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A SINGULAR FREE BOUNDARY PROBLEM

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1. Introduction

In 1961 Chernoff [1] studied the problem of sequentially testing whether the drift of a Wiener process is positive or negative, given an a priori normal distribution, and showed that this problem can be reduced to a singular parabolic free boundary problem. A description of Chernoff's formulation and reduction of the problem can also be found in [7]. Briefly, one considers a Wiener–Levy stochastic process $\chi(\tau)$ and an associated process $\xi(\tau)$ with drift μ; i.e. $\xi(\tau)=\chi(\tau)+\mu\tau$ where μ is an unknown constant whose sign is to be determined.

μ is considered as a random variable with known a priori normal distribution. The problem then of observation and periodic testing to determine the sign of ξ and hypothesize the sign of μ in such a way as to minimize the expected cost of the operation becomes one of uniformly minimizing the Bayes risk $B(\xi, \tau)$. It is assumed that the cost of an incorrect decision is proportional to $|\mu|$ and that the cost of observation is constant per unit time. Chernoff then shows that B then satisfies the equation

$$\tfrac{1}{2}B_{\xi\xi}+\frac{\xi}{\tau}B_{\xi}+B_{\tau}+1=0$$

in the continuation region and certain boundary conditions as well. Then, defining a new function $u(x, t)$ in terms of the Bayes risk $B(\xi, \tau)$ and performing a change of variables Chernoff reduces the problem to the following singular parabolic free boundary problem: find a function $u(x, t)$ and a free boundary curve $x=s(t)$ such that

$$\text{(P)}\qquad \begin{aligned} &u_t-u_{xx}=-1/(2t^2) \quad \text{for} \quad 0<x<s(t), \qquad 0<t<T,\\ &u_x(0,t)=-\tfrac{1}{2} \quad \text{for} \quad 0<t<T,\\ &u(s(t),t)=u_x(s(t),t)=0 \quad \text{for} \quad 0<t<T,\\ &s(0)=0. \end{aligned}$$

It should be noted that the conditions on u_x are incompatible at the origin and that the equation is singular at $t=0$.

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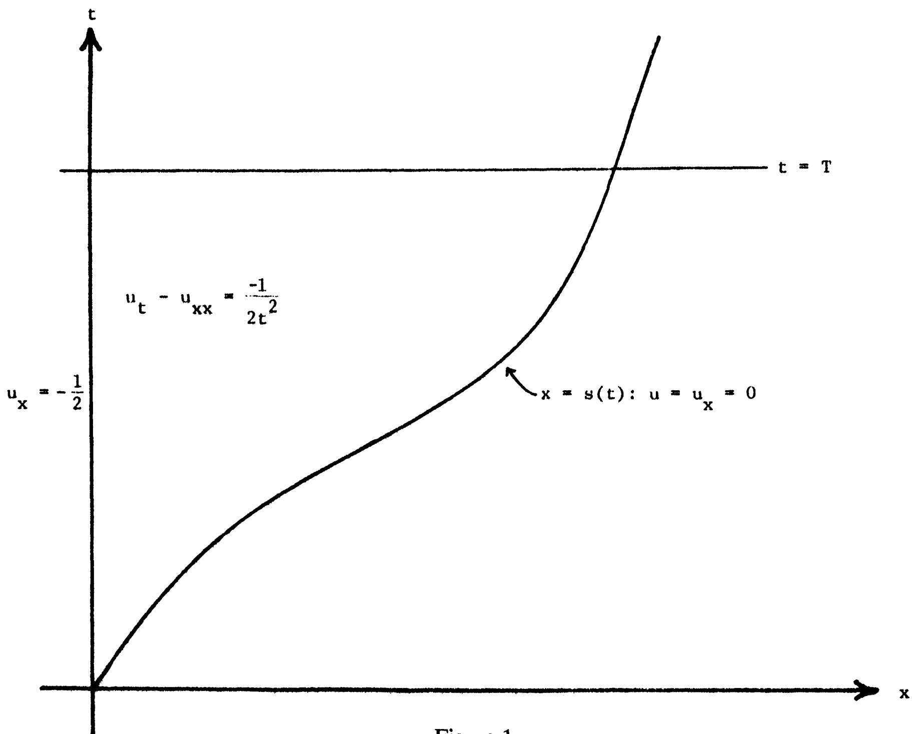

Figure 1.

There have been several studies directed at the numerical solution of this singular free boundary problem (see [6], [7] and the references cited there). However, in this paper the problem will be solved analytically by the method of penalty functions. In fact we will treat a more general class of problems where $u_t - u_{xx} = f(t)$ and $f(t)$ is negative and behaves like $-t^{-k}$ for some $k \geqslant 0$, and we will allow more general conditions on $u_x(0, t)$ as well.

In Section 2 we define the notion of a solution in the spirit of variational inequalities and prove uniqueness.

Then, in Section 3, we develop the a priori L^P and Hölder estimates that enable us to prove existence in Section 4.

To motivate the techniques used in this paper consider problem (P) above, and the following *nonrigorous* remarks. Clearly, by the maximum principle, we should expect that $-\frac{1}{2} \leq u_x \leq 0$ and therefore, since $u = 0$ along s, that $u > 0$ for $0 < x < s(t)$. Differentiating the function $u(s(t), t)$, which vanishes identically, we see that $u_t(s(t), t) = 0$. Since $u_{tx} = 0$ on $\{x = 0\}$, the maximum principle implies $u_t \geq 0$ for $0 < x < s(t)$. Next, to derive estimates of sup u, sup s consider the following simple argument which is a variant of one used in [2]: if $u(x_0, t_0) > 0$ let $Q = \{0 < x < s(t),\ 0 < t < t_0\}$ and define

$$w(x, t) = u(x, t) - \frac{1}{4t_0^2}(x - x_0)^2.$$

Since $w(x_0, t_0) > 0$ w must attain a positive maximum somewhere in $\bar{Q}$ and since $w_t - w_{xx} \leq 0$ in Q it must occur either on s or on $\{x = 0\}$. But $w \leq 0$ on s so the *positive* maximum must occur on $\{x = 0\}$, where, therefore, $w_x = \frac{1}{2}(-1 + x_0/(t_0)^2)$ must be nonpositive. Thus $x_0 \leq t_0^2$. But since $-\frac{1}{2} \leq u_x \leq 0$ it follows that $0 \leq u(x, t_0) \leq \frac{1}{2}t_0^2$ for $0 \leq x \leq s(t_0)$ so we expect that $u(x, t) \leq (\frac{1}{2})t^2$ and $s(t) \leq t^2$. Since the function $z = t^2 u$ satisfies $z_t - z_{xx} = 2tu - (\frac{1}{2} \in L^\infty$, $z = z_x = 0$ on s, $z_x = -(\frac{1}{2})t^2$ when $x = 0$, the L^p estimates of Solonnikov [9] imply that z_t and z_{xx} are in L^p for each $p > 1$. Thus u_t and u_{xx} are in L^p of regions bounded away from $t = 0$.

In Sections 3 and 4 we will make all of these remarks rigorous through the use of suitable penalty function approximations to the free boundary problem and we will prove existence. In Section 4 we also prove that the free boundary s is Holder continuous down to $t = 0$. We then prove a result about the initial growth of the free boundary when $f(t)$ behaves like $-t^{-k}$ and $k > \frac{1}{2}$. We prove that there exists a constant $\theta > 0$ such that, for each $\varepsilon > 0$, $s(t)$ initially grows faster than $(\theta - \varepsilon)t^k$ but slower than θt^k. For the special problem (P) this implies that $s(t)$ grows almost like θt^2, which agrees well with existing numerical results (see [6], [7]).

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2. Statement of the problem

Throughout the paper we let k and T denote arbitrary but fixed constants satisfying $k \geqslant 0$ and $T > 0$. The function α satisfies

$$t\alpha(t) \in C^{0,1}[0, T] \cap C^1(0, T], \tag{2.1}$$

$$t^k \alpha(t) \to 0 \quad \text{as} \quad t \searrow 0, \tag{2.2}$$

$$\alpha'(t) \leqslant 0 \quad \text{and} \quad \alpha(t) < 0 \quad \text{for} \quad t > 0. \tag{2.3}$$

The function f satisfies

$$f \in C^1(0, T], \tag{2.4}$$

$$-\infty < -c'' \leq t^k f(t) \leq -c' < 0 \quad \text{for} \quad 0 < t \leq T, \tag{2.5}$$

$$f'(t) \geq 0 \quad \text{and} \quad f(t) < 0 \quad \text{for} \quad t > 0. \tag{2.6}$$

Finally, we define

$$X = 1 + \alpha(T)/f(T). \tag{2.7}$$

Notice that we do not assume that $\alpha(t) \to 0$ as $t \searrow 0$ when $k > 0$.

