s \in \Sigma$, there is a root subgroup, U_s, of A, and U is the product of those groups, U_s, for which $s \in \Sigma^+$. For $s \in \Sigma$, let $V_s = \Omega_1(U_s)$ and let r be the positive root of highest height. Set $J = \langle V_r, V_{-r} \rangle \cong SL(2,q)$.

(3.1) *Let X be a finite group of Lie type having Lie rank at least 2 and defined over a field of characteristic 2. Suppose $Z(X) = 1$. Let $m(X)$ be the order of the Schur multiplier of X.*

(i) *If $m(X)$ is even, then $(X, (m(X))$ is one of the following*:

$(L_3(2), 2)$, $(L_3(4), 48)$, $(L_4(2), 2)$, $(Sp(4,2), 2)$, $(Sp(6,2), 2)$, $(O^+(8,2)', 4)$, $(F_4(2), 2)$, $(G_2(4), 2)$, $(U_6(2), 12)$, $({}^2E_6(2), 12)$.

(ii) *If $1 \neq m(X)$ is odd, then $(X, m(X))$ is one of the following*:

$(L_n(q), (n, q-1))$, $(U_n(q), (n, q+1))$, $(E_6(q), (3, q-1))$, $({}^2E_6(q), (3, q+1))$.

Proof. For (i) see Table 1 of [10]. To get (ii) let $\hat{X}$ be the universal group associated with X (see [15] and [16] for details). Assuming $m(X)$ odd is equivalent to the fact that $\hat{X}$ is a covering group of X. So $m(X)=|Z(\hat{X})|$ and the result follows from §8 of [15] and §9 of [16].

(3.2) (i) $t^G \cap A^{\#}\langle t\rangle \neq \emptyset$.
(ii) $C_A(t^g)^{(\infty)} \leq A^g$.

Proof. Notice that R cyclic implies K solvable, and hence $N(A)/A$ solvable. Conjugating by g we have $N(A^g)^{(\infty)} \leq A^g$, from which (ii) follows.

For the first assertion we use the results in §19 of [15]. Assume $t^g \in N(A)-AC(A)$. In most cases each involution in $t^g(C_A(t^g)^{(\infty)})$ is conjugate to t^g. For these cases the result follows from (ii) and symmetry. The exceptions are the groups $\tilde{A} \cong PSL(n, q)$, $PSU(n, q)$, $O^{\pm}(n, q)'$, all with n even, together with $\tilde{A} \cong E_6(q)$ or ${}^2E_6(q)$. In each case, t^g is in the coset of a graph automorphism of A.

First assume $\tilde{A} \cong PSL(n, q)$, $PSU(n, q)$, $E_6(q)$, or ${}^2E_6(q)$. Then (19.9) of [5] implies that when t^g is viewed as an automorphism of $\tilde{A}$, t^g is $\tilde{A}$-conjugate to σv, where σ is the involutory graph automorphism of $\tilde{A}$ and $v \in V_r$ (which we are identifying with $V_rZ(A)/Z(A)$). From the root system, Σ, we see that there is a root subgroup, I, of $\tilde{A}$ with $\langle I, I^{t^g}\rangle = I \times I^{t^g}$, and so t^g is fused to an involution $t^gC_{\tilde{A}}(t^g)$. On the other hand we know that $C_{\tilde{A}}(t^g) \cong C_{\tilde{A}}(\sigma v) = C_{\tilde{A}}(\sigma) \cap C_{\tilde{A}}(v) = T$ and $T = T^{(\infty)}$ (we use here (19.7) and (19.8) of [5], the structure of T given in (19.9) of [5], and also the fact that $q \geq 4$ if $\tilde{A} \cong PSL(4, q)$ or $PSU(4, q)$). So (i) follows from (ii) and symmetry.

Now suppose $\tilde{A} \cong O^{\pm}(n, q)'$, n even, and $t^G \cap A^{\#}t = \emptyset$. By (3.1), $\tilde{A} \cong A$ and so $A\langle t^g\rangle \cong O^{\pm}(n, q)$. We identify $A\langle t^g\rangle$ with the orthogonal group and let V be the natural module. Notation will be as in §8 of [5]. The involution t^g is of type b_l, for some $l \geq 1$, and we set $X = C_A(t^g)$. If $l=1$, then $X \cong Sp(n-2, q)$, so $X = X^{(\infty)} \leq A^g$. As in the previous paragraph, t^g is fused to an involution in $t^gX^{\#}$ (here t^g is conjugate to the graph automorphism) so the symmetry argument gives the result. From now on we take $l>1$.

We claim that $t^g \sim t^g a$ for some involution $a \in X'$. Suppose this is true. Then from the structure of $\mathrm{Aut}(A^g)$ and the fact that $X \leq N(A^g)$ we conclude that $a \in A^gK^g$. This implies (i), and so it will suffice to prove the claim.

Write the matrix for t^g in the basis $\mathcal{B}$, given in (8.3) of [5]. If $l=3$, let

$$\mathcal{B}_1 = \{x_2, x_3, x_{l+1}, x_{n-l}, x_{n-l+2}, x_{n-l+3}\}$$

and set $\mathcal{B}_2 = \mathcal{B} - \mathcal{B}_1$. If $l>3$, then set

$$\mathcal{B}_1 = \{x_2, x_3. x_4, x_5, x_{n-l+2}, x_{n-l+3}, x_{n-l+4}, x_{n-l+5}\}$$

and set $\mathcal{B}_2 = \mathcal{B} - \mathcal{B}_1$. In either case, let $V_i = \langle \mathcal{B}_i \rangle$ for $i=1,2$ and note that $V = V_1 \perp V_2$ and t^g acts on each of V_1 and V_2. If $l=3$, t^g induces a_2 on V_1 and b_1 on V_2, while if $l>3$, t^g induces a_4 on V_1 and b_{l-4} on V_2. In any case $t^g \in O(V_1) \times O(V_2)$ and it will suffice to check that $y \sim ya$ in $O(V_1)$ for some $a \in (C(y) \cap O(V_1))'$, where y is the restriction of t^g to V_1.

Suppose $l>3$. Let

$$I=\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}\quad\text{and}\quad E=\begin{pmatrix}01&&\\10&&\\&&01\\&&10\end{pmatrix}$$

In the basis $\mathscr{B}_1$ we then have

$$y=\begin{pmatrix}I&0\\I&I\end{pmatrix},\quad\text{and we set}\quad a=\begin{pmatrix}E&0\\0&E\end{pmatrix}.$$

Then ya is of type a_4 and $a\in(C(y)\cap O(V_1))'$. This follows from the facts that $C(y)\cap O(V_1)$ contains all matrices of the form $\left(\begin{smallmatrix}X&0\\0&X\end{smallmatrix}\right)$ with $X\in Sp(4,q)$ and that $E\in Sp(4,q)'$, even if $q=2$.

Now suppose $l=3$. Here $O(V_1)\cong O^+(6,q)$ and $O(V_1)'\cong SL(4,q)$. With this identification, an a_2 involution in $O(V_1)$ corresponds to a transvection in $SL(4,q)$. Checking matrices in $SL(4,q)$, we easily find a transvection a with $y\sim ya$ and $a\in(C(y)\cap O(V_1))'$. Indeed, this is possible with y, a, ya all transvections in the same direction. This proves (3.2).

