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LIE ALGEBRA COHOMOLOGY AT IRREDUCIBLE MODULES

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We will develop a procedure for producing irreducible modules for the Lie algebra of a semisimple, simply connected algebraic group at which the 1-cohomology is non-zero. Further, we will relate our computations of Lie cohomology to the cohomology of the algebraic group. The cohomology of the group may be zero at a module where the cohomology of the Lie algebra is non-zero, but there is an efficient method for augmenting the module to give a module where the cohomology of the group is non-zero.

Hochschild showed that the (restricted) 1-cohomology of a non-abelian p-Lie algebra L is non-zero at the L-module Hom (LU_L, k), where U_L is the restricted universal enveloping algebra of L [3]. In Sections 1 and 2, we show that his methods can be used in the case of the Lie algebra of a Chevalley group to produce a good supply of irreducible modules $\{V_j\}$ at which the 1-Lie cohomology is non-zero. One begins with a suitable p-semi-linear map from the Lie algebra to the trivial Lie algebra k, and uses the isomorphism

$$H^2(LU_L, k) \cong H^1(LU_L, \operatorname{Hom}(LU_L, k))$$

to obtain a 1-cohomology class with values in Hom (LU_L, k). By passing to subquotients of Hom (LU_L, k), one obtains some irreducible modules $\{V_j\}$ at which the 1-cohomology is non-zero. The highest weights of these modules are the differentials of the elements $\{-\alpha_i\}$ where $\{\alpha_i\}$ is a basis for the root system of the group relative to a maximal torus T.

The space of 1-Lie cocycles at an irreducible module is itself a module for the group. In showing in Theorem 2.2 that the cohomology spaces are non-zero at $\{V_j\}_{j=1}^{\dim(T)}$, we produce a line of 1-cocycles that is stable under the action of an appropriate Borel subgroup of the group, and show that the weight of the line under the action of T does not occur in the module V_j. Consequently, the cocycles in the line are not coboundaries. At the same time, the weight of this line gives the highest weight of a composition factor of the 1-Lie cohomology as a module for the group. As an illustration, we give the result of Sections 1 and 2 specifically for the Lie algebra of the special linear group.

A module at which the group cohomology is non-zero may be obtained economically from V_j by tensoring V_j with the dual module $(H^1(L, V_j))^*$ (Corollary to Theorem 2.2).

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In Section 3, we carry out a complete computation of the cohomology space of the Lie algebra of the special linear group Sl_3, as a module for Sl_3, at the irreducible module V_1 whose highest weight is the differential of $-\alpha_1$ for the usual basis $\{\alpha_i\}$ of the root system of Sl_3. $H^1(sl_3, V_1)$ is isomorphic as an Sl_3-module to the Frobenius power of the identity representation of Sl_3 (Theorem 3.4). The computation depends on knowing that the space of vectors in V_1 whose weight equals the weight of the line stable under the Borel sub-Lie algebra has dimension one (Lemma 3.3).

As a corollary to this theorem, we see that Sl_3 has non-zero 1-cohomology at V_1 tensored with the Frobenius power of the identity representation of Sl_3. This result shows how far one need go to realize an extension of the trivial module k by V_1 (parametrized by an element of $H^1(sl_3, V_1)$) as an sl_3-submodule of an Sl_3-module.

Notation

Let k be a field of characteristic p. Let L be a finite-dimensional p-Lie algebra over k. We also let k stand for the trivial one-dimensional L-module and for the trivial one-dimensional p-Lie algebra.

Suppose that L is the Lie algebra of an algebraic group G. The space of 1-Lie algebra cocycles $Z^1(L, V)$ with values in a G-module V may be given the structure of a G-module as follows:

$$(g \cdot f)(l) = g \cdot (f(\mathrm{Ad}\,(g^{-1})(l))) \quad \text{where} \quad g \in G, \qquad l \in L, \qquad f \in Z^1(L, V)$$

and Ad is the adjoint representation of G on L.

The expression 'p-linear' is used in place of the usual expression 'p-semi-linear'.

sl_{n+1} denotes the special linear Lie algebra, and Sl_{n+1} denotes the special linear group.

1. The cohomology of the restricted universal enveloping algebra

We will study restricted representations of a finite-dimensional p-Lie algebra L. Let U_L be the restricted universal enveloping algebra of L. Inside U_L there is the associative algebra without unit, $U_L^+ = LU_L$. First we will look at the cohomology spaces $H^n(U_L^+, V)$ of the associative algebra U_L^+ at a right U_L^+-module V.

Make V into a two-sided U_L^+-module by adding on the trivial left U_L^+-module structure. Give the space of 1-cochains $\mathrm{Hom}\,(U_L^+, V)$ the left U_L^+-module structure

$$(u * f)(u') = f(u'u) - u'f(u)\,[= f(u'u)],$$

and the right U_L^+-module structure $(f * u)(u') = f(u')u$. (We have 'exchanged' the left and right module structures given in [2].)