We have already discussed the fact that decision theory gives rise to the following problem.

Problem A. Find a nonnegative, bounded, continuous function

$u(x, t)$: $R^1 \times [0, T] \to [0, \infty)$ and a function $s(t) \in C[0, T]$ such that $s(0) = 0$ and $s(t) > 0$ for $t > 0$ such that:

(i) On the set $\Omega = \{(x, t) | 0 < x < s(t),\ t > 0\} u(x, t) > 0$ and $u(x, t)$ is a classical solution of the equation $u_t - u_{xx} = f(t)$.

(ii) u_x is continuous up to the free boundary $x = s(t)$ and up to the line $\{x = 0\}$ and $u(s(t), t) = u_x(s(t), t) = 0$ and $u_x(0, t) = \alpha(t)$ for $t > 0$.

We will solve Problem A indirectly by formulating a Problem B, solving this problem and showing that the solution also solves Problem A. The advantage of this approach is that Problem B will be stated without explicit mention of a free boundary $x = s(t)$, but part of the boundary of the set $\{u > 0\}$ will in fact be the free boundary. Also, it is relatively easy to prove uniqueness over a broad class for Problem B.

DEFINITION 2.1. We denote by $\mathcal{K}$ the set of functions $u(x, t)$ defined on $[0, \infty) \times [0, T]$ which satisfy the following conditions:

(i) $u(x, t) \in C([0, \infty) \times [0, T]) \cap L^\infty([0, \infty) \times [0, T))$.

(ii) $u(x, t) \geq 0$ on $[0, \infty) \times [0, T]$.

(iii) There exists a constant $X_u > 0$, depending on u, such that $u(x, t) \equiv 0$ if $x \geq X_u$ and $t \in [0, T]$.

(iv) For each $\tau \in (0, T)$, $u_x \in C([0, \infty) \times [\tau, T])$.

(v) u possesses a distributional (weak) derivative u_t in $L^1((0, \infty) \times (\tau, T))$ for each $\tau \in (0, T)$.

(vi) $u(x, 0) \equiv 0$ for $x \in [0, \infty)$.

Although condition (iii) implies that $\mathcal{K}$ is not closed this causes no problems since we will actually prove the existence of a solution which vanishes for $x \geq X$, where X is defined by (2.7). We would, of course, like to prove uniqueness over as large a class $\mathcal{K}$ as possible. In fact, it will become apparent that we still have existence and uniqueness if we broaden $\mathcal{K}$ so that $X_u = \infty$ in some appropriate sense and if the derivative u_x in (iv) is a weak derivative. Our formulation of conditions (iii) and (iv) is therefore a compromise in the interest of simplicity. We now define Problem B.

Problem B. Find a function $u \in \mathcal{K}$ such that the following integral inequality holds for each $0 < \tau_1 < \tau_2 \leq T$ and $v \in \mathcal{K}$:

$$(2.8) \qquad \int_{\tau_1}^{\tau_2} \int_0^X u_t(v-u) + u_x(v-u)_x \, dx \, dt + \int_{\tau_1}^{\tau_2} \alpha(t)(v-u)(0, t) \, dt \geq \int_{\tau_1}^{\tau_2} \int_0^X f(t)(v-u) \, dx \, dt$$

where $X = \min(X_u, X_v)$ (see (iii) of Definition 2.1).

Notice that, formally, a solution to Problem A is a solution to Problem B.

THEOREM 2.1 (Uniqueness). *There exists at most one solution to Problem* B.

Proof. Suppose that u and w are solutions to Problem B and let $X=\min(X_u, X_w)$. Without loss of generality we may assume that $X=X_w$. If we write (2.8) with $v=w$ and then with $u=w$ and $v=u$ and add the resulting inequalities we get

$$\int_{\tau_1}^{\tau_2}\int_0^X zz_t\,dx\,dt \le -\int_{\tau_1}^{\tau_2}\int_0^X z_x^2\,dx\,dt \le 0$$

where $z=w-u$. It follows that

$$\int_0^X (z(x,\tau_2))^2\,dx \le \int_0^X (z(x,\tau_1))^2\,dx$$

Letting $\tau_1 \searrow 0$ this implies that $\int_0^X (z(x,\tau_2))^2\,dx \le 0$ so that $u=w$ on $[0,X]\times[0,T]$. But $w\equiv 0$ on $[X,\infty)\times[0,T]$ so it suffices to prove that $u\equiv 0$ on $[X,X_u]\times[0,T]$. To show this we define two functions $v_1(x,t)$ and $v_2(x,t)$ as follows:

$$v_1(x,t)=\begin{cases} u(x,t) & \text{if } 0\le x\le X,\\ 2u(x,t) & \text{if } X\le x\le X_u,\end{cases}$$

$$v_2(x,t)=\begin{cases} u(x,t) & \text{if } 0\le x\le X,\\ \tfrac{1}{2}u(x,t) & \text{if } X\le x\le X_u.\end{cases}$$

Since $u=u_x=0$ on $x=X$ (since $u=w$ there) and since $u\in\mathcal{K}$ it is not difficult to verify that v_1 and v_2 are in $\mathcal{K}$ with $X_{v_1}=X_{v_2}=X_u$. If we write (2.8) with $v=v_1$ and $v=v_2$ we get

$$\int_{\tau_1}^{\tau_2}\int_X^{X_u} u_t u + u_x^2\,dx\,dt \ge \int_{\tau_1}^{\tau_2}\int_X^{X_u} f(t)u\,dx\,dt$$

and

$$\int_{\tau_1}^{\tau_2}\int_X^{X_u} u_t(-\tfrac{1}{2}u)+u_x(-\tfrac{1}{2}u_x)\,dx\,dt \ge \int_{\tau_1}^{\tau_2}\int_X^{X_u} f(t)(-\tfrac{1}{2}u)\,dx\,dt$$

which together imply that

$$\int_{\tau_1}^{\tau_2}\int_X^{X_u} u_t u + u_x^2\,dx\,dt = \int_{\tau_1}^{\tau_2}\int_X^{X_u} f(t)u(x,t)\,dx\,dt \le 0$$

since $u\ge 0$ and $f\le 0$. Then, as before, we deduce that $u\equiv 0$ on $[X, x_u]\times[0,T]$.

Once a solution to Problem B has been shown to exist, a solution to Problem A will be derived by setting $s(t)=\sup\{x \mid u(x,t)>0\}$.

In the next section we will establish estimates that will later be used to prove the existence of a solution to Problem B.

3. Estimates

Recall the definitions of k, T, and X. Given any $\varepsilon > 0$ we define the Problem $C(\varepsilon)$ as follows.

Problem $C(\varepsilon)$. Find a function $u^\varepsilon(x, t) \in C_{2+\alpha}(\bar{R})$ where $R = (0, X) \times (0, T)$ which satisfies:

$$u_t^\varepsilon - u_{xx}^\varepsilon + \beta^\varepsilon(u^\varepsilon) = f^\varepsilon(t) \quad \text{in } R, \tag{3.1}$$

$$u^\varepsilon(x, 0) \equiv 0 \quad \text{for} \quad 0 \le x \le X, \tag{3.2}$$

$$\frac{\partial}{\partial x} u^\varepsilon(0, t) = \zeta^\varepsilon(t) \quad \text{for} \quad 0 \le t \le T, \tag{3.3}$$

$$\frac{\partial}{\partial x} u^\varepsilon(X, t) \equiv 0 \quad \text{for} \quad 0 \le t \le T. \tag{3.4}$$

The functions β^ε, f^ε and ζ^ε are smooth functions that satisfy the conditions listed below:[2]

$$\beta^\varepsilon(t) \equiv 0, \qquad f^\varepsilon(t) \equiv f(t), \qquad \zeta^\varepsilon(t) \equiv \alpha(t) \quad \text{if} \quad t \ge \varepsilon,$$

$$\frac{d}{dt} \beta^\varepsilon(t) \ge 0, \qquad \frac{d}{dt} f^\varepsilon(t) \ge 0, \qquad \frac{d}{dt} \zeta^\varepsilon(t) \le 0 \quad \text{for all } t,$$

$$\zeta^\varepsilon(0) = 0, \qquad -\infty < \beta^\varepsilon(0) = f^\varepsilon(0),$$

$$-c'' \le t^k f^\varepsilon(t) < 0 \quad \text{for} \quad 0 \le t \quad \text{(see (2.5))}.$$

For simplicity we will suppress the superscript ε in this section. The existence of a solution to Problem $C(\varepsilon)$ follows, for example, from Theorem 7.4, Chapter V of [5].