(3.3) *If $\tilde{A}\cong L_3(q)$, then $t^G\cap A^{\#}t\neq\emptyset$.*

Proof. By (3.2), $t^G\cap A^{\#}\langle t\rangle\neq\emptyset$. Let $t^g\in A\langle t\rangle$, $t^g\neq t$. If $t^g\in A$, then choose $S\in Syl_2(A)$ with $t^g\in Z(S)$ (A has just one class of involutions). Then $S\leq N(A^g)$. For $u\in S-Z(S)$, $[u,S]=Z(S)$. As $q\geq4$, $S\cap A^g\nleq Z(S)$, so we choose $u\in S\cap A^g-Z(S)$ and conclude that $Z(S)\leq A^g$. But then $t^g\in A^g$, so $t^g\in Z(A^g)$, contradicting the assumption that $Z(A)|$ is odd.

(3.4) *Suppose that* $\operatorname{rank}(\tilde{A})=2$, $\tilde{A}\ncong L_3(q)$. *Let $s\in\Sigma^+$ be such that $U_s\nsim U_r$.*

(a) *If $\tilde{A}\ncong G_2(q)$ or $U_5(2)$, then either $t^G\cap V_r^{\#}t\neq\emptyset$ or $t^G\cap U_s^{\#}t\neq\emptyset$.*
(b) *If $\tilde{A}\cong G_2(q)$ or $U_5(2)$, then either $t^G\cap V_r^{\#}t\neq\emptyset$ or $t^G\cap U_s^{\#}\langle t\rangle\neq\emptyset$.*

Proof. Suppose $\tilde{A}$ has rank 2 and $\tilde{A}\ncong L_3(q)$. Then, either $\tilde{A}$ has 2 classes of involutions with representatives in V_r and U_s, or $\tilde{A}\cong PSp(4,q)$ and $\tilde{A}$ has a third class of involutions with representatives in $U_r^{\#}U_s^{\#}$ (see (6.1), (7.7), and §18 of [5]).

We claim that either $t^G\cap V_r\langle t\rangle\neq\emptyset$ or $t^G\cap U_s\langle t\rangle\neq\emptyset$. Suppose false. By (3.2), $\tilde{A}\cong PSp(4,q)$ and the projection of $t^G\cap A\langle t\rangle$ to A contains $U_r^{\#}U_s^{\#}$. Choose $S\in Syl_2(N(A))$ with $UR\leq S$. Further, choose S so that $t^g\in Z(S)$ and note that $Z(U)=U_rU_s$. For x an involution in UR we have $C_S(x)\geq U_rU_sR$. On the other hand, if $x\in S-UR$ is an involution, we use (19.5) of [5] to see that $m_2(C_S(x))<m_2(U_rU_sR)$.

Now, $S\in Syl_2(N(A^g))$ and the above remarks imply that $\Omega_1(UR)\leq A^gK^g$. So $\Omega_1(UR)'=U_rU_s\leq A^g$. Therefore, $t^g\in U_rU_st$ and $t^G\cap U_rU_s\langle t\rangle=U_r^{\#}U_s^{\#}t$. Now, A, A^g contain subgroups Y_1, Y_2, respectively, with $Y_1\cong Y_2\cong$

$Z_{q-1} \times Z_{q-1}$ and Y_i regular on $U_r^{\#}U_s^{\#}$, for $i=1,2$. Namely, just take 2-complements in $N_A(U_rU_s)$ and in $N_{A^g}(U_rU_s)$. Then $\langle Y_1, Y_2\rangle$ is 2-transitive on the set $U_r^{\#}U_s^{\#}t = U_r^{\#}U_s^{\#}t^g$. But this contradicts Theorem 1.1 of Hering-Kantor-Seitz [12], proving the claim.

Suppose $t^g \in V_r$ or U_s. If $t^g \in C_A(t^g)^{(2)}$ (second derived group), we argue as follows. In each case $\text{Out}(A^g)$ has cyclic Sylow 2-subgroups, and $K^g = O(K^g)R^g$. It follows that $t^g \in A^g$, contradicting $|Z(A^g)|$ odd. For the exceptional groups use the results of §18 of [5] to see that $t^g \in C_A(t^g)^{(2)}$, unless $\tilde{A} \cong G_2(q)$ and $t^g \in U_s$. So we may take $\tilde{A} \cong PSp(4,q)$, $PSU(4,q)$ or $PSU(5,q)$. First assume $t^g \in V_r$, so that t^g corresponds to a transvection. Using the fact that $q>2$, if $\tilde{A} \cong PSp(4,q)$, we check $t^g \in C_A(t^g)^{(2)}$ using (6.2) or (7.10) of [5] (or by using the Lie structure). Suppose $t^g \in U_s$. For $\tilde{A} \cong PSp(4,q)$ use the existence of the graph automorphism interchanging V_r and U_s. Finally, for $PSU(4,q)$ or $PSU(5,q)$ just use the natural embeddings of $PSp(4,q) \leq PSU(4,q) \leq PSU(5,q)$. This proves (3.4).

(3.5) *Suppose that* $\text{rank}(A) \geq 3$ *and* $\tilde{A} \not\cong O^{\pm}(n,q)'$. Then

$$t^G \cap A^{\#}\langle t\rangle \cap C(J) \neq \emptyset.$$

Proof. Suppose otherwise. By (3.2), $t^G \cap A\langle t\rangle \neq \{t\}$. Let $t^g \in A\langle t\rangle$ with $t^g \neq t$. Assume that t^g is not A-conjugate to an involution in $D \times \langle t\rangle$, where $D = O^{2'}(C_A(J))$. From (13.3), (14.3), (15.5), (16.21), and (17.18) of [5] we conclude that $\tilde{A}$ must be a classical group.

The idea of the proof to follow is this. We will choose a certain elementary abelian normal 2-subgroup, Q, of $C_A(t^g)$ and then look at the action of $N_G(Q\langle t\rangle)$ on $t^G \cap Q\langle t\rangle$. The action group will turn out to be a certain 2-transitive group or rank 3 group, and we show this to be impossible. The contradiction follows since we will know the structure of the 1-point stabilizer and the 2-point stabilizer. However, before we can do this we need to show that $Q\langle t\rangle = Q_1\langle t^g\rangle$, where Q_1 plays the same role in $C_{A^g}(t)$ as does Q in $C_A(t^g)$.

Let V be the natural module for the appropriate covering group, $\hat{A}$, of $\tilde{A}$. We have $D/Z(D) \cong PSL(n-2,q)$, $PSU(n-2,q)$ or $PSp(n-2,q)$. Also $V = V_1 \oplus V_2$ where V_1 is a 2-space (non-degenerate if V is unitary or symplectic), $\hat{J} \times \hat{D}$ acts on V_1 and on V_2, $\hat{J}$ trivial on V_2, and $\hat{D}$ trivial on V_1.