Theorem 3.1 of [2] gives an isomorphism of cohomology spaces

$$H^m(U_L^+, V) \cong H^{m-1}(U_L^+, \operatorname{Hom}(U_L^+, V)).$$

The isomorphism may be induced from the map on cochains,

$$\operatorname{Hom}((U_L^+)^{\otimes m}, V) \to \operatorname{Hom}((U_L^+)^{\otimes(m-1)}, \operatorname{Hom}(U_L^+, V)),$$

$$f \to \bar{f} \quad \text{where} \quad \bar{f}(a_1 \otimes \cdots \otimes a_{m-1})(a_m) = f(a_m \otimes a_1 \otimes \cdots \otimes a_{m-1}).$$

We will exploit the isomorphism $H^2(U_L^+, k) \cong H^1(U_L^+, \operatorname{Hom}(U_L^+, k))$ in producing non-trivial 1-cohomology classes.

1.1. *Construction.* There is a canonical linear map $\operatorname{Hom}_{p\text{-linear}}(L, k) \to H^2(U_L^+, k)$. The map is injective if $[L, L] = L$. The construction follows the method of construction used in [3] in showing that non-abelian p-Lie algebras have representations which are not completely reducible. The material in the remainder of Sections 1.1 and 1.2 comes essentially from [3].

Let $h: L \to kt = k$ be a p-linear map to the base field. Form the direct product of Lie algebras $E = L + kt$, and give E the p-map $(l, a)^{[p]} = (l^{[p]}, h(l))$. The quotient map of p-Lie algebras $E \to L$ induces an algebra map $U_E \to U_L$ and by restriction, $\phi: U_E^+ \to U_L^+$. Order a basis $l_1, \ldots, l_n$ for L and define a linear map $\psi: U_L^+ \to U_E^+$ by

$$\psi(l_1^{\alpha_1} \cdots l_n^{\alpha_n}) = l_1^{\alpha_1} \cdots l_n^{\alpha_n} \quad \text{for} \quad 0 \le \alpha_i < p.$$

ψ is a linear splitting of ϕ. We associate to E (and hence to h) a cocycle $g \in Z^2(U_L^+, k)$ as follows. The kernel of ϕ is $U_E t = U_E^+ t + kt$; let $\gamma: U_E t \to kt$ be the projection relative to this sum. Define a bilinear mapping $g: U_L^+ \times U_L^+ \to kt$ by $g(u, v) = \gamma(\psi(u)\psi(v) - \psi(uv))$. The relation $g(uv, w) = g(u, vw)$ holds for g and g is a 2-cocycle. (A different choice of linear splitting of ϕ gives a 2-cocycle which is cohomologous to g.) Map $\operatorname{Hom}_{p\text{-linear}}(L, k) \to H^2(U_L^+, k)$ by mapping h to the cohomology class of g. We will not check the linearity of this map here.

1.2. PROPOSITION. *The map*

$$\operatorname{Hom}_{p\text{-linear}}(L, k) \to H^2(U_L^+, k) \cong H^1(U_L^+, \operatorname{Hom}(U_L^+, k))$$

is injective if $[L, L] = L$.

Proof. We use the relations given in [3] to prove this proposition. Let $S = U_L^+ + kt$ be the right U_L^+-module extension of U_L^+ by kt corresponding to the 1-cocycle $\bar{g}$. Then $\bar{g}$, which is the image of h under the map of the proposition, is given by the formula $\bar{g}(u)(v) = g(v, u)$, and the module structure on S is given by

$$(v + at)u = vu + \bar{g}(u)(v)t \quad \text{for} \quad u, v \in U_L^+ \quad \text{and} \quad a \in k.$$

Suppose that there is a stable complement Q to the submodule kt. Express each element v of U_L^+ as $q(v) + r(v)$, where $q(v)$ lies in Q and $r(v)$

lies in kt. The relations of [3] show that r is zero over $[L, L]$ and that $r(l^{[p]}) = -h(l)$ for $l \in L$. Hence, under our hypothesis, $r(L) = 0$ and $h = 0$.

Remark. One can also show that the map is injective using the method of [4, §8].

2. Non-zero one-cohomology for the Lie algebra

2.0. Let G be a simply connected semisimple algebraic group over an algebraically closed field of characteristic p, with Lie algebra L. We will recall some of the structure of G and L (see [1, A] for more details).

