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TOPOLOGICAL SPACES IN WHICH BLUMBERG'S THEOREM HOLDS II

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1. This note consists of some "odds and ends" involving Blumberg's theorem. Section 2 contains an example of a Baire space with a point countable base for which Blumberg's theorem does not hold; Section 3 deals with Blumberg's theorem for linearly ordered spaces; Section 4 is concerned with a strong form of Blumberg's theorem.

2. If X denotes a set, then $\mathbf{P}(X)$ denotes the collection of all subsets of X. If $A \subset X$ and $\mathcal{F} \subset \mathbf{P}(X)$, then $\mathcal{F} \cap A$ denotes $\{F \cap A; F \in \mathcal{F}\}$ and $\mathcal{F}^*$ denotes $\mathcal{F} \sim \{\emptyset\}$. If $(X, \mathcal{T})$ is a topological space, a subset $\mathcal{P}$ of $\mathcal{T}^*$ is called a pseudo-base for $\mathcal{T}$ if every element of $\mathcal{T}^*$ contains an element of $\mathcal{P}$. A collection of sets of called σ-disjoint if it is the union of a countable set of disjoint collections. The set of real numbers is denoted by R; the set of positive integers by N.

2.1. THEOREM. *If $(X, \mathcal{T})$ is a Baire space that has either a σ-point finite or σ-locally countable pseudo-base, then the following statement, known as Blumberg's theorem, holds for X.*

2.2. If φ is a real valued function defined on X, then there is a dense subset D of X such that $\varphi \mid D$ is continuous.

Proof. This follows from [15, Proposition 1.7] and the following statements.

2.3. If $\mathcal{T}$ has a σ-locally countable pseudo-base, then it has a σ-disjoint pseudo-base.

Proof. If $\mathcal{C}$ is a locally countable subset of $\mathcal{T}^*$ and $\mathcal{U}$ is a maximal disjoint subcollection of $\mathcal{T}^*$ such that $(\mathcal{C} \cap U)^*$ is countable for every U in $\mathcal{U}$, then $\mathcal{C}' = \cup\{(\mathcal{C} \cap U)^*: U \in \mathcal{U}\}$ is a σ-disjoint subcollection of $\mathcal{T}^*$ such that every element of $\mathcal{C}$ contains an element of $\mathcal{C}'$. ■

Remark. In [5, Theorem 2.1] it is shown that $\mathcal{T}$ has a σ-disjoint pseudobase whenever it has a σ-locally countable base.

2.4. PROPOSITION [6, Theorem 3.10]. *If $(X, \mathcal{T})$ is a Baire space and $\mathcal{C}$ is a point finite subset of $\mathcal{T}^*$, then there is a dense subset D of X such that $\mathcal{C}$ is locally finite at every point of D.*

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However, as the following example shows, 2.2 need not hold for a Baire space with a point countable base.

2.5. *Example.* Let F denote the set of all functions f such that there is an ordinal α, $1 \leq \alpha < \omega_1$, such that domain $f = \{\beta : 0 \leq \beta < \alpha\}$ and range $f \subset N$. For any f in F and n in N, let $U_n(f)$ denote the set of all g in F that are extensions of f for which, if domain g $\sim$ domain f has a first element α, then $g(\alpha) \geq n$. Then $\mathcal{B} = \{U_n(f) : f \in F, n \in N\}$ is a base for a topology $\mathcal{F}$ on F; it was shown in [1, pp. 414–415] that $(F, \mathcal{F})$ is a hereditarily paracompact Hausdorff space that has a point countable base.

The following statement is readily verified.

2.6. If $\mathcal{P} = \{U_1(f) : f \in F\}$, then $\mathcal{P}$ is a pseudo-base for $\mathcal{F}$ such that if $\mathcal{C}$ is a non-empty countable subset of $\mathcal{P}$ that has the finite intersection property, then $\bigcap \mathcal{C} \in \mathcal{P}$.

Statement 2.6 implies that F is a Baire space (actually that F is pseudo-complete [15, page 460]). The following statement follows from 2.6.

2.7. Suppose $(F, \mathcal{F})$ is a dense subspace of the regular space $(Y, \mathcal{U})$; then 2.2 fails to hold for $(Y, \mathcal{U})$ if and only if some element of $\mathcal{U}^*$ is the union of $\leq 2^{\aleph_0}$ nowhere dense sets.

Because F has cardinality $2^{\aleph_0}$ and $(F, \mathcal{F})$ has no isolated points, 2.7 implies that 2.2 does not hold for F. (Actually, 2.6 implies that F is strongly non-Blumberg in the sense of [7].) If $(F, \mathcal{F})$ is a dense subspace of the regular Lindelöf space $(Y, \mathcal{U})$, then a slight modification of the argument in [15, Proposition 1.11] shows that no element of $\mathcal{U}^*$ is the union of $\leq \aleph_1$ nowhere dense sets. Hence, if $2^{\aleph_0} = \aleph_1$, it follows from [15, Proposition 1.4] that 2.2 holds for Y. In this connection, it should be mentioned that 2.2 does not hold for any Souslin line [14].