Let x be the projection in A of t^g. The only way x can fail to be A-conjugate to an involution in D is for $\dim([V,\hat{x}]) = l$, where $l = [n/2]$ (see (4.2), (6.1), and (7.7) of [5]). So this must occur for each such element $t^g \in A\langle t\rangle - \{t\}$. If $\tilde{A} \cong PSp(n,q)$ with $n \equiv 0 \pmod 4$, then $\hat{x}$ may be of type a_l or c_l (in the notation of §7 of [5]). In this case choose $\hat{x}$ to be of type a_l, if possible. We define an elementary abelian 2-group, $Q \leq A$, such that

$$\hat{Q} = C_{\hat{A}}([V,\hat{x}]) \cap C_{\hat{A}}(V/[V,\hat{x}]).$$

Then $N_A(Q)$ is a parabolic subgroup of A. If $t^h \in Q^{\#}\langle t\rangle$, then $[V,\hat{x}] = [V,\hat{y}]$, where y is the projection of t^h to A. Using the results of §§4–7 of [5] we see that, except in one situation, x and y are conjugate in $N_A(Q)$. The exceptional case is when $\tilde{A} \cong PSp(n, q)$ for $n \equiv 0 \pmod 4$, one of x or y is of type a_l and the other of type c_l.

Let $C = C_G(\langle t, t^g\rangle)$ and note that $C \leq C(t^g) \cap N(A)$. We will use the following facts about C. First, $O_2(C) = O_2(C_A(t^g)) \times O_2(C_K(t))$. This can be checked using the results in §§4–7 and §18 of [5]. What is relevant is the action of an involutory outer automorphism on $C_A(t^g)/O_2(C_A(t^g))$. The other remark is that unless $\hat{x}$ is of type c_l we have $Q\langle t\rangle = \Omega_1(Z(O_2(C)))$. (In fact $Q = O_2(C_A(t^g))$ unless $\dim(V)$ is odd (§§4–7 of [5]).) Now set $Q_0 = Q\langle t\rangle$ and $Q_1 = Q_0 \cap A^g$.

Case 1. Suppose there does not exist $h \in G$ with $t^h \in A\langle t\rangle$ and projecting to an involution of type c_l, where $l = n/2$. As above $N_A(Q)$ is transitive on $t^G \cap Q$ and on $t^G \cap Q^{\#}t$ (one of which may be empty). As $C_A(t^g) \leq C(t) \cap N(A^g)$, t induces an inner automorphism on A^g, and by symmetry, $C_A(t^g) \sim C_{A^g}(t)$. From the above we have $Q_1 = Z(O_2(C_{A^g}(t)))$ and $N_{A^g}(Q_1)$ transitive on the involutions in Q_1 that are A^g-conjugate to the projection of t. Then $Y = \langle N_A(Q_0), N_{A^g}(Q_0)\rangle$ acts on Q_0, on $\Omega = t^Y$, and Y^Ω is a 2-transitive group or a rank 3 group.

If Y^Ω is 2-transitive, consider $C_Y(t)^\Omega$, the 1-point stabilizer. If $\tilde{A} \cong PSL(n, q)$, then $C_Y(t)^\Omega$ contains a normal subgroup, X, with X a central product of two copies of $SL(l, q)$ (see §4 of [5]). For $SL(l, q) \neq SL(2, 2)$, this contradicts O'Nan [11]. If $SL(l, q) = SL(2, 2)$, this contradicts O'Nan [12]. Suppose $G \cong PSU(n, q)$ or $PSp(n, q)$. Here $C_Y(t)$ contains a normal subgroup X with $X^\Omega \cong GL(l, q^2)$ or $GL(l, q)$, respectively. From (6.2), (7.9), and (7.10) of [5] we see that Y^Ω satisfies the hypotheses of Theorem (2.1). We conclude $X^\Omega \cong L_2(4)$. But then $l = 2$, $n \leq 5$, whereas we have assumed rank $(A) \geq 3$.

Therefore Y^Ω is a rank 3 group. In particular, Y is transitive on $Q(t) \cap t^G$ and $t^G \cap Q \neq \emptyset \neq t^G \cap Qt$. Moreover, for each $t^g \in Q\langle t\rangle - \{t\}$, $t \sim t^g t$. As $x \sim t$ we may assume that $t^g = x$. Let $t^h \in Q - \{t^g\}$. Then considering $t^h \in Q_0 \leq A^g\langle t^g\rangle$ we have $t^h t^g \sim t^g$, by symmetry. However, it is easy to check (see (6.2) and (7.9) of [5]) that t^h can be chosen so that $t^h t^g$ is a transvection in A. This is a contradiction.

Case 2. Suppose that $\hat{x}$ is of type c_l. Then $\tilde{A} \cong PSp(n, q)$ and $n \equiv 0 \pmod 4$. Notation is as in §7 of [5]. With Q as before, $N_A(Q)$ induces $SL(l, q)$ on Q. This case differs from Case 1 because here $T_l \cap A \neq O_2(C_A(t^g))$ (see (4.3) of [5] for the definition of T_l). In fact $C_A(x)/Q$ is isomorphic to the centralizer of a transvection is $Sp(l, q)$. In particular $|O_2(C_A(t^g))| = q^{l-1}|Q|$.

Let $K_1 = O(K) \cap C(t)$. From (7.11) of [5] we have

$$O_{2'2}(C) = O_2(C_A(t^g)) \times K_1 R.$$

By symmetry, $C_{A^g}(t)$ contains a normal elementary subgroup $\bar{Q}_1$ with $N_{A^g}(\bar{Q}_1)$ inducing $SL(l, q)$ on $\bar{Q}_1$. We set $Q_2 = \bar{Q}_1 R^g$ and claim that $Q_2 = \bar{Q}_1\langle t^g\rangle = Q_0$. First, we note that, by symmetry, $\bar{Q}_1 \leq O_{2'2}(C)$, so Q_2 projects into $O_2(C_A(t^g))$ when considered as a subgroup of AK. Suppose we can show that Q_2 projects into Q. Then $Q_2 \leq Q \times K_1 R$ and by orders $Q_2 = (QR)^k$ for some $k \in K_1$. If $|R| = 2$, then we are done. If $|R| > 2$, then $\Omega_1(\Phi(QR)) = \langle t\rangle$, whereas $\Omega_1(\Phi(Q_2)) = \langle t^g\rangle$. This is a contradiction. So we need only show that Q_2 projects into Q.

Suppose false and let $\bar{Q}_2$ denote the projection of Q_2 to A. Then $\bar{Q}_2 \trianglelefteq C_A(t^g)$ and from (7.11) of [5] we conclude that either

$$\bar{Q}_2 Q/Q \leq Z(O^{2'}(C_A(t^g))/Q)$$

and is of order at most q or $\bar{Q}_2 Q = O_2(C_A(t^g))$. In either case, $\bar{Q}_2 \cap Q \leq Z(\bar{Q}_2 Q)$. For notation and computations use (4.3), (7.11), and (7.12) of [5].

In the first case $|Q: Z(Q\bar{Q}_2)| \leq q$ and some element, u, of $\bar{Q}_2$ satisfies

$$X(u) = \begin{pmatrix} I & & \\ & I & \\ x & & I \end{pmatrix}, \quad x \neq 0.$$

An easy computation shows that $|Q: C_Q(u)| > q$, so this case is out. In the other case, $\bar{Q}_2 \cap Q \leq Z(O_2(C_A(t^g))) = P$. However, computing, we check that $Q \cap P$ consists of matrices of the form

$$\begin{pmatrix} I_l & 0 \\ M & I_l \end{pmatrix} \quad \text{where } M = \begin{pmatrix} r & & \\ \mu & rI_{l-2} & \\ y & \xi & r \end{pmatrix}$$

and r, y, μ, ξ satisfy the conditions of (7.12) of [5]. Checking orders, we have a contradiction, establishing the claim.