There is a complex semisimple Lie algebra $\mathbf{g}$ and a Z-form $\mathbf{g}_Z$ of $\mathbf{g}$ such that $\mathbf{g}_Z \otimes k = L$. $\mathbf{g}_Z$ arises in the following way. Let $\mathbf{h}'$ be a Cartan subalgebra of $\mathbf{g}$ and let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a set of simple roots for the root system ϕ of $\mathbf{g}$, relative to $\mathbf{h}'$. Let $\{X_\alpha\}_{\alpha \in \phi} \cup \{[X_{\alpha_i}, X_{-\alpha i}]\}_{\alpha_i \in \Delta}$ be a Chevalley basis for $\mathbf{g}$ [1, A, 1.2]. This basis spans a Z-form $\mathbf{g}_Z$ of $\mathbf{g}$. $\{[X_{\alpha_i}, X_{-\alpha_i}]\}_{\alpha_i \in \Delta}$ spans a Z-form $\mathbf{h}'_Z$ of $\mathbf{h}'$. Let $\mathbf{h} = \mathbf{h}'_Z \otimes k$. In $L = \mathbf{g}_Z \otimes k$, denote $X_{\alpha_i} \otimes 1$ by X_i, $X_{-\alpha_i} \otimes 1$ by Y_i, and $[X_i, Y_i]$ by H_i.

We locate some objects inside G (see [1, A, 3.2, 3.3]). There are 1-parameter subgroups $\{X_i(t)\}_{t \in k}$, $\{Y_i(t)\}_{t \in k}$, $i = 1, \ldots, n$, and a maximal torus T with the following properties:

(1) T normalizes each 1-parameter subgroup and operates on $\{X_i(t)\}_{t \in k}$ (resp. $\{Y_i(t)\}_{t \in k}$) by a character α_i of T (resp. $-\alpha_i$), i.e., for $a \in T$, $aX_i(t)a^{-1} = X_i(\alpha_i(a)t)$.

(2) There is a morphism from Sl_2 to G that maps

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{to} \quad X_i(t)$$

and

$$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \quad \text{to} \quad Y_i(t).$$

The differential of the morphism maps

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ to } X_i \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ to } Y_i.$$

(3) The Lie algebra of $\{X_i(t)\}_{t \in k}$ is kX_i and the Lie algebra of $\{Y_i(t)\}_{t \in k}$ is kY_i, and the Lie algebra of T is $\mathbf{h}$.

(4) The differential of the character α_i of T is the root $\alpha_i \in \mathbf{h}^*$ of the line kX_i, obtained from the root of X_{α_i} by tensoring with k. We will denote both the character and the root by α_i.

(5) $\{X_i(t)\}_{t \in k}$ $i = 1, \ldots, n$ and T generate a Borel subgroup B of G.

There is the decomposition of $\mathbf{g}_Z = \mathbf{n}_Z^+ \oplus \mathbf{h}_Z \oplus \mathbf{n}_Z^-$, where the first factor is the sum of the positive root spaces and the last is the sum of the negative root spaces. Let $\mathbf{n}^+ = \mathbf{n}_{\mathbf{Z}}^+ \otimes k$, and let $\mathbf{n}^- = \mathbf{n}_{\mathbf{Z}}^- \otimes k$. Then L has the decomposition $L = \mathbf{n}^+ \oplus \mathbf{h} \oplus \mathbf{n}^-$, and $\mathbf{b} = \mathbf{n}^+ \oplus \mathbf{h}$ is the Lie algebra of B.

We will assume that L is a Lie algebra such that $\mathbf{n}^+$ is generated by $\{X_i\}$ $i = 1, \ldots, n$. $L = sl_{n+1}$ is such a Lie algebra in any characteristic.

LEMMA. *$\{\alpha_i\}_{i=1}^n$ generates a free abelian subgroup of the character group of T with rank n.*

Proof. This follows easily from the fact that G has finite center.

2.1. Let L be the Lie algebra of the simply connected, semisimple group G and suppose that $\mathbf{n}^+$ is generated by $\{X_i\}_{i=1}^n$. Choose ordered bases for $\mathbf{n}^+$ and for $\mathbf{n}^-$ consisting of root vectors. Order L by placing some ordered basis for $\mathbf{h}$ first, the ordered basis for $\mathbf{n}^-$ second, and that for $\mathbf{n}^+$ last.

Fix some non-zero p-linear map $f: L \to k$ which is zero on $\mathbf{n}^+ + \mathbf{h}$. We require some information about the 1-cocycle $g: U_L^+ \to \operatorname{Hom}(U_L^+, k)$ which was associated with f in Section 1.1, (which was called $\bar{g}$ there).

(1) $g(\mathbf{n}^+ + \mathbf{h}) = 0$. One may check that $\psi(u)\psi(l) - \psi(ul) = 0$ for $l \in \mathbf{n}^+ + \mathbf{h}$, using the fact that $f(\mathbf{n}^+ + \mathbf{h}) = 0$. Thus, $g(\mathbf{n}^+ + \mathbf{h}) = 0$.

