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THE LOCALLY FREE CLASSGROUP OF THE SYMMETRIC GROUP

BY

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1. Introduction and statement of results

Let S_n denote the symmetric group on n letters, and let $\mathrm{Cl}\,(\mathbf{Z}S_n)$ be the classgroup of finitely generated locally free $\mathbf{Z}S_n$ modules. $\mathrm{Cl}\,(\mathbf{Z}S_n)$ has been studied in [5], [6] and [11]. (See also [12]). By use of two sets of homomorphisms on $\mathrm{Cl}\,(\mathbf{Z}S_n)$ namely those developed by A. Fröhlich in [1], and those developed by the author in [9], we are able to describe $\mathrm{Cl}\,(\mathbf{Z}S_n)$, up to a two group, in terms of two groups of polynomials.

I should like to express my thanks to S. Ullom who originally suggested to me that the representation theory of the symmetric group might be particularly applicable to the calculation of the locally free classgroup.

Let $\mathbf{Z}$ denote the ring of rational integers and let $\mathbf{Q}$ be the field of rationals. If l is a prime of $\mathbf{Z}$, we define $\mathbf{Z}_l$ to be the ring of l-adic integers and $\mathbf{Q}_l$ to be the rational l-adic field. If Γ is a finite group we let R_Γ be the ring of virtual characters of Γ. For any ring R we denote the group of units of R by R^*.

Let $\Lambda_n^{(m)}$ be the additive group of symmetric polynomials of degree n over $\mathbf{Z}$, in the m variables $x_1, x_2 \cdots x_m$.

We have a homomorphism of groups $\Lambda_n^{(m+1)} \to \Lambda_n^{(m)}$ given by setting $x_{m+1}=0$. We let $\Lambda_n = \lim_{\leftarrow} \Lambda_n^{(m)}$ (the limit being taken with respect to the above projective system). For each rational prime l we set $\Lambda_{n,l} = \mathbf{Z}_l \otimes_{\mathbf{Z}} \Lambda_n$.

In the usual way we identify the conjugacy classes of S_n with the partitions of n (via cycle structure). If π is a partition of n then $|\pi|$ denotes the number of elements in the conjugacy class π. For $a \in \mathbf{Z}$, π^a denotes that conjugacy class to which the ath powers of elements of π belong.

If π is a partition of n, $n = r_1 + \cdots + r_k$, then we define the symmetric polynomial $\sigma_\pi^{(m)} \in \Lambda_n^{(m)}$ by setting

$$\sigma_\pi^{(m)} = \prod_{i=1}^{k} (x_1^{r_i} + \cdots x_m^{r_i}).$$

We set $\sigma_\pi = \lim_{\leftarrow} \sigma_\pi^{(m)}$.

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For $l \neq 2$, we define

$$\Xi_{n,l} = \left\{ \sum_{\pi} a_\pi \sigma_\pi \;\middle|\; \sum a_\pi \sigma_\pi \in \Lambda_{n,l};\ a_\pi \in \mathbf{Q}_l;\ \text{for each } \pi, \left(l a_\pi - \sum_{\pi_i \mid \pi_i^l = \pi} a_{\pi_i} \right) \in \mathbf{Z}_l \right\}$$

If $l = 2$, we define

$$\Xi_{n,2} = \left\{ \sum_{\pi} a_\pi \sigma_\pi \;\middle|\; \sum a_\pi \sigma_\pi \in \Lambda_{n,2};\ a_\pi \in \mathbf{Q}_2;\ \text{for each } \pi, \left(2 a_\pi - \sum_{\pi_i \mid \pi_i^l = \pi} a_{\pi_i} \right) \in \tfrac{1}{2}\mathbf{Z}_2 \right\}$$

We let $s(l)$ denote the number of l-singular conjugacy classes of S_n (i.e. those π such that $\pi^l \neq \pi$). Our main result is:

THEOREM 1. *There is an exact sequence of abelian groups*

$$1 \to H \to \mathrm{Cl}\,(\mathbf{Z}S_n) \to \prod_{l \leq n} \frac{\Lambda_{n,l}}{\Xi_{n,l}} \times \frac{\prod_{l \leq n} (\mathbf{F}_l^*)^{s(l)}}{G} \to 1$$

where G (resp. H) is an elementary two group (resp. an abelian two group) of rank less than or equal to the number of conjugacy classes of S_n.