Since we now have $|R| = 2$, we necessarily have $t \notin C_G(t)'$. On the other hand we will show $x \in C_A(x)'$, which will imply $t^G \cap A = \emptyset$. Actually, we show $x \in C_{A_0}(x)'$, where $A_0 = O^{\pm}(n, q)'$, viewed as a subgroup of A. We do this in order to handle a similar configuration arising in the proof of (3.6). So consider V equipped with a quadratic form for which A stabilizes the underlying bilinear form. Write $V = V_1 \perp \cdots \perp V_k$ where $k = l/2$ and each V_i is an x-invariant 4-space. We write x_i for the restriction of x to V_i, and we may assume that each x_i is a c_2 involution in $O(V_i)$. For each $i = 1, \ldots, k$ there is an involution $y_i \in O(V_i)$ and a transvection $t_i \in O(V_i)$ with $[x_i, y_i] = 1$ and $[t_i, y_i] = x_i$. So fix $i \in \{1, \ldots, k\}$ and choose $j \neq i$. Then $t_i t_j \in O(V)'$ and $[t_i t_j, y_i] = x_i$ implies $x_i \in C_{A_0}(x)'$. Hence $x \in C_{A_0}(x)'$, as claimed.

It follows from the above remarks that $Y = \langle N_A(Q_0), N_{A^g}(Q)\rangle$ is 2-transitive on $\Omega = t^Y = t^G \cap Q_0$. Let $S = Y^\Omega$ and, fixing $\alpha \neq \beta$ in Ω, set $S_0 = O_2(S_{\alpha\beta})$. Since $N_S(S_{\alpha\beta}) \leq N_S(S_0)$, we have $N_S(S_0) > N_{S_\alpha}(S_0)$. Let Δ denote the set of fixed points of S_0 on Ω. Then $N_{S_\alpha}(S_0)^\Delta$ contains a normal elementary abelian 2-group, D, extended by $Sp(l-2, q)$. Here $|D|=q^{l-2}$ and is semiregular on $\Delta-\{\alpha\}$. So D is a strongly closed subgroup of a Sylow 2-subgroup of $N_S(S_0)^\Delta$ and we apply the main theorem of [9] to conclude $l-2=q=2$. At this point one can obtain a contradiction by applying Sylow's theorem to S.

Case 3. Here $\tilde{A}\cong PSp(n, q)$, $n=0 \pmod 4$, x is of type a_l for $l=n/2$, but for some $t^h\in A\langle t\rangle$, t^h projects to an involution, y, in A of type c_l. We may choose y so that $C_A(t^h)=C_A(y)\leq C_A(t^g)$. Indeed, choosing a basis for V as in (7.6) (3) of [5] we take

$$\hat{y}=\begin{pmatrix} I & 0 \\ I & I \end{pmatrix} \quad \text{and} \quad \hat{x}=\begin{pmatrix} I & 0 \\ M & I \end{pmatrix} \quad \text{for} \quad M=\begin{pmatrix} 1 & \cdot & & \\ & & \cdot & \\ 1 & & \cdot & 1 \end{pmatrix}.$$

As before, consider $Y=\langle N_A(Q_0), N_{A^g}(Q_0)\rangle$ acting on Q_0. Let $\Omega=t^Y$ and $S=Y^\Omega$. If the projections of the involutions in Ω to A are all of the same type, then we use the argument of Case 1 or of Case 2. So suppose this is not the case. Let $\alpha=t$, $\beta=t^h$, and $\gamma=t^g$, with $\beta, \gamma\in\Omega$. As in Case 2, $t^h\notin Q$ and we set $S_0=O_2(S_{\alpha\beta})$. The embedding of $E(S_{\alpha\gamma})$ in $E(S_\alpha)$ and in $E(S_\gamma)$ is the natural embedding of $Sp(l, q)$ in $SL(l, q)$. Except for the case $l=4$ these embeddings determine the embedding of S_0 in $E(S_\alpha)$ and in $E(S_\gamma)$. In the case of $l=4$, $E(S_{\alpha\gamma})$ could be twisted by a graph automorphism of $Sp(4, q)$. But in all cases

$$N(S_0)\cap E(S_\gamma)\not\leq S_\alpha.$$

So we consider $\langle N(S_0)\cap E(S_\alpha), N(S_0)\cap E(S_\gamma)\rangle$ acting on the fixed points of S_0 on Ω. As in Case 2 we have a contradiction to the main theorem of [9], unless $l=4$ and $q=2$.

For the exceptional case, argue as follows. First check that a_4 and c_4 involutions in $Sp(8, q)$ are each in the derived group of their centralizer. As $q=2$, $N_G(A)=AK$ and $t\notin C_G(t)'$. Thus $t^G\cap A=\emptyset$ and Y acts on Ω as a rank 3 group. The orbit of t^g under $C_Y(t)$ has length $|GL(4,2)\colon Sp(4,2)|=28$ and the orbit of t^h under $C_Y(t)$ has length 420. So $|\Omega|=449$, and we obtain a contradiction from Sylow's theorem. This completes the proof of (3.5).

Next, we prove an analogue of (3.5) for the orthogonal groups. If $\tilde{A}\cong O^\pm(n, q)'$, then by (3.1), $A\cong O^\pm(n, q)'$. Also, $O^{2'}(C_A(J))=J_0\times D_0$, where $J_0\sim J$ in A and $D_0\cong O^\pm(n-4, q)'$.

(3.6) *Suppose* $\tilde{A}\cong O^\pm(n, q)'$. *Then, either*

(i) $t^G\cap A\langle t\rangle\cap C(JJ_0)\neq\{t\}$

or

(ii) *there exists* $x\in C(t)$ *such that* $A\langle x\rangle\cong O^\pm(n, q)$, x *induces a transvection on* A, *and there exists* $t\neq t^g\in C(t)\cap C(x)$; *moreover* $t^g\in D\times\langle x\rangle\times\langle t\rangle$, *where* $Sp(n-2, q)\cong D=C_A(x)$.

Proof. First suppose that

$$t^G \cap C(t) \cap (N(A) - AC(A)) \neq \emptyset.$$

Choose $t^h \in C(t) - AC(A)$. If $A\langle t^h\rangle \not\cong O^{\pm}(n, q)$, then t^h induces a field or graph-field automorphism on A and $A \cong O^+(n, q)'$. By (19.1) and (19.6) of [5] we see that all involutions in $C_A(t^h)t^h$ are fused. Since $n \geq 8$, $C_A(t^h) = C_A(t^h)^{(\infty)} \leq A^h$, and so (i) holds. Suppose then that $A(t^h) \cong O^{\pm}(n, q)$. Then t^h induces an involution of type b_l on A, and therefore centralizes a transvection $x \in A\langle t^h\rangle$. So here (ii) holds. We assume from now on that $t^G \cap C(t) \leq AC(A)$. Let x be the projection of t^g to A. Then x is of type a_l or c_l, assuming (i) false; the possibilities are given in the following table.