(2) $k \cdot g(Y_j)$ is $\mathbf{b}$-stable for each j. The cocycle condition for g and the relations in L, $[X_i, Y_j] = 0$, for $i \neq j$, $[X_j, Y_j] = H_j$, and $[H_i, Y_j] = -\alpha_j(H_i)Y_j$ lead to the equalities $0 = X_i \cdot g(Y_j)$, for $i \neq j$, $0 = X_j \cdot g(Y_j)$, and $H_i \cdot g(Y_j) = -\alpha_j(H_i)g(Y_j)$. Since $\{X_i\}_{i=1}^n$ generates $\mathbf{n}^+$ as a Lie algebra and since $\{H_i\}_i$ spans $\mathbf{h}$, $kg(Y_j)$ is $\mathbf{b}$-stable with weight $-\alpha_j$.

We proceed to produce some non-zero 1-cohomology classes for L with values in irreducible modules. Let V denote the left-U_L^+-submodule of $\operatorname{Hom}(U_L^+, k)$ generated by $g(L) = g(\mathbf{n}^-)$. V may be generated by $\{g(Y_j)\}_{j=1}^n$ since $\{Y_j\}_{j=1}^n$ generate $\mathbf{n}^-$ as a Lie algebra. Choose a maximal proper submodule W of V. The 1-cocycle

$$\bar{g}: U_L^+ \xrightarrow{g} V \to V/W$$

has values in an irreducible module; we will show that the cohomology class of $\bar{g}$ is non-zero. We show in fact that the Lie algebra cocycle obtained by restricting $\bar{g}$ to L represents a non-zero element of $H^1(L, V/W)$.

Let $\{\alpha_i\}_{i=1}^n$ be the basis for the root system of G relative to T that is given in Section 2.0.

PROPOSITION. *$H^1(L, V/W)$ is non-zero if the characters $\{\alpha_i\}_{i=1}^n$ have distinct differentials.*

This condition is satisfied when $\mathbf{h} \cap \operatorname{Center}(L) = 0$, for instance.

Proof of the proposition. There is an index j_0 for which $g(Y_{j_0})$ does not lie in W. Then $k \cdot \bar{g}(Y_{j_0})$ is a $\mathbf{b}$-stable line of vectors in V/W of weight $-\alpha_{j_0}$. The line of vectors in an irreducible L-module that is $\mathbf{b}$-stable is unique and the $\mathbf{h}$-weight of this line determines the module up to isomorphism [1, A, §6]. Therefore, V/W is the irreducible L-module of highest weight $-\alpha_{j_0}$.

Since G is simply connected, there is a representation of G on V/W whose differential is the given representation of L on V/W. Let $l \subset V/W$ be a B-stable line. Then l is $\mathbf{b}$-stable, and so $l = k\bar{g}(Y_{i_0})$. Denote the T-weight of $\bar{g}(Y_{j_0})$ by λ.

We proceed to show that $k \cdot \bar{g}$ is a B-stable line in $Z^1(L, V/W)$. We have $\bar{g}(Y_i) = 0$ for $i \neq j_0$ since the unique $\mathbf{b}$-stable line in V/W has weight $-\alpha_{j_0}$ and since the weights $\{\alpha_i\}_{i=1}^n$ are distinct elements of $\mathbf{h}^*$ by the hypothesis. $\bar{g}$ is determined up to scalar multiplication by the conditions $\bar{g}(\mathbf{n}^+ + \mathbf{h}) = 0$ and $\bar{g}(Y_i) = 0$ for $i \neq j_0$.

The one parameter subgroups $\{X_i(t)\}_{t\in k}$ act trivially on $\bar{g}$. Since $[X_i, Y_j] = 0$ for $i \neq j$, Y_j commutes with $X_i^m/m!$ in the Kostant Z-form U_Z of the universal enveloping algebra of the complex Lie algebra $\mathbf{g}$ [1, A]. Consequently, on the representation space of U_Z, that gives rise to the simply connected group G under reduction modulo p, the operator $X_i(t)$ commutes with the operator coming from Y_j. Therefore, $\mathrm{Ad}\,(X_i(t))(Y_j) = Y_j$, where Ad is the adjoint representation of G on L. Since $X_i(t)$ acts trivially on $\bar{g}(Y_j)$, $X_i(t) \cdot \bar{g} = \bar{g}$ at Y_j for $i \neq j$. Furthermore, we have

$$\mathrm{Ad}\,(X_j(t))(Y_j) = Y_j + tH_j - t^2 X_j$$

by 2.0(2), and the corresponding computation for Sl_2 and sl_2; hence, $X_j(t) \cdot \bar{g} = \bar{g}$ at Y_j. Since $X_i(t) \cdot \bar{g}$ and $\bar{g}$ are both zero on $\mathbf{n}^+ + \mathbf{h}$, we have that $X_i(t) \cdot \bar{g} = \bar{g}$.