In Section 5 we firstly show how to calculate the order of certain elementary symmetric polynomials in the quotient group $\Lambda_{n,l}/\Xi_{n,l}$, and then we show that $\Lambda_{n,l}/\Xi_{n,l}$ can be interpreted as the classes of certain induced Swan modules.

Remark. For actual computation it is worth remarking that for any $k \geq n$, the natural projection $\Lambda_n \to \Lambda_n^{(k)}$ is in fact an isomorphism of abelian groups, so that $\Lambda_{n,l}/\Xi_{n,l}$ can be turned into a quotient of two polynomial groups in a finite number of variables.

2. Components of $\mathrm{Cl}\,(\mathbf{Z}S_n)$

We now recall certain results on classgroups. The main reference is [1], and in general we preserve the notation of [1] and [9]. From Appendix II of [1] we have an isomorphism of groups

$$\mathrm{Cl}\,(\mathbf{Z}S_n) \xrightarrow{\sim} \frac{\prod_{l \leq n} \mathrm{Hom}\,(R_{S_n}, \mathbf{Z}_l^*)}{\mathrm{Hom}\,(R_{S_n}, \pm 1) \prod_{l \leq n} \mathrm{Det}\,(\mathbf{Z}_l S_n^*)} \tag{2.1}$$

(where we regard ± 1 as embedded diagonally in $\prod_{l \leq n} \mathbf{Z}_l^*$).

Remark. In going from the isomorphism given in [1] to 2.1 we are using two facts. Firstly that all complex representations of S_n can be achieved over $\mathbf{Q}$ (see 13.1 of [7]). Secondly that $D(\mathbf{Z}S_n) = \mathrm{Cl}\,(\mathbf{Z}S_n)$, because the classgroup of any maximal order of $\mathbf{Q}S_n$ is the product of several classgroups of $\mathbf{Z}$.

For $l\neq 2$ (resp. for $l=2$) we define r_l (resp. r_2) to be the reduction mod l (resp. mod 4) homomorphism given by the composite

$$\mathrm{Det}\,(\mathbf{Z}_l S_n^*) \hookrightarrow \mathrm{Hom}\,(R_{S_n}, \mathbf{Z}_l^*) \to \mathrm{Hom}\,(R_{S_n}, \mathbf{F}_l^*) \quad \text{if} \quad l\neq 2$$

(resp. $\mathrm{Det}\,(\mathbf{Z}_2 S_n^*) \hookrightarrow \mathrm{Hom}\,(R_{S_n}, \mathbf{Z}_2^*) \to \mathrm{Hom}\,(R_{S_n}, \pm 1_2)$ if $l=2$ where 1_2 denotes the "one" of $\mathbf{Q}_2$; we make this distinction between "ones" because in this section we shall need to distinguish between 1_2, and 1 diagonally embedded in $\prod \mathbf{Z}_l^*$).

For $l\neq 2$ we set

$$C_l^{(1)} = \frac{\mathrm{Hom}\,(R_{S_n}, \mathbf{F}_l^*)}{\mathrm{Im}\,(r_l)}, \quad C_l^{(2)} = \frac{\mathrm{Hom}\,(R_{S_n}, 1+l\mathbf{Z}_l)}{\mathrm{Ker}\,(r_l)},$$

and if $l=2$ we define

$$C_2^{(1)} = \frac{\mathrm{Hom}\,(R_{S_n}, \pm 1_2)}{\mathrm{Im}\,(r_2)}, \quad C_2^{(2)} = \frac{\mathrm{Hom}\,(R_{S_n}, 1+4\mathbf{Z}_2)}{\mathrm{Ker}\,(r_2)}.$$