	$n \equiv 0 \pmod 4$	$n \equiv 2 \pmod 4$
$O^+(n, q)'$	$a_{n/2}, c_{n/2}$	$a_{(n-2)/2}, c_{(n-2)/2}$
$O^-(n, q)'$	$a_{(n-4)/2}, c_{n/2}$	$a_{(n-2)/2}, c_{(n-2)/2}$

If possible, choose t^g so that x is of type a_l.

Consider A acting on the natural module, V, for $O^{\pm}(n, q)$ and set $V_0 = [V, t^g]$. If x is of type a_l, then V_0 is totally singular, while if x is of type c_l, V_0 contains a unique totally singular $(l-1)$-subspace, V_1 (see (8.4) of [5]). Let $Q = C_A(V_0) \cap C_A(V/V_0)$. Then Q is an elementary 2-group. If $y \in Q$ is A-conjugate to x, then $[V, y] = V_0$, so $x \sim y$ in $N_A(Q)$. The arguments here will be similar to those of (3.5) for the case $\tilde{A} \cong PSp(n, q)$.

We begin with the following observations. If $y \in N(A)$ with $A\langle y\rangle \cong O^{\pm}(n, q)$, and if x is of type a_l, with $l < n/2$, or of type x_l, then (ii) holds. For in these cases x centralizes a transvection in $A\langle y\rangle$ (consider an orthogonal decomposition of V into a 2-space and an $(n-2)$-space). So in the presence of a graph automorphism of A, we may assume that x is of type a_l, with $l = n/2$. In the latter case $N(Q) \cap N(A)$ does not contain an involution acting as a graph automorphism on A. Let $C = \langle t^G \cap C(\langle t, t^g\rangle)\rangle$. Then $C \leq AC(A) \cap A^gC(A^g)$. We check that

$$O_2(C(\langle t, t^g\rangle) = O_2(C) \times O_2(C_K(t) \leq O_2(C)R,$$

and using (10.2) of [5] we have

$$C/O_2(C) \cong 0^{2'}(C_A(t^g)/O_2(C_A(t^g))).$$

Say $l = n/2$. The arguments here are similar to those in the proof of (3.5) for $\tilde{A} \cong PSp(n, q)$. However, note that here we cannot have $t \neq t^h \in Q\langle t\rangle$ with t^h projecting to an involution of type different from that of x. So Case 3 in (3.5) does not occur here. The analogue of Case 1 goes as before, but in Case 2 things are a bit different. Again we obtain a 2-transitive group $S = Y^{\Omega}$. But here a computation shows that, for $\alpha \in \Omega$, S_α is an extension of a parabolic subgroup of $L(l, q)$ corresponding to the stabilizer of a hyperplane of the usual module for $SL(l, q)$. We also have the structure of $S_{\alpha\beta}$ for

$\alpha \neq \beta \in \Omega$ (see (8.8) of [5].) Using O'Nan [12] we obtain a contradiction. So now assume that $l \leq (n-2)/2$.

Assume first that x is of type $a_{(n-2)/2}$. Then, from (8.6) of [5] or from the Lie structure of A, we compute

$$Q = Z(O_2(C_A(t^g))) \quad \text{and} \quad Q_0 = Q\langle t\rangle = \Omega_1(Z(C)).$$

The usual argument shows that $Y = \langle N_A(Q_0), N_{A^g}(Q_0)\rangle$ induces a 2-transitive group or rank 3 group on t^Y. The 2-transitive case is out by (2.1). In the rank 3 case we may assume $t^g \in Q$. Choose a basis for V as in (8.2) of [5]. Then t^g has matrix form

$$\begin{pmatrix} 1 & & & \\ & \cdot & & \\ & & \cdot & \\ I_l & & & 1 \end{pmatrix}$$

Let t^h be the element of Q with matrix form

$$\begin{pmatrix} 1 & & & \\ & \cdot & & \\ & & \cdot & \\ M & & & 1 \end{pmatrix} \text{where } M = \begin{pmatrix} & & 11 \\ & & 01 \\ & I_{l-4} & \\ 11 & & \\ 01 & & \end{pmatrix}$$

Then $t^g t^h$ has type a_2. But $t^g \sim t^g t \sim t$, and by symmetry $t^g t^h \sim t^g$. Therefore, $l = 2$, whereas $n \geq 10$ here.

Next, assume that x has type $a_{(n-4)/2}$. As above, consider the groups Q_0 and Y, with Y 2-transitive or rank 3 on t^Y. The rank 3 case does not occur for $n = 8$. This is because x is then of type a_2, so $x \in C_A(x)'$. We are assuming there are no graph automorphisms in this case, so $t \notin C_G(t)'$. Therefore $t \not\sim x$. So in the rank 3 case $l \geq 4$ and the argument of the previous paragraph gives a contradiction. In the 2-transitive case the contradiction follows from (2.1) (as above), except when $l = 2$. So now assume $l = 2$ and let $I = O_2(C)$. Then x is a 2-central involution in A, $I \cap A$ is special with center of order q, and

$$C/I = (C_1/I) \times (C_2/I) \cong L_2(q) \times L_2(q^2).$$

Considering the embedding of C in $N(A^g)$, we see that g may be assumed to normalize C. So recalling the definition of $U \in Syl_2(A)$, we may assume $g \in N(U\langle t\rangle)$, and hence $g \in N(I_1)$ for $I_1 = U\langle t\rangle \cap C_1$. Notice that I_1 is just the product of I with a root subgroup of A.

We will consider the group $W_1 = U\langle t\rangle \cap C_G(I_1')$. Computation within the Lie structure of A shows that $W_1 = W\langle t\rangle$, where $W = O_2(P)$ and $P = N_A(W)$ is the stabilizer of a singular 1-space of V. Now $S = \langle P, g\rangle$ acts on W_1 and we consider $\theta = t^S \cap W_1$. The involutions in W are of type a_2 and c_2, so since (i) is false, each involution in t^S projects to an involution of type a_2. As usual, $x \in C_A(x)'$ implies $t^S \subseteq Wt$. Therefore, S^θ is 2-transitive and $|\theta| = 1 + n$, where $n = (q^3+1)(q^2-1)$ is the number of a_2 involutions in W. Also $C(t) \cap S^\theta$ contains a cyclic normal subgroup of order $q-1$. So for $q > 2$ this is against Theorem 3 of [3].

For $q = 2$ we use a special argument as follows. First note that S^θ normalizes $W = \langle t^h t^k : t^h, t^k \in Wt \rangle$. We have $P = WL$ for $L \cong SO^-(6, 2)$. Let a be an involution in S^θ interchanging t and t^g. Then setting $t = \alpha$, $t^g = \beta$, we have a stabilizing $S^\theta_{\alpha\beta}$, where $S^\theta_{\alpha\beta}$ is the extension of an elementary group of order 2^4 by $SO^-(4, 2) \cong A_5$. Let $Z \in Syl_5(S^\theta_{\alpha\beta})$. Then $|N(\mathbf{Z}) \cap S^\theta_\alpha| = 20$ and Sylow's theorem (applied to S^θ) gives $|N(Z)| = 60$. From the action of Z on W we have an element of order 3 in $S - S_\alpha$ centralizing Z and irreducible on the klein group $C_W(Z)$. Now, L preserves a non-degenerate quadratic form on W and, of course, the associated alternating bilinear form. The non-zero singular vectors in W are just the a_2-involutions. Viewing $Z \leq SO^-(6, 2) \leq Sp(6, 2)$ we see that the above mentioned 3-element is necessarily in $Sp(6, 2)$ It is then easy to conclude that $S^\theta \cong Sp(6, 2)$. But then $|S^\theta : S^\theta_\alpha| = 56 > |\theta|$, a contradiction.