LEMMA 3.1. *If u^ε is a solution to Problem $C(\varepsilon)$ then the following inequalities hold on R:*

$$\frac{\partial}{\partial t} u^\varepsilon(x, t) \ge 0, \tag{3.5}$$

$$\alpha(T) \le \frac{\partial}{\partial x} u^\varepsilon(x, t) \le 0, \tag{3.6}$$

$$0 \le u^\varepsilon(x, t). \tag{3.7}$$

Also, if $0 < \varepsilon < t \le T$ then

$$0 \le u^\varepsilon(x, t) \le \varepsilon + A\alpha^2(t) t^k \quad \textit{for} \quad 0 \le x \le X \tag{3.8}$$

[2] For the duration of the paper we shall assume, for simplicity of expositon, that the functions α and f are in C^∞. Otherwise we would simply define the functions β^ε, ζ^ε and f^ε differently and proceed along the same lines.

where $A = 1/(2c')$ *(see* (2.5) *for the definition of* c'*). In particular, there exists a constant* $M > 0$ *not depending on* ε, *such that*

(3.9) $$0 \le u^\varepsilon(x, t) \le M \quad \textit{for} \quad (x, t) \in R \quad \textit{for} \quad 0 < \varepsilon < T$$

Proof. To prove (3.5) we differentiate equation (3.1) with respect to t and set $\partial u^\varepsilon/\partial t = v$ to obtain

$$v_t - v_{xx} + \beta'(u)v = \frac{\partial}{\partial t} f^\varepsilon \ge 0 \quad \text{in} \quad R.$$

Since $v(x, 0) = f^\varepsilon(0) - \beta^\varepsilon(0) \ge 0$, $v_x(0, t) = d\zeta^\varepsilon(t)/dt \le 0$ and $v_x(X, t) = 0$ it follows from the maximum principle that $v(x, t) \ge 0$ in R. Inequality (3.6) follows from the maximum principle, applied to the equation which results from differentiating equation (3.1) with respect to x. Then (3.7) follows from (3.5) and (3.2).

To prove (3.8), fix $0 < \tau \le T$ and let $0 < \varepsilon < \tau$. Then let

(3.10) $$X_0 \equiv \frac{\alpha(\tau)}{f(\tau)} \le \frac{\alpha(T)}{f(T)} < X.$$

Recall that $f^\varepsilon(\tau) = f(\tau)$ and $\zeta^\varepsilon(\tau) = \alpha(T)$ since $\varepsilon < \tau$. We define functions v and z by

(3.11) $$v(x, t) = \varepsilon + c(x - X_0)^2 \quad \text{where} \quad c = -f(\tau)/2,$$

(3.12) $$z(x, t) = v(x, t) - u^\varepsilon(x, t).$$

Let $S = (0, X_0) \times (0, \tau)$ and let $\Omega \equiv \{(x, t) \mid u(x, t) > \varepsilon\} \cap S$. Then $z_t - z_{xx} \ge 0$ on Ω, since $\beta(x) \equiv 0$ for $x \ge \varepsilon$ implies $z_t - z_{xx} = -2c - f^\varepsilon(t) \ge -2c - f^\varepsilon(\tau) = -2c - f(\tau) = 0$. For $0 < t < \tau$ we also have

$$z_x(0, t) \le -2cX_0 - \alpha(\tau) = 0 \quad \text{and} \quad z_x(X_0, t) = -u_x(X_0, t) \ge 0.$$

Since $z \ge 0$ at boundary points of Ω in S, where $u^\varepsilon = \varepsilon$, we can use the maximum principle to conclude that $z \ge 0$ in Ω. Thus $u \le \varepsilon + c(x - X_0)^2$ in Ω and since $u \le \varepsilon$ on $S \setminus \Omega$ it follows that

(3.13) $$u(x, t) \le \varepsilon + cX_0^2$$

holds on S. But since $u_x \le 0$ (see (3.6)), (3.13) holds for $0 < x < X$, $0 < t < \tau$. Therefore, using (2.5) we see that

$$u(x, t) \le \varepsilon + cX_0^2 = \varepsilon - \frac{1}{2} \frac{(\alpha(\tau))^2}{f(\tau)} \le \varepsilon + \frac{\alpha^2(\tau)}{2c'} \tau^k$$

so that $0 < \varepsilon < \tau$ implies

(3.14) $$u^\varepsilon(x, t) \le \varepsilon + (\alpha^2(\tau)/(2c'))\tau^k \quad \text{on} \quad (0, X) \times (0, \tau).$$

This proves (3.8), and also (3.9). ■

Lemma 3.1 facilitates the proof of the next lemma. From now on we will always assume that $\varepsilon < T$.

LEMMA 3.2. *If u^ε is a solution to Problem* C(ε) *then, for any integer $p>1$,*

$$\int_0^T \int_0^X (t^{k+1}\beta^\varepsilon(u^\varepsilon))^p \, dx\, dt \le C \tag{3.15}$$

where C depends on p but not on ε.

Proof. It suffices to consider p to be an even integer. Let $s=k+1$ and let $\alpha>0$ be an arbitrary constant and define $\xi(t)=(T-t)^\alpha$. Then

$$I \equiv \int_0^T \int_0^X t^{sp}\xi(t)\beta^p(u)\, dx\, dt = \int_0^T \int_0^X t^{sp}\xi(t)\beta^{p-1}(u)[f^\varepsilon(t)-u_t+u_{xx}]\, dx\, dt.$$

By expanding we get three integrals which we denote I_1, I_2, and I_3. Then

$$\begin{aligned} I_2 &\equiv -\int_0^T \int_0^X t^{sp}\xi(y)\beta^{p-1}(u)u_t\, dx\, dt \\ &= \int_0^T \int_0^X u\frac{\partial}{\partial t}\{t^{sp}\xi(t)\beta^{p-1}(u)\}\, dx\, dt \\ &= \int_0^T \int_0^X u\{spt^{sp-1}\xi(t)\beta^{p-1}(u)+t^{sp}\frac{\partial}{\partial t}(\xi(t)\beta^{p-1}(u))\}\, dx\, dt \\ &\le \int_0^T \int_0^X ut^{sp}\frac{\partial}{\partial t}(\xi(t)\beta^{p-1}(u))\, dx\, dt \end{aligned}$$

(because $u\ge 0$ and $\beta(u)\le 0$ and p is even)

$$\le M\int_0^T \int_0^X t^{sp}p\frac{\partial}{\partial t}(\xi(t)\beta^{p-1}(u))\, dx\, dt$$

(because $u\le M$ (see (3.9)) and all factors in the integrand are nonnegative)

$$= -M\int_0^T \int_0^X spt^{sp-1}\xi(t)\beta^{p-1}(u)\, dx\, dt.$$

If we apply Young's inequality $a^{p-1}b\le\eta((p-1)/p)a^p+b^p/(p\eta^{p-1})$ we see that

$$\begin{aligned} I_2 &\le spMT^{(s-1)}\int_0^T \int_0^X \xi(t)^{1/p}(t^s\xi(t)^{1/p}|\beta(u)|)^{p-1}\, dx\, dt \\ &\le spMT^k\int_0^T \int_0^X \eta((p-1)/p)t^{sp}\xi(t)(\beta(u))^p+(1/p)\eta^{1-p}\xi(t)\, dx\, dt \\ &= \eta s(p-1)MT^k\int_0^T \int_0^X t^{sp}\xi(t)(\beta(u))^p\, dx\, dt + s\eta^{1-p}MT^K\int_0^T \int_0^X \xi(t)\, dt\, dx. \end{aligned}$$