$k \cdot \bar{g}$ is a T-stable line of weight $\lambda - (-\alpha_{j_0})$. Since T stabilizes kY_i and $\mathbf{n}^+ + \mathbf{h}$ under the adjoint representation, $t \cdot \bar{g}$ is also zero at Y_i for $i \neq j_0$ and at $\mathbf{n}^+ + \mathbf{h}$, for $t \in T$. The value of $t \cdot g$ at Y_{j_0} is $(\lambda - (-\alpha_{j_0}))(t)\bar{g}(Y_{j_0})$. Therefore, $\bar{g}$ and $t \cdot \bar{g}$ differ by the scalar $(\lambda - (-\alpha_{j_0}))(t)$, and the claim is established.

$\bar{g}$ is not a coboundary. In fact, V/W has no vector of T-weight $\lambda - (-\alpha_{j_0})$. If $\lambda - (-\alpha_{j_0})$ were a T-weight of V/W, then the character $-\alpha_{j_0} = \lambda - (\lambda - (-\alpha_{j_0}))$ would be a linear combination of $\{\alpha_i\}_{i=1}^n$ with non-negative integral coefficients. However, $\{\alpha_i\}_{i=1}^n$ generates a free abelian subgroup of the character group T^x, by the lemma in Section 2.0. This completes the proof of the proposition.

2.2. Theorem. *Let $\{\alpha_i\}_{i=1}^n$ be the basis for the root system of G relative to T. Suppose that these characters have distinct differentials. The* 1*-Lie algebra cohomology $H^1(L,\ \ )$ is non-zero at the irreducible L-modules $\{V_j\}_{j=1}^n$ of highest weights $\{-\alpha_j\}_{j=1}^n$.*

Proof. Let $f_j : L \to k$ be a p-linear map where $f_j(\mathbf{n}^+ + \mathbf{h}) = 0$, $f_j(Y_j) \neq 0$, and $f_j(Y_i) = 0$ for $i \neq j$. Let g_j be the 1-cocycle with values in $\mathrm{Hom}\,(U_L^+, k)$ which is associated with f_j in Section 1.2, and let V be the U_L^+-submodule generated by $g_j(L)$. If W is a maximal proper submodule of V, then $V/W \cong V_j$ as L-modules and $H^1(L, V_j) = H^1(L, V/W) \neq (0)$, by the proposition.

The group cohomology $H^1(G, V_j)$ may very well be zero (see Section 3). However:

COROLLARY. *Under the hypothesis of the theorem, the cohomology space $H^1(G, V_j \otimes (H^1(L, V_j))^*)$ is non-zero.*

Proof. Since V_j is irreducible and non-trivial as an L-module, the canonical map $V_j \to B^1(L, V_j)$ from V_j to the space of one-coboundaries is an isomorphism of G-modules. Therefore,

$$0 \to B^1(L, V_j) \to Z^1(L, V_j) \xrightarrow{\pi} H^1(L, V_j) \to 0 \qquad (*)$$

gives a G-module extension of $H^1(L, V_j)$ by V_j. This is a non-trivial G-module extension since it is non-trivial as an L-module extension. In fact, since $H^1(L, V_j)$ is always trivial as an L-module, we have that, for any $f \in Z^1(L, V_j) - B^1(L, V_j)$, $V_j + k \cdot f$ is an L-submodule of $Z^1(L, V_j)$. The one Lie cohomology class associated to the extension $0 \to V_j \to V_j + k \cdot f \to k \to 0$ is (f), which is non-zero. Since this L-extension is non-trivial, $(*)$ is a non-trivial L-extension. Therefore the space

$$H^1(G, V_j \otimes (H^1(L, V_j))^*) \cong \mathrm{Ext}_G\,(H^1(L, V_j), V_j)$$

is non-zero.

2.3. EXAMPLE. $Sl_{n+1,k}$ and $sl_{n+1,k}$.

Let $\mathbf{h}$ be the diagonal sub-Lie algebra of sl_{n+1} and let $\alpha_j \in \boldsymbol{h}^*$ have the value $\alpha_j((h_i)) = h_j - h_{j+1}$ at (h_i), the diagonal matrix with entries $h_1, \ldots, h_{n+1}$. Let T be the diagonal subgroup of Sl_{n+1} and let α_j be the character on T with values $\alpha_j((t_i)) = t_j t_{j+1}^{-1}$ at the diagonal matrix with entries $t_1, \ldots, t_{n+1}$.

LEMMA. (a) *The characters $\{\alpha_j\}_{j=1}^n$ have distinct differentials for all $sl_{n+1,k}$ in all characteristics except for sl_3 in characteristic 3.*

(b) *The characters $\{\alpha_j\}_{j=1}^n$ have linearly independent differentials if and only if characteristic $k \nmid n+1$.*

Proof. (a) This may be checked easily.