The remaining case is when x is necessarily of type c_l, for $l = (n-2)/2$. Again we set $Q_0 = Q\langle t \rangle$. As in (3.5), $x \in C_A(x)'$, and as before, $t \notin C_G(t)'$. Therefore $t^G \cap A = \emptyset$. From (8.8) of [5] we have

$$C/O_2(C) \cong Sp(l-2, q) \times SL(2, q).$$

We will show that Q_0 can be recovered from the abstract structure of C. Once this is done we will have $Y = \langle N_A(Q_0), N_{A^g}(Q_0) \rangle$ 2-transitive on $t^G \cap Q_0$, at which point the earlier arguments for x of type $c_{n/2}$ give a contradiction.

To recover Q_0 from C argue as follows. First assume $l-2 > 2$. Let H_1 be the complete preimage of the $SL(2, q)$ factor of $C/O_2(C)$. This group is well defined as $l-2 > 2$. Let H_0 be a $(q+1)$-Hall subgroup of H_1. Then H_0 is determined, up to conjugacy, within C. Also, it is easy to check that $O^{2'}(C_A(H_0)) \cong SO^\mp(2l, q)$ and $C_C(H_0)/H_0$ has the structure of the centralizer of a c_l involution in $SO^\mp(2l, q)$. The arguments in Case 2 of (3.5) show that Q is determined from the abstract structure of $C_C(H_0)$. This shows that for $l-2 > 2$, Q_0 is determined by the abstract structure of C.

Finally we assume $l-2 = 2$, $l = 4$. Then

$$C/O_2(C) \cong SL(2, q) \times SL(2, q).$$

If $q > 2$, then $C/O_2(C)$ contains precisely two proper normal subgroups, while if $q = 2$, $C/O_2(C)$ contains precisely two normal subgroups of order 3. Let C_1 and C_2 denote the preimages of these factors. One checks that if C_1 corresponds to the factor centralizing Q, then $|Z(C_1)| > |Z(C_2)|$. So $C_1 = H_1$ is determined by C, and we can choose H_0 as before. This completes the proof of (3.6).

4. Standard subgroups

For this section we assume the notation of §3. The results in this section are aimed at showing that for a suitable subgroup, $X \leq A$, the group $C_G(X)$ contains $E(C_A(X))$ as a standard subgroup.

(4.1) *Notation.* Assume A has Lie rank at least 3. We define a subgroup of $N(A)$ as follows. If $A \not\cong O^{\pm}(n, q)'$, then let X be a $(q+1)$-Hall subgroup of $J \cong SL(2, q)$. If $\tilde{A} \cong O^{+}(8, q)'$, let $X = O_{2'}(F)$, where $F \leq \tilde{A}$ is the stabilizer of a non-degenerate 2-space having index 1, of the natural orthogonal space for $\tilde{A}$. Here X is cyclic of order $q-1$ (recall, $q > 2$ here) and $E(C_A(X)) \cong O^{+}(6, q)'$. Finally, suppose $\tilde{A} \cong O^{\pm}(n, q)'$, but $\tilde{A} \not\cong O^{+}(8, q)'$. In the notation of (3.6), let X be a $(q+1)$-Hall subgroup of JJ_0 if (3.6)(i) holds and $X = \langle x \rangle$ if (3.6)(i) fails to hold. In all cases set $D = E(C_A(X))$.

Let $\bar{J} = J$, if $\tilde{A} \not\cong O^{\pm}(n, q)'$ and $\bar{J} = J \times J_0$, otherwise. Set $D_0 = E(C_A(\bar{J}))$. If $\tilde{A} \not\cong O^{\pm}(n, q)'$, then $\bar{J} = J = \langle V_r, V_{-r} \rangle$ and D_0 is the Levi factor in $C_A(V_r)$ that is generated by root subgroups for certain roots in Σ.

(4.2) *There exists $t^g \neq t$ with $t^g \in C(X) \cap C(t)$ and $C_A(t^g)$ 2-constrained.*

Proof. If $\tilde{A} \not\cong O^{\pm}(n, q)$, then this follows immediately from (3.5). Suppose $A \cong O^{\pm}(n, q)'$, but $A \not\cong O^{+}(8, q)'$. Then (3.6) gives the existence of $t^g \neq t$ with $t^g \in C(t) \cap C(X)$. The only way $C_A(t^g)$ could fail to be 2-constrained is that $A\langle t^g \rangle \cong O^{\pm}(n, q)$ with t^g corresponding to a transvection. If this occurs consider $C_A(t^g) \cong Sp(n-2, q)$. Then $C_A(t^g) = C_A(t^g)^{(\infty)} \leq A^g$ and $t^g \sim t^g a$ for a an involution in the center of a Sylow 2-subgroup of $C_A(t^g)$. Using the symmetry between $N(A^g)$ and $N(A)$ we get the result.

Finally, suppose $A \cong PSO^{+}(8, q)$ and let V be the natural module for A. Choose t^g according to (3.6) and argue as above that we are done if t^g induces a transvection on A. So $C_A(t^g)$ is 2-constrained and it suffices to show that t^g can be chosen to centralize a non-degenerate 2-space of V with index 1. If (3.6)(i) holds, this follows as t^g will centralize the 4-space $[V, JJ_0]$. Suppose (3.6)(i) fails to hold. Choose $t^g \in D \times \langle x \rangle \times \langle t \rangle$ as in (3.6) (ii). Assuming the result false, we see that t^g must induce an inner automorphism of type a_4 or c_4. The former is impossible as t^g centralizes the transvection x. So all such involutions project to involutions in A of type c_4 and we can use the argument in the fourth paragraph of the proof of (3.6) to get a contradiction.

(4.3) *If $\tilde{A} \not\cong O^{\pm}(n, q)'$, then $D = D_0$. If $\tilde{A} \cong O^{\pm}(n, q)'$, but $\tilde{A} \not\cong O^{+}(8, q)'$, and if* (3.6)(i) *holds, then $D = D_0$.*

Proof. If $\tilde{A} \cong O^{\pm}(n, q)'$ (in fact if $A/Z(A)$ is any classical group), then this can be checked directly by considering the natural module for A. Otherwise, we argue as follows. We first will show that for $g \in A$, $X \leq J^g$

implies $J^g = J$. This implies that $N_A(X) \le N(J) \le N(D_0)$ and the result follows.

Suppose, then, that $g \in A$ and $X \le J^g$. If $A/Z(A) = G(q)$ is an untwisted Chevalley group, consider $X \le G(q^2) = Z$. If $G(q)$ is twisted, let Z be the Chevalley group from which $A/Z(A)$ is constructed (e.g. $A/Z(A) \cong {}^2E_6(q)$, $Z = E_6(q^2)$). Then $J \le J_1 \le Z$, where $J_1 \cong SL(2, q^2)$ is generated by root subgroups. Now X is contained in a Borel subgroup of J_1 and hence a Borel subgroup of Z. So $N_Z(X)$ is easily determined using the Bruhat decomposition (see (4.2) of [4]), and one sees that $J_1^g = J_1$. That is X is contained in a unique conjugate of J_1. So $J^g \le J_1$ and we have $J^g = J$, as desired.