Thus, for any $\eta>0$,

$$I_2\le \eta s(p-1)MT^kI+s\eta^{1-p}MT^{\alpha+s}X/(\alpha+1). \tag{3.16}$$

Proceeding, we see that

$$I_3=\int_0^T\int_0^X t^{sp}\xi(t)\beta^{p-1}(u)u_{xx}\,dx\,dt$$

$$=\int_0^T\left\{t^{sp}\xi(t)\beta^{p-1}(u)u_x\Big|_{x=0}^{X}-\int_0^X t^{sp}\xi(t)(p-1)\beta^{p-2}(u)\beta'(u)u_x^2\,dx\right\}dt.$$

Then

$$I_3\le 0. \tag{3.17}$$

Continuing, we get

$$I_1=\int_0^T\int_0^X t^{sp}\xi(t)\beta^{p-1}(u)f^{\varepsilon}(t)\,dx\,dt$$

$$\le\int_0^T\int_0^X t^{sp}\xi(t)\,|\beta(u)|^{p-1}\,(t^{-k}c'')\,dx\,dt$$

$$=Tc''\int_0^T\int_0^X (t^{s}\xi(t)^{1/p}\,|\beta(u)|)^{p-1}\xi^{1/p}\,dx\,dt$$

$$\le T\eta c''((p-1)/p)I+\eta^{1-p}c''XT^{\alpha+2}/((\alpha+1)(p)).$$

Using this inequality, together with (3.16), (3.17) and the fact that $I=I_1+I_2+I_3$, we get

$$I\le\eta\gamma_1 I+\eta^{1-p}\gamma_2T^{\alpha}/(\alpha+1)$$

where γ_1, γ_2 depend on k, p, M, T, and c'' but not on ε. Letting $\eta=1/(2\gamma_1)$ we get

$$\int_0^T\int_0^X t^{sp}\beta^p(u)(\xi(t)/T^{\alpha})\,dx\,dt\le 2\gamma_2\eta^{1-p}/(\alpha+1).$$

Using the Lebesgue Bounded Convergence Theorem to let $\alpha\searrow 0$ we obtain

$$\int_0^T\int_0^X t^{sp}\beta^p(u)\,dx\,dt\le 2\eta^{1-p}\gamma_2. \quad\blacksquare \tag{3.18}$$

LEMMA 3.3. *If u is a solution to Problem* C(ε) *then, for each* $1\le p<\infty$,

$$\|t^{k+1}u_t\|_{L^p(R)},\qquad \|t^{k+1}u_{xx}\|_{L^p(R)}\le C \tag{3.19}$$

where C does not depend on ε.

Proof. Consider the function $z(x,t)=t^su(x,t)$ where $s=k+1$. According to equations (3.1)–(3.4) we have

$$z_t-z_{xx}=t^sf(t)-t^s\beta(u)+st^ku\quad\text{in } R, \tag{3.20}$$

$$z(x,0)=0\quad\text{for}\quad 0\le x\le X, \tag{3.21}$$

$$z_x(0,t)=t^s\zeta^{\varepsilon}(t)\quad\text{for}\quad 0\le t\le T, \tag{3.22}$$

$$z_x(X,t)=0\quad\text{for}\quad 0\le t\le T. \tag{3.23}$$

Theorem 17, p. 122 of Solonnikov [9], and Lemmas 3.1 and 3.2 imply that

(3.24) $$\|z_t\|_{L^p(R)}, \quad \|z_{xx}\|_{L^p(R)} \leq C(p)$$

where $C(p)$ does not depend on ε. This proves the result. ■

LEMMA 3.4. *For any monotone sequence $\{\varepsilon_n\}$ converging to zero, if u^n denotes the solution to Problem* C(ε^n), *then there exists a subsequence, which we again denote $\{\varepsilon^n\}$, and a function $u(x, t)$ such that $u(x, t)$ satisfies:*

(3.28) $$u \in L^\infty(\bar{R}) \cap C(\bar{R}) \quad \text{and} \quad u(x, t) \leq A\alpha^2(t)t^k$$

in $\bar{R}$ where $A = 1/(2c')$ and

(3.29) $$u(x, t) \geq 0 \text{ in } \bar{R} \quad \text{and} \quad u(x, 0) = 0 \quad \text{for} \quad x \in [0, X];$$

(3.30) $$u_x \in C_\alpha(R_\tau) \cap L^\infty(R)$$

for each $\tau \in (0, T)$, where $\alpha \in (0, 1)$ depends on τ and C_α is the space of functions which are Hölder continuous with respect to x (exponent α) and t (exponent $\alpha/2$), and $R_\tau = (0, X) \times (\tau, T)$;

(3.31) $$u_x(0, t) = \alpha(t), \quad u_x(X, t) = 0 \quad \text{for} \quad t \in (0, T);$$

(3.32) $$\alpha(T) \leq u_x(x, t) \leq 0 \quad \text{in} \quad R;$$

for some $\beta \in (0, 1)$ and $C > 0$,

(3.33) $$|u(x, \hat{t}) - u(x, t)| \leq C\,|\hat{t} - t|^\beta$$

for all $(x, \hat{t})$, (x, t) in $(0, X) \times (\tau, T)$ where C and β depend on τ; and u possesses weak derivatives

(3.34) $$u_t, u_{xx} \in L^p((0, X) \times (\tau, T))$$

for each $\tau \in (0, T]$, $p > 1$; and, for each $\tau > 0$,

(3.35) $$u^n \to u \quad \text{uniformly in } R,$$

(3.36) $$u^n_x \to u_x \quad \text{uniformly in } [0, X] \times [\tau, T],$$

(3.37) $$u^n_t \to u_t \quad \text{weakly in } L^p((0, X) \times (\tau, T)),$$

(3.38) $$u^n_{xx} \to u_{xx} \quad \text{weakly in } L^p((0, X) \times (\tau, T)).$$

Proof. Let τ be an arbitrary number in $(0, T)$ and define $S = (0, X) \times (\tau, T)$. By Lemma 3.3 of [5] and Lemmas 3.1 and 3.3 we see that

$$\begin{aligned} \sup |u^n| + \sup |u^n_x| + \langle u^n \rangle_{t,S}^{1-\varepsilon/2} + \langle u^n_x \rangle_{x,S}^{1-\varepsilon} + \langle u^n_x \rangle_{t,S}^{(1-\varepsilon)/2} \\ \leq C_1(\|u^n\|_{L_p(S)} + \|u^n_t\|_{L_p(S)} + \|u^n_{xx}\|_{L_p(S)} \\ \leq C_2 \end{aligned}$$

where C_1 and C_2 depend on p and τ but not on n, and $\varepsilon = 3/p$. Here

$$\langle u \rangle_{t,S}^\theta = \sup |u(x, \hat{t}) - u(x, t)|/|\hat{t} - t|^\theta; \qquad \langle u \rangle_{x,S}^\theta = \sup |u(\hat{x}, t) - u(x, t)|/|\hat{x} - x|^\theta$$

where the sup is taken over $(x, \hat{t})$, $(\hat{x}, t)$ and (x, t) in S. By Ascoli's lemma it is clear that some subsequence of $\{u^n\}$ converges uniformly together with its derivatives $\{u_x^n\}$ on $(0, X)\times(\tau, T)$ for each $\tau\in(0, T)$ to a function $u(x, t)$ satisfying (3.29)–(3.33). All of the other claims, except (3.28), (3.29), and (3.35) are also clear. That $u\in L^\infty(\bar{R})$ follows from (3.9), and that $u\in C(\bar{R})$ and that (3.35) holds are consequences of (3.8) and (3.5). In fact, $u(x, t)\leq (1/(2c'))\alpha^2(t)t^k$ for $(x, t)\in\bar{R}$ follows from (3.8) and proves that u is bounded and that $u(x, 0)\equiv 0$.

4. Existence

We are now in a position to prove the existence of a solution to Problem B.