Remarks. (1) It was shown in [11, Theorem 1] that F has no dense developable subspace; it follows easily from 2.6 that no dense subset of F has a quasi-G_δ-diagonal.

(2) Suppose $(X, \mathcal{T})$ is a hereditarily paracompact Hausdorff Baire space. If X has a dense G_δ subset that is metrizable, it is clear that 2.2 holds for X. Therefore, if (i) X has a quasi-G_δ-diagonal and (ii) there is a countably compact regular Hausdorff space Y such that X is a dense Borel subset of Y, then 2.2 holds for X. Must 2.2 hold if either (i) or (ii) fails to hold (both (i) and (ii) fail to hold form $(F, \mathcal{F})$)?

(3) Suppose X is a first countable Hausdorff space such that (a) 2.2 holds for X and (b) X is the union of $\leq 2^{\aleph_0}$ nowhere dense subsets. Must X have a dense metrizable subspace? If (b) is deleted from the first sentence, then the answer to the resulting question is "no", as the following example shows. If, in 2.5, ω_1 is replaced by the first ordinal of cardinality $2\mathfrak{c}$, then the resulting space is a completely regular, Hausdorff first countable space such that 2.2

holds for X and every metrizable subspace of X is nowhere dense in X.

(4) In $ZF+AC$, is there a compact Hausdorff space that is the union of $\leq 2^{\aleph_0}$ nowhere dense subsets for which 2.2 does not hold? This question might accurately be called the real "Blumberg problem".

Added in proof. Example (2.11) of H. R. Bennett's paper *A note on point countability in linearly ordered spaces* (Proc. Amer. Math. Soc., vol. 28 (1971) pp. 598–606) is a linearly ordered Baire space with a point countable base for which 2.2 does not hold.

3. In [14], it is shown that if X is a linearly ordered topological space (LOTS) such that 2.2 holds for X and X is the union of $\leq 2^{\aleph_0}$ nowhere dense sets, then X has a σ-disjoint pseudo-base. The following statement augments this.

3.1. THEOREM. *Suppose $(X, \mathcal{T})$ is a Baire LOTS with no isolated points. Then the following statements are equivalent.*

(1) *Blumberg's theorem holds for X and X is the union of $\leq 2^{\aleph_0}$ nowhere dense sets.*

(2) *X has a dense subset of the first category.*

(3) *X has a dense metrizable subspace.*

(4) *X has a σ-disjoint pseudo-base.*

Proof. (1) $\Rightarrow$ (2). Let f be a real valued function on X such that $f^{-1}(r)$ is nowhere dense for every r in R, and let D be a dense subset of X such that $f \mid D$ is continuous. Define, by induction, a sequence $(\mathcal{U}_n)_{n\in N}$ such that, for each n in N, $\mathcal{U}_n$ is a disjoint collection of closed intervals $[a, b]$ such that $(a, b)\neq\emptyset$ and $\operatorname{diam}(f[D\cap(a, b)])\leq n^{-1}$, $\cup\mathcal{U}_n$ is dense in X, and $\mathcal{U}_{n+1}$ refines $\{(a, b): [a, b]\in\mathcal{U}_n\}$. Then

$$\cup\{X\sim\cup\{(a, b); [a, b]\in\mathcal{U}_n\}: n\in N\}$$

is a dense subset of X that is of the first category.

(2) $\Rightarrow$ (3). Suppose $(F_n)_{n\in N}$ is a non-decreasing sequence of closed nowhere dense subsets of X such that $\cup\{F_n: n\in N\}$ is dense in X. Let $\mathcal{U}_n$ denote the collection of maximal open subintervals of $X\sim F_n$. Then $\cup\{\mathcal{U}_n: n\in N\}$ is a σ-disjoint pseudo-base for $\mathcal{T}$ and $Y=\cap\{\cup\mathcal{U}_n: n\in N\}$ is a dense subset such that $\mathcal{T}$ satisfies the first axiom of countability at every point of Y. (If X has no gaps, then Y is metrizable.) By [16, Theorem 2.6] Y has a dense metrizable subspace.

(3) $\Rightarrow$ (4). This follows from [15, Proposition 1.9(1)].