(4.4) *Assume that* $\tilde{A} \not\cong O^{\pm}(n, q)'$ *or* $\tilde{A} \cong O^{\pm}(n, q)'$, *but* $\tilde{A} \not\cong O^{+}(8, q)'$, *and* (3.6)(i) *holds. Let* $Y = O(C_G(X))$ *and let bars denote images in* $C_G(X)/Y$. *Then* $\bar{D}$ *is standard in* $\overline{C_G(X)}$ *and* $\bar{R} \in Syl_2(C(\bar{D}) \cap \overline{C_G(X)})$.

Proof. We first claim $\bar{R} \in Syl_2(C(\bar{D}) \cap \overline{C_G(X)})$. Otherwise, there is a 2-element $u \in C_G(X)$ such that $\bar{u} \in C(\bar{D})$ and $u \in N(R) - R$. Then $u \in C(t) \le N(A)$, so $u \in N(D)$. As $\bar{u} \in C(\bar{D})$, we have $u \in C(D) \cap N(A)$. However, $O^{2'}(N(A) \cap C(D)) \le \bar{J}K$. This is a contradiction and proves the claim. The rest of the lemma follows easily.

(4.5) *With hypothesis as in* (4.4), $\bar{D} \not\trianglelefteq \overline{C_G(X)}$.

Proof. Suppose false. Then $RO(C(X)) \trianglelefteq C_G(X)$. The idea is this. Let $I \le C_{A^g}(X)$ be t-invariant, where t^g is as in (4.2). Then

$$I \le N(RO(C(X))) \quad \text{and} \quad I^t = I.$$

Therefore, $[I, t] \le I \cap O(C(X))$, and if I is quasi-simple, then $[I, t] = 1$. For example, suppose X and X^g are conjugate in A^g. Then $I = E(C_{A^g}(X)) \cong D$, so $I \le C(t)$, and $I = I^{(\infty)} \le N(A)^{(\infty)} = A$. This forces $I = D$, whereas $t^g \notin C(D)$. So X and X^g are not A^g-conjugate. In fact, we can argue:

(*) For no $a \in A$ is $X^a \le C(t^g)$ and $X^aK^g \sim X^gK^g$ in A^g.

The rest of the proof will be concerned with either providing a suitable $I \le A^g$ or contradicting (*). Let $C = C_G(\langle t, t^g \rangle)$ and $C_1 = O^{2'}(C)$.

First suppose $\tilde{A}$ is an exceptional group and $q > 2$. Then $J = \bar{J} = \bar{J}^{(\infty)} \le N(A^g)^{(\infty)} = A^g$, so $X \le A^g$. From the description of centralizers in (13.3), (14.3), (15.5), (16.20), (17.15), and §19 of [5] we see that $t \in A^gK^g$ and $C_A(t^g) \cong C_{A^g}(t)$. It follows that we may choose g to normalize $\langle t, t^g \rangle$. Let $N_A(U) = B \le P$ be the minimal parabolic subgroup of A subject to $P \ge C_A(t^g)$. This parabolic subgroup is obtainable from (13.2), (14.2), (15.4), (16.19), and (17.14) of [5] and in each case $P = P^{w_0}$, where w_0 is the word of greatest length in the fundamental reflections $\{s_1, \ldots, s_k\}$. Then $L = L^{w_0}$, where $L = O^{2'}(L_1)$ and L_1 is the Levi factor of P. Since $g \in N(C)$, $g \in N(Y)$, where $Y = C^{(\infty)} \le A^g \cap A$. Conjugating by an element $y \in Y$, we have $X^y \le L^g$. Now set $I = \langle Z, Z_\circ^{w^g} \rangle$, for $Z = Z(C_{A^g}(t))$. The group Z is given explicitly in [5] in terms of the root system Σ (all carried to A^g, via g) and in each case I is quasi-simple with $|Z(I)|$ odd. As t projects to an involution in I we have a contradiction to the first paragraph.

Say $\tilde{A}$ is a classical group and let $Z = Z(O_2(C_1))$. One checks that

$$O^2(C_{C_1}(Z)) = \langle X^{C_1}\rangle O(C_K(t)).$$

This can be computed from the results in §§4–8 of [5] or by passing to the Lie structure and computing within certain parabolic subgroups of A. Let $C_2 = O^2(C_{C_1}(Z))$, so that $C_2 = \langle X^{C_1}\rangle O(C_K(t))$. Now look at C_2/C_3, where

$$C_3 = O_2(C_2)O(C_2) = O_2(\langle x^{C_1}\rangle)O(C_K(t)).$$

In most cases the class of XC_3/C_3 is uniquely determined by the structure of C_2/C_3 (for example, in most cases, $\bar{J} \leq C_2$ and the class of $\bar{J}C_3/C_3$ is uniquely determined (see Timmesfeld [17])). In these cases we read all of this in $C(t^g) \leq N(A^g)$ and contradict (*). The exception is when $C_2/C_3 \cong Sp(4, q)$ and $\bar{J} = J \cong SL(2, q)$. But here, choose a Hall subgroup, $\tilde{X}$, of C_2 containing X, set $I = E(C_{A^g}(\tilde{X}))$ and contradict the first paragraph of the proof, unless $A \cong Sp(6, 2)$. In the latter case, first argue that we may take $g \in N(\langle t, t^g\rangle)$. Then g acts on $O^{2'}(C(\langle t, t^g\rangle)) = L$ and induces an outer automorphism on $L/O_2(L) \cong Sp(4, 2)$. Considering the action of $L\langle g\rangle$ on $O_2(L)$, we have a contradiction.

If $\tilde{A}$ is an exceptional group with $q = 2$ and if $|Z(C_{A^g}(t)| > 2$, then $A \cong F_4(2)$ (see [5] and [6]) and we can argue as in the second paragraph. For all other cases we will contradict (*). Let $Y = C_A(t^g)$ and let bars denote images in $Y/O_2(Y)$. Then $O^{2'}(\bar{Y}) = \bar{Y}_1\bar{Y}_2$, a central product of Chevalley groups, where notation is chosen so that $\bar{J} \leq Y_2$.