(b) Suppose that $\sum_i a_i\alpha_i = 0$. At $(h_j)_{j=1}^{n+1}$,

$$0 = \sum_i a_i\alpha_i((h_j)) \quad \text{and} \quad (a_1 + a_n)h_1 + \sum_{i=2}^{n} (a_i - a_{i-1} + a_n)h_i = 0$$

for all $(h_1, \ldots, h_n) \in k^n$. Therefore the following relations hold:

$$a_1 + a_n = 0, \qquad a_i - a_{i-1} + a_n = 0 \quad \text{for} \quad i = 2, \ldots, n. \qquad (*)$$

The sum of these relations is $(n+1)a_n = 0$. If characteristic $k \nmid n+1$, then $a_n = 0$, and furthermore, $a_i = 0$ for all i by $(*)$.

Conversely, if characteristic $k \mid n+1$, then $\sum_{i=1}^n (n-i+1)\alpha_i = 0$.

Let $\{\lambda_j\}_{j=1}^n$ be the characters on T given by $\lambda_j((t_i)) = t_1 \cdots t_j$. The character $-\alpha_j$ equals $\lambda_{j+1} - 2\lambda_j + \lambda_{j-1}$, and as an Sl_{n+1}-module, the sl_{n+1}-module of highest weight $-\alpha_j$ has highest T-weight $\lambda = \lambda_{j+1} + (p-2)\lambda_j + \lambda_{j-1}$. The line spanned by the cohomology class of $\bar{g}_j$ in $H^1(sl_{n+1}, V_j)$ is B-stable of weight $p \cdot \lambda_j$ (except possibly for sl_3 in characteristic 3). Consequently, $H^1(sl_{n+1}, V_j)$ has a G-composition factor of highest weight $p \cdot \lambda_j$, and in particular, $H^1(sl_{n+1}, V_1)$ has a composition factor isomorphic to the first Frobenius power $\mathrm{Id}^{(p)}$ (see [4]) of the identity representation Id of Sl_{n+1} on k^{n+1}. In the next section, we show that if $p \nmid 3$, then $H^1(sl_3, V_1)$ is isomorphic as an Sl_3-module to $\mathrm{Id}^{(p)}$.

3. The one-cohomology of Sl_3 and sl_3

3.1. Lemma. *Let G be an affine algebraic group with Lie algebra L, and let T be a torus in G with Lie algebra $\mathbf{h}$. Let V be a finite-dimensional U_L^+-module and let V^L be the space of L-invariants. Then the canonical image of $H^1(L, V)$ in $H^1(\mathbf{h}, V)$ lies in the image of $H^1(\mathbf{h}, V^L)$ in $H^1(\mathbf{h}, V)$.*

Proof. First we show that $H^1(\mathbf{h}, V^{\mathbf{h}}) = H^1(\mathbf{h}, V)$.

Let G^1 be the kernel of the Frobenius morphism of G. Since V is a unitary U_L-module, V is a G^1-module (see [4]). V is completely reducible as an $\mathbf{h}$-module, since $\mathbf{h}$ and T^1 stabilize the same subspaces and T^1 is a diagonalizable group scheme. Let V_λ be the subspace of V of vectors of $\mathbf{h}$-weight λ. Then $H^1(\mathbf{h}, V) = H^1(\mathbf{h}, V^{\mathbf{h}}) + \sum_{\lambda \neq 0} H^1(\mathbf{h}, V_\lambda)$, and we must show that $H^1(\mathbf{h}, V_\lambda) = 0$ for $\lambda \neq 0$. We may suppose that the dimension of V_λ is one in giving the demonstration.

The kernel of any non-zero cocycle f with values in V_λ equals the kernel of λ. In fact, for any $t \in \text{kernel } f$ and $t' \in \mathbf{h}$,

$$0 = f([t, t']) = t \cdot f(t') - t' \cdot f(t) = t \cdot f(t') = \lambda(t) f(t').$$

Therefore, $t \in \ker(\lambda)$, and $\ker(f) = \ker(\lambda)$. Let $t \in \mathbf{h} - \ker(f)$ and let ∂^0 be the zero-th coboundary operator for Lie algebra cohomology. f is the coboundary $\partial^0((1/\lambda(t))f(t))$.

Therefore, in proving the lemma, we may suppose that $f: L \to V$ is a one-cocycle that maps $\mathbf{h}$ into $V^{\mathbf{h}}$.

The diagonalizable group T^1 acts completely reducibly on L via the adjoint representation; hence, so does $\mathbf{h}$ via the inner action ad. Let $L = \sum_{\lambda \in \mathbf{h}^*} L_\lambda$ be the decomposition into weight spaces. For $l \in L_\lambda$ and $t \in \mathbf{h}$, the application of f to the relation $[t, l] = \lambda(t) l$ gives the relation

$$(t - \lambda(t)) \cdot f(l) = l \cdot f(t). \qquad (*)$$

Let s be the λ-component of $f(l)$. Since $f(t)$ lies in $V^{\mathbf{h}}$, $l \cdot f(t)$ has weight λ and

$$(t - \lambda(t)) \cdot s = l \cdot f(t).$$

Therefore, $l \cdot f(t) = 0$ and $f(t)$ lies in V^L.