THEOREM 4.1. *Suppose that* (2.1)–(2.7) *all hold. Then there exists a solution $u(x, t)$ to Problem* B. *Furthermore, $u(x, \cdot)$ is an increasing function for each $x\in[0, X]$ and $u(\cdot, t)$ is a decreasing function for each $t\in[0, T]$.*

Proof. Let $\{\varepsilon^n\}$ be a sequence such that $0<\varepsilon_{n+1}<\varepsilon_n<T$ and such that the solutions u^n to Problem $C(\varepsilon^n)$ converge to a function u as described in Lemma 3.4. Let $0<\tau_1<\tau_2\leq T$, $v\in\mathcal{K}$, and $\delta>0$ be arbitrary and let $w(x, t)=v(x, t)+\delta$. If we write (3.1) for u^n, multiply both members by $(w-u^n)$ and integrate by parts over $(0, X)\times(\tau_1, \tau_2)$ where $Y=\min(X_v, X)$ we find that

$$(4.1)\quad \int_{\tau_1}^{\tau_2}\int_0^Y u_t^n(w-n^n)\,dx\,dt-\int_{\tau_1}^{\tau_2} u_x^n(w-u^n)(Y, t)\,dt$$
$$+\int_{\tau_2}^{\tau_2}\alpha(t)(w-u^n)(0, t)\,dt+\int_{\tau_1}^{\tau_2}\int_0^Y u_x^n(w-u^n)_x\,dx\,dt$$
$$+\int_0^Y\int_{\tau_1}^{\tau_2}\beta^n(w)(w-u^n)\,dx\,dt$$
$$-\int_0^Y\int_{\tau_1}^{\tau_2}(\beta^n(w)-\beta^n(u^n))(w-u^n)\,dx\,dt$$
$$=\int_0^Y\int_{\tau_1}^{\tau_2} f(t)(w-u^n)\,dx\,dt$$

if n is sufficiently large (so that $\zeta_n(t)\equiv\alpha(t)$, and $f_n(t)\equiv f(t)$ for $t\in(\tau_1, \tau_2)$). Let us label these integrals consecutively so that (4.1) reads

$$(4.2)\qquad I_1+I_2+I_3+I_4+I_5+I_6=J.$$

Consider I_2:

$$(4.3)\qquad I_2=-\int_{\tau_1}^{\tau_2} u_x^n(Y, t)(w-u^n)(Y, t)\,dt.$$

There are two cases: either $Y=X_v$ or $Y=X$. If $Y=X_v$ then $w(Y,t)=\delta$, $w_x(Y,t)=0$, $u^n(Y,t)\geq 0$ and $u_x^n(Y,t)\leq 0$ so that $I_2\leq\alpha(T)(\tau_2-\tau_1)\delta$ (where we have used (3.6)). On the other hand, if $Y=X$ then $u_x^n(Y,t)=0$ and $I_2=0<\alpha(T)(\tau_2-\tau_1)\delta$. In any case, we have

$$I_2\leq\alpha(T)(\tau_2-\tau_1)\,\delta. \tag{4.4}$$

The monotonicity of β^n implies

$$I_6\leq 0 \tag{4.5}$$

and from (4.2)–(4.5) we get

$$I_1+\alpha(T)(\tau_2-\tau_1)\,\delta+I_3+I_4+I_5\geq J. \tag{4.6}$$

By Lemma 3.4 it is clear that passage to the limit as $n\to\infty$ is possible in (4.6). This yields

$$\int_{\tau_1}^{\tau_2}\int_0^Y u_t(w-u)\,dx\,dt+\alpha(T)(\tau_2-\tau_1)\,\delta+\int_{\tau_1}^{\tau_2}\alpha(t)(w-u)(0,t)\,dt + \int_{\tau_1}^{\tau_2}\int_0^Y u_x(w-u)_x\,dx\,dt\geq\int_0^Y\int_{\tau_2}^{\tau_2} f(t)(w-u)\,dx\,dt \tag{4.7}$$

since $w\geq\delta>0\Rightarrow I_5\to 0$ as $n\to\infty$. The integral inequality of Problem B now follows by letting $\delta\to 0$.

To show that $u\in\mathcal{K}$ it suffices to show that $u(X,t)=0$ for $t\in[0,T]$. To do this, we define

$$\Omega=\{(x,t)\in R\mid u(x,t)>0\}.$$

Since $u\in C(\bar{R})$, Ω is an open set and, by the Schauder estimates (see [3]), it follows that $u\in C_{2+\alpha}(\Omega)$. Also, from this and Lemma 3.4 we find that

$$u_t-u_{xx}=f(t)\quad\text{in}\quad\Omega\subset R, \tag{4.8}$$

$$u(x,0)=0\quad\text{for}\quad x\in[0,X], \tag{4.9}$$

$$u_x(0,t)=\alpha(t)\quad\text{for}\quad t\in(0,T), \tag{4.10}$$

$$u_x(X,t)=0\quad\text{for}\quad t\in(0,T). \tag{4.11}$$

Let $(x_0,t_0)\in\Omega$ and define a function $w(x,t)$ by

$$w(x,t)=u(x,t)-c(x-x_0)^2\quad\text{where}\quad c=-f(T)/2.$$

Let $Q=\Omega\cap\{0<t<t_0\}$. Then $w_t-w_{xx}=f(t)+2c\leq f(T)+2c=0$ in Q. At boundary points of Ω in R we have $u=0$ and $w\leq 0$. Also,

$$w_x(X,t)=u_x(X,t)-2c(X-x_0)=-2c(X-x_0)\leq 0.$$

By the maximum principle w cannot take a maximum in Q. But $w\in C(\bar{Q})$ and $w(x_0,t_0)=u(x_0,t_0)>0$ so that w must achieve a positive maximum somewhere on the parabolic boundary of Q. One easily deduces from the above considerations that the maximum must occur at some point $(0, t^*)$ where $0 \le t^* \le T$. But then $w_x(0, t^*) \le 0$ and $0 \ge u_x(0, t^*) + 2cx_0 \ge \alpha(T) + 2cx_0$ so that

$$x_0 \le -\alpha(T)/2c = \alpha(T)/f((T) \le X - 1.$$

Thus $\Omega \cap \{X-1 < x < X\} = \phi$ which proves that $u(x, t) \equiv 0$ for $x \ge X-1$ so that $u \in \mathcal{K}$. The other assertions of the theorem follow easily from (3.5) and (3.6). ■

We will now show that the solution u to Problem B gives rise to a solution $\{u, s\}$ to Problem A.

THEOREM 4.2. *Let u be the solution to Problem* B *and define* $\Omega = \{(x, t) \in R \mid u(x, t) > 0\}$. *Then there exists a function* $s(t) \in C[0, T] \cap C^{1/2-\gamma}(\tau, T)$ *for each* $\tau \in (0, T)$ *and* $\gamma \in (0, \frac{1}{2})$, *such that:*

$$\Omega = \{(x, t) \mid 0 < x < s(t)\}, \tag{4.12}$$

$$s(t) \le A\alpha(t)t^k \; (\Rightarrow s(0) = 0), \tag{4.13}$$

where $A = -1/c'$,

(4.14) *s is a monotone increasing function and $s(t) > 0$ for $t \in (0, T)$,*

and

$$u(s(t), t) = u_x(s(t), t) = 0 \quad \textit{for} \quad t \in (0, T). \tag{4.15}$$

Proof. Define $s(t) = \max_{0<x<X}\{t \mid u(x, t) > 0\}$. Since $u \ge 0$ on R and $u_x(0, t) = \alpha(t) < 0$ it follows that $s(t) > 0$ for each $t \in (0, T)$. Therefore, since $u(X, t) = 0$, by Theorem 4.1, we have

$$0 < s(t) < X \quad \text{for} \quad t \in (0, T). \tag{4.16}$$

The monotonicity of s is clear because $u(\cdot, t)\searrow$ and $u(x, \cdot)\nearrow$. To prove (4.13) we observe that in the proof of Theorem 4.1 it is possible to take $c = -f(t_0)/2$ instead of $c = -f(T)/2$ and we then deduce from (2.5) that

$$x_0 \le \alpha(t_0)/f(t_0) \le (-1/c')\alpha(t_0)t_0^k$$

whenever $(x_0, t_0) \in \Omega$, which implies (4.13). Since (4.15) is a direct result of Theorem 4.1 we need only to prove that s is locally Hölder continuous, and this will be accomplished by an argument similar to the maximum principle argument of the proof of Theorem 4.1. Let $0 < t_1 < t_2 < T$ and let $x_1 = s(t_1)$ and $Q = ((x_1, X) \times (t_1, t_2)) \cap \Omega$ where we assume that $s(t_2) > s(t_1)$ (since if $s(t_2) = s(t_1)$ then $s(t) \equiv s(t_1)$ for $t \in [t_1, t_2]$ and s is locally Hölder continuous in (t_1, t_2)). Let $x_1 < x_2 < s(t_2)$, which implies $u(x_2, t_2) > 0$, since $u(\cdot, t)$ is a decreasing function. Let