We now have A an exceptional group and $q = 2 = |Z(C_A(t^g))|$. In most cases we argue by setting $Y = C^{(\infty)} = C_A(t^g)^{(\infty)} = C_{A^g}(t)^{(\infty)}$, noting that $J^a \leq Y$ for some $a \in A$ and that $Y/O_2(Y)$ has just one conjugacy class of (2, 3, 4)-root involutions (Timmesfeld [17]). In these cases we contradict (*) immediately. The exceptions are as follows, where we list the isomorphism type of A and the notation for the projection of t^g to A as given in [5]: $(E_6(2), z)$, $(E_8(2), z)$ $(E_7(2), u)$, $({}^2E_6(2), v)$. For the last case note that there is an error in (14.3)(iii) of [5], corrected in [6]. The essential change is that

$$C_A(v)/O_2(C_A(v)) \cong L_2(q) \times U_3(q) \quad \text{and} \quad |Z(C_A(v))| = q = 2.$$

Also, the $L_2(q)$ factor is covered by J^a for some $a \in A$. Let

$$Y_1 = O_{2,3,2}(C) = C_A(v)O_{3,2}(C_K(t)),$$

and check that

$$O^{2'}(Y_1/O_2(Y_1) \cap C((Y_1/O_2(Y_1))^{(2)})) = J^aO_2(Y_1)/O_2(Y_1).$$

So this factor is determined by the abstract structure of C and we again contradict (*). Similarly, if $\tilde{A} \cong E_6(2)$ or $E_8(2)$, then

$$C/O_2(C) \cap C(Y/O_2(Y)) = J^a O_2(C)/O_2(C) \quad \text{for some} \quad a \in A.$$

Finally, assume $\tilde{A} \cong E_7(2)$. Here $Y/O_2(Y) \cong F_4(2)$ and $X^a \leq J^a \leq Y$, for some $a \in A$. We may choose g to normalize $\langle t, t^g \rangle$, hence $g \in N(C) \cap N(Y)$. The only difficulty is when g induces on $Y/O_2(Y)$ an element in the coset of a graph automorphism of $F_4(2)$. However, checking the action of fundamental reflections of $F_4(2)$ on $O_2(Y)$ we see that Y admits no such automorphism. This completes the proof of (4.5).

We need analogues of (4.4) and (4.5) when $A \cong O^+(8, q)'$ or $A \cong O^{\pm}(n, q)'$ and (3.6)(ii) holds.

(4.6) *Assume* $\tilde{A} \cong O^+(8, q)'$ *or* $\tilde{A} \cong O^{\pm}(n, q)'$ *and* (3.6)(i) *does not hold. Let* $Y = O(C_G(X))$ *or* $O(C_G(X))X$, *respectively, and let bars denote images in* $C_G(X)/Y$. *Then* $\bar{D}$ *is standard in* $\overline{C_G(X)}$ *and* $\bar{R} \in Syl_2(C(\bar{D}) \cap \overline{C_G(X)})$.

Proof. For $\tilde{A} \cong O^+(8, q)'$ this follows as in (4.4). Suppose

$$\tilde{A} \cong O^{\pm}(n, q)', \quad \tilde{A} \not\cong O^+(8, q)',$$

and (3.6)(i) fails to hold. This is also similar to (4.4), although there is a difference. Namely, in trying to show

$$\bar{R} \in Syl_2(C(\bar{D}) \cap \overline{C_G(X)})$$

we assume otherwise and obtain an element $u \in N(R\langle x \rangle) - R\langle x \rangle$ such that $u \in C(X) \cap C(\bar{D})$. So it is possible that $t^u = tx$. If this happens consider $N_G(A^u)$. As $n \geq 8$, $D = D^{(\infty)} \leq A^u$ and $A^u\langle t \rangle \cong O^{\pm}(n, q)$ with t inducing a transvection on A^u. If $1 \neq d \in D$ is in a root group of $D \cong Sp(n-2, q)$ for a short root, then $x \sim dx$ by an element of A, so $t^u \sim t^u d$. By symmetry, (3.6) (i) does hold, contrary to our assumption.

(4.7) *With hypotheses and notation as in* (4.6), $\bar{D} \ntrianglelefteq \overline{C_G(X)}$.

Proof. If $\tilde{A} \cong O^+(8, q)'$ (where $g > 2$), then the arguments of (4.5) apply. So assume $\tilde{A} \not\cong O^+(8, q)'$. Then $X = \langle x \rangle$ with $A\langle x \rangle \cong O^{\pm}(n, q)$ and x a transvection. Assuming the result false, let $C = C_{A^g}(X)$. Then $[C, t] \leq C \cap Y$, and so either $t \sim tx$ or

$$[C, t] \leq C \cap O(C_G(X)) \leq O(C) = 1$$

(see (3.1)). In the first case consider $D \times \langle tx \rangle \leq C(tx)$ and argue that (3.6)(i) holds, which is not the case. Therefore, $[C, t] = 1$ and $C_{A^g}(x) \leq C_{A^g}(t)$.

The results of §10 of [5] imply that either $x \sim t \pmod{C(A^g)}$ or x corresponds to an involution of type b_l with $l = n/2$ and t corresponds to an involution of type a_{l-1}. As mentioned in the proof of (8.12) of [5], each involution in $O^{\pm}(n, q)$ centralizes a transvection except for the one case of a_l

involutions in $O^+(2l, q)$. Consequently, t^g cannot project to an $a_{n/2}$ involution in A.

First suppose that t^g can be chosen in $C(X) \cap AC(A)$. Let

$$\Delta = t^G \cap C(t) \cap C(t^g).$$

Since (3.6)(i) is false we necessarily have $\Delta \subseteq \langle x^g \rangle A^g K^g$ (see §19 of [5]). Assume also that $x \sim t \pmod{C(A^g)}$, so that $C_{A^g}(x) = C_{A^g}(t)$. From this we conclude that $\Delta \subseteq C_G(x)$. Now view this in $N(A)$ and apply (10.6)–(10.8) of [5] to conclude that t^g induces an involution of type b_l on A. This contradicts the choice of t^g. Therefore, the earlier remarks give $t \in A^g C(A^g)$ projecting to an a_{l-1} involution, x corresponding to a b_l involution, and $l = n/2$. Since $C_{A^g}(t) \leq N(A)$, we easily see that t^g must project to an a_{l-1} involution in A.

Let W be the natural module for A and let y be the projection of t^g to A. Then $W_0 = [W, y]$ is a singular $(l-1)$-space and

$$Q_1 = Z(O_2(C_A(t^g)\langle x \rangle)) = C(W_0) \cap C(W/W_0).$$

Let $Q = Q_1 \times \langle t \rangle$. One checks that $Q \cap t^G$ consists of t together with involutions projecting to involutions in A of type a_{l-1}. Now consider $N = \langle N_A(Q), N_{A^g}(Q) \rangle$ acting on $t^G \cap Q = \Omega$. At this point we argue as in Case 1 of the proof of (3.5), using the permutation group N^Ω. The only difference is that in the case where N^Ω is rank 3 on Ω we first choose $t^g \in A$ and then notice that there is an element $t^h \in Q \cap A$ with $t^h t^g$ of type a_2. This leads to a contradiction as in Case 1 of (3.5).

Now assume that it is not possible to choose $t^g \in AC(A)$ and $t^g \in C(X)$. So $A\langle x \rangle \cong O^+(n, q)$ and each involution $t \neq t^h \in AC(A)$ projects to an involution in A of type a_l, where $l = n/2$. Choose $t \neq t^h \in AC(A)$ (possible by (3.2)(i)) and let W be the natural module for A. If y is the projection of t^h to A, we have $[W, y] = W_0$, a singular l-space. As above let $Q_1 = C_A(W_0) \cap C_A(W/W_0)$ and $Q = Q_1 \times \langle t \rangle$. Then Q is elementary abelian. As $C_A(t^h) \leq N(A^h) \cap C(t)$, it follows that t projects to an involution of type a_l in A^h. This time set $N = \langle N_A(Q), N_{A^h}(Q) \rangle$ and obtain a contradiction by considering N^Ω, where $\Omega = t^G \cap Q$. This completes the proof of (4.7).

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