3.2. PROPOSITION. *Let $sl_{n+1,k}$ be a special linear Lie algebra other than sl_3 in characteristic 3. Let V_1 be the irreducible sl_{n+1}-module of highest weight $-\alpha_1$, and let $V_{-\alpha_1}$ be the space of vectors in V_1 of weight $-\alpha_1$. If the dimension of $V_{-\alpha_1}$ is one, then $H^1(sl_{n+1}, V_1)$ is isomorphic as an Sl_{n+1}-module to the Frobenius power of the identity representation of Sl_{n+1} on k^{n+1}.*

Proof. $H^1(sl_{n+1}, V_1)$ has the composition factor $\mathrm{Id}^{(p)}$ of dimension $n+1$, by Section 2.3. It will suffice then to show that $\dim_k H^1(sl_{n+1}, V_1) \leq n+1$.

First we show that the kernel of the restriction mapping

$$\pi: H^1(sl_{n+1}, V_1) \to H^1(\mathbf{n}^- + \mathbf{h}, V_1)$$

has dimension ≤ 1. (This fact does not require the hypothesis.) Represent a cohomology class in the kernel of π by a one-cocycle $f: sl_{n+1} \to V_1$ which is 0 on $\mathbf{n}^- + \mathbf{h}$. The computation

$$0 = f([Y_i, X_j]) = Y_i \cdot f(X_j) - X_j \cdot f(Y_i) = Y_i \cdot f(X_j)$$

shows that $f(X_j)$ is a lowest weight vector in V_1. The lowest weight of V_1 is the image $w \cdot (-\alpha_1) = \alpha_n$ of the highest weight $-\alpha_1$ under the opposite involution w in the Weyl group of sl_{n+1}. Therefore, the weight of $f(X_j)$ is α_n; at the same time, the weight of $f(X_j)$ is α_j by the computation

$$f(\alpha_j(t)X_j) = f([t, X_j]) = t \cdot f(X_j) - X_j \cdot f(t) = t \cdot f(X_j) \quad \text{for} \quad t \in \mathbf{h}.$$

Thus, we see that $f(X_j) = 0$ for $j \neq n$, and that f is determined by its value at X_n alone. Since $f(X_n)$ lies in the space of lowest weight vectors, which has dimension one, the dimension of the kernel of π is ≤ 1.

Second we show that $H^1(\mathbf{n}^- + \mathbf{h}, V_1)$ has dimension $\leq n$. The Weyl group of sl_{n+1} acts transitively on $-\Delta = \{-\alpha_i\}_{i=1}^n$. Therefore, there is an element of W (realized as an element of Sl_{n+1}) that transforms $V_{-\alpha_1}$ into $V_{-\alpha_i}$; consequently, each $V_{-\alpha_i}$ has dimension 1. Since V_1 is an irreducible and nontrivial sl_{n+1}-module, $V_1^{\mathbf{h}} = (0)$. By the lemma in Section 3.1, we may represent a cohomology class in $H^1(\mathbf{n}^- + \mathbf{h}, V_1)$ by a cocycle f which is 0 on h. Therefore, $f(Y_j)$ is a vector of weight $-\alpha_j$, and f is determined by the family

$$\{f(Y_j) \in V_{-\alpha_j}\}_{j=1}^n.$$

$H^1(\mathbf{n}^- + \mathbf{h}, V_1)$ has dimension $\leq n$, since each $V_{-\alpha_j}$ has dimension 1.

3.3. LEMMA. *Let V_1 be the irreducible sl_3-module of highest weight $-\alpha_1$. The dimension of $V_{-\alpha_1}$ is 1 if $p \neq 3$.*

Proof. Take the irreducible sl_{n+1}-module V of highest Sl_{n+1}-weight $\lambda_{j+1} + (p-2)\lambda_j + \lambda_{j-1} = p\lambda_j - \alpha_j$. The opposite involution w, which maps $(h_i)_i$ in $\mathbf{h}$ to $(h_{n+2-i})_i$, transforms the highest weight $p\lambda_j - \alpha_j$ into the lowest weight

$$w \cdot (p\lambda_j - \alpha_j) = \alpha_{n+1-j} - p\lambda_{n+1-j}.$$

The difference $(p\lambda_j-\alpha_j)-w\cdot(p\lambda_j-\alpha_j)=p(\lambda_j+\lambda_{n+1-j})-(\alpha_j+\alpha_{n+1-j})$ restricts the possible weights of V to a certain range. For $j=1$,

$$p(\lambda_1+\lambda_n)-(\alpha_1+\alpha_n)=(p-1)\alpha_1+p\alpha_2+p\alpha_3+\cdots+p\alpha_{n-1}+(p-1)\alpha_n$$

limits the possible Sl_{n+1}-weights of V to those of the form

$$\alpha=(p\lambda_1-\alpha_1)-\sum_i a_i\alpha_i,$$

$0\leq a_1<p$, $0\leq a_i\leq p$ for $1<i<n$, $0\leq a_n<p$, by [1, A, 5.3].