(4) $\Rightarrow$ (1). It follows from [15, Proposition 1.7] that 2.2 holds for X. And the argument used to prove [13, Theorem 3.6] shows that X is the union of $\leq 2^{\aleph_0}$ nowhere dense sets (see also [17]). ∎

Remarks. (i) If $(X, \mathcal{T})$ is any topological space, then there are U, V in $\mathcal{T}$ such that $U\cap V=\emptyset$, $U\cup V$ is dense in X, U is the union of $\leq 2^{\aleph_0}$ nowhere dense sets, and no non-empty open subset of V is the union of $\leq 2^{\aleph_0}$ nowhere dense sets. Then 2.2 holds for V [15, Proposition 1.4], so 2.2 holds for X if and only if it holds for U. Thus 3.1 is more general than it initially appears. Of course, whether or not U is non-empty may depend on what axioms for set theory are used.

(ii) In 3.1, the hypothesis that X is a LOTS can be replaced by the hypothesis that X is a subspace of a LOTS (a GO space, [9]).

(iii) If X is either (a) locally connected or (b) a finite product of LOTS, then (1) implies (2).

(iv) If $(X, \mathcal{T})$ is a Baire LOTS with no isolated points such that X has a dense metrizable subspace, it may fail to have a dense G_δ metrizable subspace, even if it is compact, separable, and first countable. For let $\mathcal{T}$ be the topology on R generated by $\{[a, b): a, b \in R, a < b\}$. Then [9, (2.9) and (7.2)] there is a compact LOTS $(Y, \mathcal{U})$ such that $(R, \mathcal{T})$ is a dense subset of $(Y, \mathcal{U})$. If $(Y, \mathcal{U})$ had a dense G_δ metrizable subspace, then so would $(R, \mathcal{T})$; but every metrizable subspace of $(R, \mathcal{T})$ is countable.

4. We consider briefly the following statement.

4.1. Blumberg's Theorem (Strong Form). *If $(Y, \mathcal{U})$ is a second countable space and $f: (X, \mathcal{T}) \to (Y, \mathcal{U})$, then there is a dense subset D of X such that $f \mid D$ is continuous.*

4.2. Theorem. *If $(X, \mathcal{T})$ is a Baire space with a σ-disjoint pseudobase, then 4.1 holds for X.*

This is clear because the proofs of the theorem in [3] and [15, Proposition 1.7] remain true if R is replaced by any second countable space.

4.3. Theorem. *For any topological space $(X, \mathcal{T})$, the following statements are equivalent.*

(1) *Statement 4.1.*

(2) *If $\mathcal{W}$ is any second countable topology on X, then there is a $\mathcal{T}$-dense subset D of X such that $\mathcal{W} \cap D \subset \mathcal{T} \cap D$.*

(3) *If $(Z, \mathcal{V})$ is a second countable space and $g: Z \to \mathbf{P}(X)$, then there is a dense subspace D of X such that $g[V] \cap D \in \mathcal{T} \cap D$ for all V in $\mathcal{V}$.*

Proof. (1) $\Rightarrow$ (2). Apply (1) to the identity mapping $i: (X, \mathcal{T}) \to (X, \mathcal{W})$.

(2) $\Rightarrow$ (3). Let $\mathcal{B}$ be a countable base for $\mathcal{V}$ and let $\mathcal{W}$ denote the topology generated by $\{X\} \cup \{g[B]: B \in \mathcal{B}\}$. By (2), There is a $\mathcal{T}$ dense subset D of X such that $\mathcal{W} \cap D \subset \mathcal{T} \cap D$; D is the required set.

(3) $\Rightarrow$ (1). If $(Y, \mathcal{U})$ is second countable and $f: X \to Y$, apply (3) to the function g defined by $g(y) = f^{-1}(y)$ for every y in Y. ∎

4.4. Corollary. *Suppose 4.1 holds for X. Suppose that for each n,*

$$f_n: (X, \mathcal{T}) \to (Y_n, \mathcal{U}_n) \quad \textit{and} \quad g_n: (Z_n, \mathcal{V}_n) \to (X, \mathcal{T}),$$

where $(Y_n, \mathcal{U}_n)$ and $(Z_n, \mathcal{V}_n)$ are second countable. Then there is a dense subset D of X such that for all n, $f_n \mid D$ is continuous and $g_n[V] \cap D \in \mathcal{T} \cap D$ for all V in $\mathcal{V}_n$.

Proof. For each n in N, let $\mathcal{B}_n$ be a countable base for $\mathcal{U}_n$ and $\mathcal{C}_n$ be a countable base for $\mathcal{V}_n$. Let $\mathcal{W}$ be the topology on X generated by

$$\bigcup\{f_n^{-1}[\mathcal{B}_n] \cup g_n[\mathcal{C}_n]: n \in N\},$$

and apply 4.3(2). ∎

In [12] is it proven that the conclusion of 4.4 holds for any proper first countable Baire space X. It is easy to see that every proper first countable space has a σ-disjoint pseudo-base (in fact, a T_1 proper first countable space is a Nagata space); hence 4.4 is a generalization of the result in [12].

We conclude with a question. If Blumberg's theorem holds for X, does 4.1 hold for X?

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