$$w(x, t) = u(x, t) - c(x - x_2)^2$$

where $c = -f(t_2)/2$. Then $w_t - w_{xx} \le f(t_2) + 2c = 0$ on Q and $w(x_2, t_2) = u(x_2, t_2) > 0$. As before, there must be a point (x_1, t^*) where $t_1 \le t^* \le t_2$ such that $w(x_1, t^*)>0$. Hence $0<u(x_1, t^*)-c(x_2-x_1)^2$. But, by (3.33) there exist positive constants β and $\tilde{c}$ such that $u(x_1, t^*)\le\tilde{c}\,|t^*-t_1|^\beta$ so that

$$0<\tilde{c}\,|t_2-t_1|^\beta-c(x_2-x_1)^2.$$

Since this holds for all $x_2<s(t_2)$ we get

$$(s(t_2)-s(t_1))^2\le\frac{\tilde{c}}{c}\,|t_2-t_1|^\beta$$

or

$$0\le s(t_2)-s(t_1)\le\sqrt{\tilde{c}/c}\,(t_2-t_1)^{\beta/2}.$$

A review of Lemma 3.4 shows that β can be taken to be any constant in $(0, 1)$ but that $\tilde{c}$ will depend on t_1 and β. Thus $s(t)\in c^{1/2-\gamma}(t, T)$ for each $\gamma\in(0,\frac{1}{2})$ and the theorem is proved.

THEOREM 4.3. *If $k\ge1$ and $t^k\alpha(t)\in C^{0,1}[0, T]$ then $s\in C^\delta[0, T]$ for each $\delta\in(0,\frac{1}{2})$.*

Proof. The proof of Lemma 3.2, in the case $k\ge1$, can be modified to give

$$\int_0^T\int_0^X(t^k\beta^\varepsilon(u^\varepsilon))^p\,dx\,dt\le C(p). \tag{4.17}$$

In fact, if we go back to the estimation of I_2 in Lemma 3.2, with $s=k$, we get

$$I_2\le kpMT^{k-1}\int_0^T\int_0^X(t^k\xi(t)^{1/p}\,|\beta(u)|)^{p-1}\xi(t)^{1/p}\,dx\,dt$$

which implies that

$$I_2\le\eta k(p-1)MT^{(k-1)}I+k\eta^{1-p}MT^{\alpha+k}X/(\alpha+1)$$

holds instead of (3.16). Also, I_1, and I_3 can be estimated to yield (4.17). As in Lemma 3.3 it then follows that t^ku_t and t^ku_{xx} are bounded in the L^p norm on R for each $p>1$ and therefore by Lemma 3.3 of [4] (see the proof of Lemma 3.4) that

$$\langle t^ku\rangle_{t,R}^{1-3/(2p)}\le C(p). \tag{4.18}$$

Now suppose that $u(x_0, t_0)>0$ and let $0<t^*<t_0$ and $x^*=s(t^*)$. Let Q denote the open set

$$\{(x, t)\colon x^*<x<s(t)\quad\text{and}\quad t^*<t<t_0\}$$

We will suppose that t_0 is sufficiently small that

$$\alpha^2(t)t^{2k-1}<(c')^2/k\quad\text{for}\quad 0\le t\le t_0 \tag{4.19}$$

since the Hölder continuity of s for large t was established in Theorem 4.2.

By (4.18), there exists a constant B, depending on k and p but not on t^*, such that

$$z(x^*, t) \leq B(t - t^*)^\theta \quad \text{for} \quad t \geq t^* \tag{4.20}$$

where $\theta = 1 - 3/(2p)$ and $z(x, t) = t^k u(x, t)$.

We now use an argument we have used several times before. Let $\zeta(x, t) = (c'/4)(x - x_0)^2$ and define $w(x, t) = z(x, t) - \zeta(x, t)$. Then, using (3.28) we get

$$\begin{aligned} w_t - w_{xx} &= t^k f(x, t) + k t^{k-1} u(x, t) + c'/2 \\ &\leq -c'/2 + k\alpha^2(t) t^{2k-1}/(2c') \leq 0 \quad \text{on } Q \end{aligned}$$

(where we have used (4.19)). Since $w \leq 0$ on s, the maximum principle implies that a positive maximum of z in Q is attained at some point (x^*, t) with $t^* < t \leq t_0$. Thus $\zeta(x^*, t) \leq z(x^*, t)$ which implies that

$$x_0 - x^* \leq 2\sqrt{B/c'}(t_o - t^*)^\delta$$

where $\delta = (\frac{1}{2}) - 3/(4p)$. Recalling that $x^* = s(t^*)$ and letting $x_0 \uparrow s(t_0)$ proves the result, since s is monotone. ■

LEMMA 4.4. *Suppose that* $0 \leq k < 1$ *and* $\alpha(t) t^{-s} \in L^\infty(0, T)$ *for some* $s > \frac{3}{4} - k$. *Then*

$$\int_\tau^T \int_0^X (t^k \beta_\varepsilon(u^\varepsilon))^2 \, dx \, dt \leq \frac{c(\varepsilon\tau^{k-1} + 1)}{1 - 2\varepsilon\tau^{k-1}} \tag{4.21}$$

where $c > 0$ *does not depend on* ε *if* $0 < \varepsilon < \min(1, \tau, \frac{1}{2}\tau^{1-k})$.

Proof. Let us first remark that with no loss of generality we may assume that

$$\varepsilon\, |\beta^\varepsilon(0)| \leq C_1 \tag{4.22}$$

holds for some constant C_1. To see this note that the condition $\beta^\varepsilon(0) = f^\varepsilon(0) = 2f(\varepsilon)$ is consistent with the other assumptions concerning β^ε and f^ε. However, under this assumption we deduce that

$$\varepsilon\, |\beta^\varepsilon(0)| = -\varepsilon\beta^\varepsilon(0) = -2\varepsilon f(\varepsilon) \leq 2\varepsilon(c''\varepsilon^{-k}) \leq 2c'' \quad \text{for} \quad 0 \leq \varepsilon < 1.$$

Also, ζ^ε satisfies those hypotheses stated for $\alpha(t)$.

The proof now proceeds along the lines of the proof of Lemma 3.2. Let $\tau \in (0, T)$ and define

$$I \equiv \int_\tau^T \int_0^X t^{2k} \beta^2(u) \xi(t) \, dx \, dt \tag{4.23}$$

where $\xi(t) = (T - t)^\alpha$ and $\alpha \in (0, 1)$ is arbitrary. Here β denotes β_ε and u denotes u_ε, a solution of Problem C(ε). Then

$$I = I_1 + I_2 + I_3 \tag{4.24}$$

where

$$I_1 = \int_\tau^T \int_0^X \xi(t) t^{2k} \beta(u) f(t)\, dx\, dt, \tag{4.25}$$

$$I_2 = -\int_\tau^T \int_0^X \xi(t) t^{2k} \beta(u) u_t\, dx\, dt, \tag{4.26}$$

$$I_3 = \int_\tau^T \int_0^X \xi(t) t^{2k} \beta(u) u_{xx}\, dx\, dt. \tag{4.27}$$

As before, it follows easily that $I_3 \le 0$ and $I_1 \le (1/3)I + C$ where C does not depend on α, τ, or ε. Thus

$$I \le 2I_2 + C \tag{4.28}$$

and it remains to estimate I_2. We get

$$\begin{aligned} I_2 &\le \int_\tau^T \int_0^X u \frac{\partial}{\partial t}(\xi(t) t^{2k} \beta(u))\, dx\, dt \\ &\le \int_\tau^T \int_0^X (\varepsilon + A\zeta^2(t) t^k) t^{2k} \frac{\partial}{\partial t}(\xi(t)\beta(u))\, dx\, dt \quad \text{(by (3.8))} \\ &\equiv J_1 + J_2 \quad \text{where } \zeta = \zeta_\varepsilon. \end{aligned}$$

Then

$$\begin{aligned} J_1 &= \varepsilon \int_\tau^T \int_0^X t^{2k} \frac{\partial}{\partial t}(\xi(t)\beta(u))\, dx\, dt \\ &= \varepsilon \int_0^X \left\{ -\tau^{2k}\xi(\tau)\beta(u(x,\tau)) - \int_\tau^T 2k t^{2k-1} \xi(t)\beta(u)\, dt \right\} dx \\ &\equiv J_1^* + J_1^{**}. \end{aligned}$$