Let G_1 be the sub-group of $\mathrm{Hom}\,(R_{S_n}, \pm 1)$ which corresponds to $\mathrm{Im}\,(r_2)$ under the canonical isomorphism $\mathrm{Hom}\,(R_{S_n}, \pm 1) \cong \mathrm{Hom}\,(R_{S_n}, \pm 1_2)$, and let G_1' (resp. G_1'') be the natural projection of G_1 into

$$\prod_{2<l\leq n} \mathrm{Hom}\,(R_{S_n}, \mathbf{Z}_l^*) \quad \Bigl(\text{resp.} \prod_{2<l\leq n} \mathrm{Hom}\,(R_{S_n}, \mathbf{F}_l^*)\Bigr).$$

For brevity we let

$$H_1 = \mathrm{Hom}\,(R_{S_n}, \pm 1), \quad H_{1_2} = \mathrm{Hom}\,(R_{S_n}, \pm 1_2), \quad D_l = \mathrm{Det}\,(\mathbf{Z}_l S_n^*).$$

Projection into $\mathrm{Hom}\,(R_{S_n}, \mathbf{Z}_2^*)$ and reduction mod (4) yields a commutative diagram

$$\begin{array}{ccccccc}
1 \to \mathrm{Ker}\,(r_2) \times G_1' \prod\limits_{2<l\leq n} D_l & \to & H_1 \prod\limits_{l\leq n} D_l & \to & H_{1_2} & \to & 1 \\
 & & \uparrow & & & & \\
1 & \to & H_1 & \to & H_{1_2} & \to & 1.
\end{array}$$

So by the Snake Lemma we have an isomorphism

$$\frac{H_1 \prod\limits_{l\leq n} D_l}{H_1} \cong \mathrm{Ker}\,(r_2) \times G_1' \prod_{2<l\leq n} D_l.$$

However, for each $l\neq 2$, we have $D_l \cong \mathrm{Ker}\,(r_l) \times \mathrm{Im}\,(r_l)$ (since $\mathrm{Im}\,(r_l)$ has order prime to l, whilst $\mathrm{Ker}\,(r_l)$ is a pro-l-group). This decomposition induces an isomorphism

$$\frac{H_1 \prod\limits_{l\leq n} D_l}{H_1} \cong \prod_{l\leq n} \mathrm{Ker}\,(r_l) \times G_1'' \prod_{2<l\leq n} \mathrm{Im}\,(r_l).$$

Similarly we have an isomorphism

$$\frac{\prod_{l\le n} \operatorname{Hom}(R_{S_n}, \mathbf{Z}_l^*)}{H_1}$$
$$\cong \operatorname{Hom}(R_{S_n}, 1+4\mathbf{Z}_2)\times \prod_{2<l\le n} \operatorname{Hom}(R_{S_n}, 1+l\mathbf{Z}_l)\times \prod_{2<l\le n} \operatorname{Hom}(R_{S_n}, \mathbf{F}_l^*).$$

So from (2.1) we have an isomorphism

$$\mathrm{Cl}(\mathbf{Z}S_n)\cong \prod_{l\le n} C_l^{(2)}\times \frac{\prod_{2<l\le n} C_l^{(1)}}{G_2}$$

where G_2 denotes the image of $G_1''\prod_{2<l\le n}\operatorname{Im}(r_l)$ in $\prod_{2<l\le n} C_l^{(1)}$. Because G_2 is an elementary two group whose rank is bounded by the $\mathbf{Z}$-rank of R_{S_n} (i.e. by the number of conjugacy classes of S_n), in order to prove Theorem 1 it is sufficient to show:

THEOREM 2. *For each prime $l\le n$,*

(i) *if $l\ne 2$ then $C_l^{(1)}\cong (\mathbf{F}_l^*)^{(s(l))}$,*
(ii) *if $l\ne 2$ then $C_l^{(2)}\cong \Lambda_{n,l}/\Xi_{n,l}$,*
whilst if $l=2$ we have an exact sequence

$$1\to \tilde{H}\to C_2^{(2)}\to \Lambda_{n,2}/\Xi_{n,2}\to 1$$

where $\tilde{H}$ is an abelian 2-group of rank at most the number of conjugacy classes of S_n.