When $n=2$, the possible weights of a vector v in V have the form $\alpha=(p\lambda_1-\alpha_1)-(a_1\alpha_1+a_2\alpha_2)$, $0\leq a_1$, $a_2\leq p-1$. Since the sl_3-weights are figured modulo p, v has sl_3-weight $-\alpha_1$ if and only if α and $-\alpha_1$ are congruent modulo p, that is, if $a_1\alpha_1+a_2\alpha_2=0$ as an sl_3-weight. By the lemma of Section 2.3, α_1 and α_2 are independent over k if $p\neq 3$. Thus, the only vectors of sl_3-weight $-\alpha_1$ in V_1 are those in the one-dimensional highest Sl_3-weight space. This completes the proof of the lemma.

3.4. THEOREM. *At the irreducible sl_3-module V_1 of highest weight $-\alpha_1$, $H^1(sl_3, V_1)$ is isomorphic as an Sl_3-module to the Frobenius power of the identity representation of Sl_3 on k^3 if the characteristic of k is not* 3.

Proof. Proposition 3.2 and Lemma 3.3 establish this theorem.

3.5. Comparison of the cohomology of Sl_3 with that of sl_3. The highest Sl_3-weight of V_1 is $\lambda_2+(p-2)\lambda_1$.

COROLLARY 1. *The group cohomology $H^1(Sl_3, V_1)$ is zero if $p\neq 3$.*

Proof. Let Sl_3^1 be the kernel of the Frobenius morphism of Sl_3. The exact sequence of group schemes $1\to Sl_3^1\to Sl_3\to Sl_3/Sl_3^1\to 1$ induces an exact sequence of cohomology spaces

$$1\to H^1(Sl_3/Sl_3^1, V_1^{Sl_3^1})\to H^1(Sl_3, V_1)\to H^1(Sl_3^1, V_1)^{Sl_3}, \qquad (*)$$

by [4, Lemma 5.1], once one adds to that lemma the fact that the canonical map

$$H^1(Sl_3, V_1)\to H^1(Sl_3^1, V_1)$$

has its image in the Sl_3-fixed part. The term $H^1(Sl_3/Sl_3^1, V_1^{Sl_3^1})$ is zero since $V_1^{Sl_3^1}=V_1^{sl_3}=(0)$. Once we show that the term $H^1(Sl_3^1, V_1)^{Sl_3}$ is (0), we will know that $H^1(Sl_3, V_1)$ is (0).

There is a canonical injection $H^1(Sl_3^1, V_1)\hookrightarrow H^1(sl_3, V_1)$ which corresponds to the map

$$\mathrm{Ext}_{Sl_3^1}(k, V_1)\to \mathrm{Ext}_{sl_3}(k, V_1)$$

that takes an Sl_3^1-extension to its underlying differential extension. This map is a map of Sl_3-modules, and so,

$$H^1(Sl_3^1, V_1)^{Sl_3}\hookrightarrow H^1(sl_3, V_1)^{Sl_3}=((\mathrm{Id})^{(p)})^{Sl_3}=(0).$$

COROLLARY 2. *$H^1(Sl_3, V_1 \otimes (\mathrm{Id}^{(p)})^*)$ is non-zero if the characteristic of k is not* 3.

Proof. Theorem 3.4 and the corollary to Theorem 2.2 establish this corollary.

Remark. One can show that the dimension of $H^1(Sl_3, V_1 \otimes (\mathrm{Id}^{(p)})^*)$ is one if $p \neq 3$.

Remark. Theorem 3.4 holds with sl_4 in place of sl_3 if the characteristic of k is not 2. In fact, Lemma 3.3 may be easily established for sl_4 when $p \neq 2$.

Remark. If the characteristic of k is 3, $H^1(sl_3, V_1)$ also has a composition factor that is isomorphic to the trivial module k. This factor is the image of the one-dimensional module $H^1(Sl_3, V_1)$ under the canonical map from $H^1(Sl_3, V_1)$ to $H^1(sl_3, V_1)$. (See [5, Table 4.5] for the fact that $H^1(Sl_3, V_1)$ has dimension one when the characteristic of k is 3.)

3.6. Questions. (a) Is $H^1(sl_{n+1}, V_1)$ isomorphic as an Sl_{n+1}-module to the Frobenius power of the identity representation of the group if the characteristic of k does not divide $n+1$?

(b) More generally, is $H^1(L, V_j)$ irreducible except possibly if $p \mid \dim(h)+1$? (V_j is the module given in Theorem 2.2.)

(c) At which irreducible modules other than those produced in Theorem 2.2 does L have non-zero one cohomology?

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