But, by (4.22) $J_1^* \le \varepsilon \tau^{2k} \xi(\tau) |\beta^\varepsilon(0)| X \le C$ so that

$$J_1^* \le C. \tag{4.29}$$

Estimating J_1^{**} we get

$$\begin{aligned} J_1^{**} &= -2k\varepsilon \int_\tau^T \int_0^X t^{2k-1} \xi(t) \beta(u)\, dx\, dt \\ &\le 2\varepsilon k \tau^{k-1} \int_\tau^T \int_0^X t^k |\beta(u)| \xi(t)\, dx\, dt \\ &\le 2\varepsilon k \tau^{k-1} \left\{ \int_\tau^T \int_0^X \eta t^{2k} \xi(t) \beta^2(u) + (1/(4\eta))\xi\, dx\, dt \right\} \end{aligned}$$

for each $\eta > 0$. Choosing $\eta = 1/2k$ we get

$$J_1^{**} \le \varepsilon \tau^{k-1}(I + C). \tag{4.30}$$

Thus, combining these results we see that

$$J_1 \leq \varepsilon\tau^{k-1}(I+C)+C. \tag{4.31}$$

We shall now estimate J_2. Since we assume that $\alpha(t)t^{-s} \in L^\infty$ we can choose the ζ_ε so that $\zeta_\varepsilon^2(t) \leq Bt^s$ where $B<0$ does not depend on ε. Using this fact and extending the integral in J_2 to $(0, X)\times(0, T)$ we see that

$$\begin{aligned} J_2 &\leq AB\int_0^T\int_0^X t^{2s+3k}\frac{\partial}{\partial t}(\xi(t)\beta(u))\,dx\,dt \\ &= -AB\int_0^T\int_0^X (2s+3k)\xi(t)\beta(u)t^{2s+3k-1}\,dx\,dt \\ &\leq C'\int_0^T\int_O^X \xi(t)(\beta^2(u)t^{2(k+1)}+\tfrac{1}{4}t^{4(s+k-1)})\,dx\,dt \end{aligned} \tag{4.32}$$

where $C'=AB|2s+3k|$. Using Lemma 3.2 and the fact that $s>\frac{3}{4}-k$ it follows that $J_2 \leq C$ where C does not depend on ε or τ. From (4.31) we get

$$I_2 \leq \varepsilon\tau^{k-1}(I+C)+C \tag{4.33}$$

which, by (4.28) implies

$$I \leq 2\varepsilon\tau^{k-1}(I+C)+C \tag{4.34}$$

and the result follows by letting $\alpha \to 0$.

LEMMA 4.5. *If the hypotheses of Lemma 4.4 hold and $t^k\alpha(t) \in W^{1,2}(0, T)$ then*

$$t^k u_t,\ t^k u_{xx} \in L^2((0, X)\times(0, T))$$

and $t^k u(x, t)$ is Hölder continuous in t (exponent $\frac{1}{4}$).

Proof. Let u^n and u be the functions of Lemma 3.4 and let $S=(0, X)\times(\tau, T)$ for an arbitrary constant $\tau \in (0, T)$. Let $z^n=t^k u^n$. Then z^n satisfies

$$(z^n)_t-(z^n)_{xx}=t^k f^n(t)-t^k\beta_n(u^n)+kt^{k-1}u^n \equiv \tilde{f}$$

in S and $z_x(0, t)=t^k\zeta^n(t)$, $z_x(X, t)=0$ for $\tau<t<T$. By Theorem 9.1 of [5] or Theorem 17 of [7] we see that

$$\|z^n\|_{L^2(S)}+\|z_t^n\|_{L^2(S)}+\|z_{xx}^n\|_{L^2(S)} \leq C(\|\tilde{f}\|_{L^2(S)}+|z^n(\cdot, \tau\|_{W^{1,2}(0,X)}+\|t^k\zeta^n(t)\|_{W^{1,2}(\tau,T)}).$$

By (2.5), Lemma 3.1 and Lemma 4.1 it follows that

$$\|\tilde{f}\|_{L^2(S)}^2 \leq C+C(\varepsilon_n\tau^{k-1}+1)/(1-2\varepsilon_n\tau^{k-1})+C\|t^{k-1}u^n\|_{L^2(S)}^2$$

where C depends neither on n nor τ. Thus, there exists a function $\Sigma(x, t)$ in $L^2(S)$ possessing weak derivatives Σ_t and Σ_{xx} in $L^2(S)$ such that some subsequence of z^n (again denoted z^n) converges weakly in $L^2(S)$ along with z_t^n and z_{xx}^n to Σ, Σ_t, and Σ_{xx} respectively. Also, from the L^2 estimates above we get

$$\|\Sigma\|+\|\Sigma_t\|+\|\Sigma_{xx}\|\le C+C\|t^{k-1}u\|^2+\|z(\cdot,\tau)\|_{W^{1,2}(0,X)}$$

where $\|\cdot\|=\|\cdot\|_{L^2(S)}$. But by (3.28) and the fact that $\alpha^2(t)\le Bt^s$, for some $B>0$ and $s>\frac{3}{4}-k$ we get

$$t^{k-1}u\le ABt^{2k-1+2s}$$

which is bounded by assumption. By (3.30) it is clear that $\|z(\cdot,\tau)\|_{W^{1,2}}$ does not depend on τ. Thus

$$\|\Sigma|+\|\Sigma_t\|+\|\Sigma_{xx}\|\le C$$

where C does not depend on τ and where $\|\cdot\|=\|\cdot\|_{L^2(S)}$. However, it is clear from (3.35)–(3.38) that $\Sigma=t^k u$ and that

$$\Sigma_t=\frac{\partial}{\partial t}(t^k u)\quad\text{and}\quad \Sigma_{xx}=\frac{\partial}{\partial x^2}(t^k u)\quad\text{(weak derivatives) a.e. in } S.$$

Thus

$$\|(t^k u)_t\|+\|t^k u_{xx}\|\le C \tag{4.35}$$

where C does not depend on τ and where $\|\cdot\|=\|\cdot\|_{L^2((0,X)\times(\tau,T))}$. Fatou's lemma implies that (4.35) holds with $\|\cdot\|=\|\cdot\|_{L^2((0,X)\times(0,T))}$ and this, together with Lemma 3.3 of [5] proves the lemma.

THEOREM 4.6. *If $0\le k<1$ and the hypotheses of Lemma 4.5 hold, then $s(t)\in C^{1/8}[0,T]$.*

Proof. Lemma 4.5 establishes (4.18) with $p=2$ and the rest of the proof is identical to that portion of the proof of Theorem 4.3 which follows (4.18) since $s>\frac{3}{4}-k$ implies $\alpha^2(t)t^{2k-1}\to 0$ as $t\to 0$.

LEMMA 4.7. *Let α_0, c_0, and k be positive constants with $k>\frac{1}{2}$. Then for each constant θ_0 satisfying*

$$(2/3)(\alpha_0 c_0)<\theta_0<(\alpha_0/c_0)$$

there exists a positive constant τ, depending on α_0, c_0, θ_0, and k, and classical solution $u(x,t)$ to the problem

$$u_t-u_{xx}\le -c_0t^{-k}\quad\textit{for}\quad 0<x<s(t),\quad 0<t<\tau, \tag{4.36}$$

$$u_x(0,t)=-\alpha_0\quad\textit{for}\quad 0<t<\tau, \tag{4.37}$$

$$u(s(t),t)=u_x(s(t),t)=0\quad\textit{for}\quad 0<t<\tau, \tag{4.38}$$

where

$$s(t)=\theta_0 t^k. \tag{4.39}$$

Proof. We shall omit the zero subscripts of α_0, c_0, and θ_0. Let $a(t)$ and

$b(t)$ be functions given by

(4.40) $$a(t) = (\tfrac{1}{2})ct^{-k},$$

(4.41) $$b(t) = ((\alpha - c\theta)(3\theta^2)t^{-2k}),$$

and notice that both functions are nonnegative. We define

(4.42) $$u(x, t) = a(t)(s(t) - x)^2 + b(t)(s(t) - x)^3$$

for $0 < x < s(t)$ and $0 < t < 1$. By writing $u = as^2(1-\xi)^2 + bs^3(1-\xi)^3$ where $\xi = x/s$ it is easy to see that u is bounded for $0 < x < s(t)$, $0 < t < 1$. It is also easy to check that u satisfies (4.37) and (4.38). By direct computation we get

(4.43) $$(u_t - u_{xx} + ct^k)/(s\eta) = \alpha\eta^2 + \beta\eta + \gamma \equiv \varphi(\eta)$$

where

$$\eta = (1 - x/s) \in (0, 1), \qquad \alpha = (-2k/3)(\alpha - c\theta)t^{-1} < 0,$$

$$\beta = (\alpha - (3/2)c\theta)kt^{-1}, \qquad \gamma = kc\theta t^{-1} - (2/\theta^2)(\alpha - c\theta)t^{-2k}.$$

Thus $\varphi(\eta)$ is a convex parabola with vertex at

$$\eta = \eta^* = -\beta/(2\alpha) = \frac{3}{8}\frac{(2\alpha - 3c\theta)}{\alpha - c\theta}.$$

By hypothesis $\theta > (2/3)(\alpha/c)$ so that $\eta^* \leq 0$ and therefore the result will be established once we show that $\varphi(0) \leq 0$ for small t. But

$$\varphi(0) = \gamma = t^{-2k}(kc\theta t^{2k-1} - (2/\theta^2)(\alpha - c\theta))$$

which, because we assume $k > \frac{1}{2}$ and $\alpha > c\theta$, is clearly negative for all $0 < t < \tau$ where τ depends on k, c, θ and α. ■

Our choice of the function $u(x, t)$ was inspired by a lecture given by Alan Soloman [8].

THEOREM 4.8. *Let $u(x, t)$ be a solution to Problem* B *with $\alpha(t) \leq -\alpha_0 < 0$ and $k > \frac{1}{2}$. Then for each sufficiently small $\varepsilon > 0$ there exists a constant $\tau > 0$ depending on ε such that*

(4.44) $$[(\alpha_0/c'') - \varepsilon]t^k \leq s(t) \quad \textit{for} \quad 0 < t < \tau.$$

Proof. Let $c_0 = c''$ and $\theta_0 = (\alpha_0/c_0) - \varepsilon$ where $\varepsilon < \alpha_0/(3c_0)$ in Lemma 4.7 and denote the solution of (4.36)–(4.39) by (u^*, s^*). Also let τ be the constant τ of Lemma 4.7.

We shall compare u and u^* in the domain

$$D = \{(x, t) \mid 0 < t < \tau, 0 < x < \hat{s}(t)\} \quad \text{where} \quad \hat{s}(t) = \min(s(t), s^*(t)).$$

Let $\hat{s} = I \cup II$ where $I = \{\hat{s} = s\}$ and $II = \{\hat{s} = s^*\}$ and let $P = u - u^*$. Then P satisfies

$$P_t - P_{xx} \geq f(t) + c''t^{-k} \geq 0 \quad \text{in } D.$$

Since it is easily seen that $u^* \in C(\bar{D})$ it follows that $P \in C(\bar{D})$ and attains a minimum in $\bar{D}$. If this minimum is negative then it must be attained either on $x=0$ or on $x=\hat{s}$, by the maximum principle. But $P_x(0,t)=\alpha(t)+\alpha_0 \leq 0$, $P_x \geq 0$ on I and $P \geq 0$ on II since, on I, $u=u_x=0$ and $\tilde{u} \geq 0$, $\tilde{u}_x \leq 0$ and, on II, $u^*=u^*_x=0$ and $u \geq 0$, $u_x \leq 0$. Thus a negative minimum cannot be attained anywhere in $\bar{D}$ and hence $P \geq 0$ in $\bar{D}$. Hence $u^*(x,t) \leq u(x,t)$ in $\bar{D}$ and in particular $u^* \leq u$ on $\hat{s}$. But since $uu^* \equiv 0$ on $\hat{s}$ it must be that $u^* \equiv 0$ on $\hat{s}$ and that $s^*(t) \leq s(t)$ for $0 \leq t \leq \tau$; for, if for some t we have $s(t) < s^*(t)$ then $\hat{s}(t) < s^*(t)$ and $u^*(\hat{s}(t),t) > 0$ since $u^*(\cdot,t)$ is a strictly decreasing function and $u^*(s^*(t),t)=0$. But by (4.39) we see that $s^*(t)=[(\alpha_0/c'')-\varepsilon]t^k$ for $0 \leq t < \tau$ and the result follows. ■

Theorems 4.2 and 4.8 together imply the following corollary.

COROLLARY 4.9. *Let the assumptions of Theorem* 4.8 *hold. Then for* $0 \leq t < \tau$,

$$[(\alpha_0/c'')-\varepsilon]t^k \leq s(t) \leq [-\alpha(t)/c']t^k. \tag{4.45}$$

In particular if $\alpha(t)=-\alpha_0$ *then* $[(\alpha_0/c'')-\varepsilon]t^k \leq s(t) \leq [\alpha_0/c']t^k$ *for* $0 \leq t < \tau$. ■

Thus we have proved that $s(t)$ grows initially like t^k if $k > \frac{1}{2}$.

Remark 4.10. For the original transformed optimal stopping time problem of Chernoff we had $\alpha_0 = \frac{1}{2}$, $c''=c'=1$ and $k=2$. Thus we get $(1-\varepsilon)t^2 \leq s(t) \leq t^2$ for $0 \leq t < \tau$ where τ depends on ε. This agrees well with the results of various numerical approximations (see [6], [7]).

The method of Lemma 4.7 seems to fail to provide a useful comparison function when $k \leqq \frac{1}{2}$. However the next lemma and theorem give lower bounds on the initial growth of the free boundary when k is small.

LEMMA 4.11. *Let* $k \geq 0$ *and* c, θ, α, ε, $\beta > 0$ *be constants and let* $\gamma = k+\beta+\varepsilon$ *and suppose that* $\gamma > \frac{1}{2}$, $k+\varepsilon < \beta$, *and* $\theta < \alpha/c$. *Then there exists a classical solution of the problem*

$$u_t - u_{xx} \leq -ct^{-k} \quad \text{for} \quad 0<x<s(t), \quad 0<t<\tau, \tag{4.46}$$

$$u_x(0,t) = -\alpha t^\beta \quad \text{for} \quad 0<t<\tau, \tag{4.47}$$

$$u(s(t),t) = u_x(s(t),t) = 0 \quad \text{for} \quad 0<t<\tau \tag{4.48}$$

where

$$s(t) = \theta t^\gamma \tag{4.49}$$

and $\tau > 0$ *is a constant.*

Proof. We proceed as in the proof of Lemma 4.7 except that now we define

$$a(t)=(c/2)t^{-k}>0, \tag{4.50}$$

$$b(t)=At^{\beta-2\gamma}-Bt^{-k-\gamma} \tag{4.51}$$

where $A=\alpha/(3\theta^2)$ and $B=c/(3\theta)$. Clearly A and B are positive and $u(x,t)$ defined by (4.42) is bounded for $0<x<s(t)$ and $t>0$ sufficiently small.

After appropriately modifying (4.43) one easily deduces that

$$\eta^*=\frac{1}{4\theta}\left[\frac{ckt^{\varepsilon}-6\theta\gamma A+6\theta\gamma Bt^{\varepsilon}}{-A(2k+\beta+2\varepsilon)+B(k+\gamma)t^{\varepsilon}}\right]\to\left(\frac{3}{2}\right)\frac{\gamma}{k+\varepsilon+\gamma}>1$$

as $t\searrow 0$. It follows that φ is a convex parabola with $\varphi(1)\leq 0$ for small t, and thus completes the proof.

THEOREM 4.12. *Suppose that $u(x,t)$ is a solution of Problem* B *and that $\alpha(t)\leq -\alpha_0 t^{\beta}$ where α_0 and β are positive constants and $\beta>\max(\frac{1}{4},k)$. Then for each γ satisfying $\max(\frac{1}{2},k+\beta)<\gamma$ and each $\theta\in(0,\alpha_0/c)$ there is a $\tau>0$ such that $\theta t^{\gamma}\leq s(t)$ for $0\leq t<\tau$.*

Proof. Without loss of generality $\gamma<2\beta$. Let $\varepsilon=\gamma-k-\beta$ in Lemma 4.11 and proceed as in the proof of Theorem 4.8.

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