3. Proof of Theorem 2(i)

We now recall certain facts and definitions from the theory of modular representations. The main reference is [7]. Let Γ be an arbitrary finite group. We define $R_{\Gamma,l}$ to be the Grothendieck group of finitely generated $\mathbf{F}_l\Gamma$-modules. From 15.2 of [7] we have the decomposition homomorphism $d_l\colon R_{\Gamma,\mathbf{Q}_l}\to R_{\Gamma,l}$ where $R_{\Gamma,\mathbf{Q}_l}$ is the Grothendieck group of $\mathbf{Q}_l\Gamma$-modules.

We define $P_{\mathbf{Z}_l}$ (resp. $P_{\mathbf{F}_l}$) to be the Grothendieck group of finitely generated projective $\mathbf{Z}_l\Gamma$ (resp. $\mathbf{F}_l\Gamma$) modules. Reduction mod l yields an isomorphism $\theta\colon P_{\mathbf{Z}_l}\xrightarrow{\sim} P_{\mathbf{F}_l}$.

From 15.3 of [7] we have an injective homomorphism

$$e_l\colon P_{\mathbf{F}_l}\to R_{\Gamma,\mathbf{Q}_l}.$$

It is well known that $\operatorname{Im}(e_l)\cap\operatorname{Ker}(d_l)=(0)$ and that

$$(R_{\Gamma,\mathbf{Q}_l}\colon \operatorname{Im}(e_l)+\operatorname{Ker}(d_l))=l^N \quad \text{for some } N\ge 0.$$

So because $(|\mathbf{F}_l^*|, l)=1$ we have an isomorphism

(3.1) $$\operatorname{Hom}(R_{\Gamma,\mathbf{Q}_l}, \mathbf{F}_l^*)\cong \operatorname{Hom}(\operatorname{Ker}(d_l), \mathbf{F}_l^*)\times \operatorname{Hom}(\operatorname{Im}(e_l), \mathbf{F}_l^*).$$

Now Fröhlich has shown in Appendix III of [1] that under the homomorphism given by the composite of restriction to $\mathrm{Ker}\,(d_l)$ and reduction mod l, $\mathrm{Det}\,(\mathbf{Z}_l\Gamma^*) \to 1$. Thus

$$(3.2)\qquad \frac{\mathrm{Hom}\,(R_{\Gamma,\mathbf{Q}_l}, \mathbf{F}_l^*)}{r_l(\mathrm{Det}\,(\mathbf{Z}_l\Gamma^*))|_{R_{\Gamma,\mathbf{Q}_l}}} \xrightarrow{\sim} \frac{\mathrm{Hom}\,(\mathrm{Im}\,(e_l), \mathbf{F}_l^*)}{r_l(\mathrm{Det}\,(\mathbf{Z}_l\Gamma^*))|_{\mathrm{Im}(e_l)}} \times \mathrm{Hom}\,(\mathrm{Ker}\,(d_l), \mathbf{F}_l^*).$$

From 16.1 of [7] we have that the Cartan homomorphism $c_l = d_l \circ e_l$ is an injection,

$$(3.3)\qquad c_l : P_{\mathbf{F}_l} \to R_{\Gamma,l},$$

and further c_l has a finite cokernel of l-power order, $(R_{\Gamma,l}\colon \mathrm{Im}\,(c_l)) = l^M$ say.

We now suppose that all complex representations of Γ are achievable over $\mathbf{Q}$; we may identify R_Γ with $R_{\Gamma,\mathbf{Q}_l}$. It is well known that

$$\mathrm{Ker}\,(d_l) = \{\chi \in R_{\Gamma,\mathbf{Q}_l} \mid \chi(\gamma) = 0 \text{ for all } l\text{-regular } \gamma \in \Gamma\}.$$

Hence $\mathrm{Ker}\,(d_l)$ is a free abelian group of rank equal to the number of l-singular conjugacy classes of Γ, $s(l)$. Therefore we have an isomorphism of abelian groups

$$(3.4)\qquad \mathrm{Hom}\,(\mathrm{Ker}\,(d_l), \mathbf{F}_l^*) \cong (\mathbf{F}_l^*)^{(s(l))}.$$

So in order to prove Theorem 2(i), from (3.2) we are required to show

$$(3.5)\qquad r_l(\mathrm{Det}\,(\mathbf{Z}_l\Gamma^*))|_{\mathrm{Im}\,e_l} = \mathrm{Hom}\,(\mathrm{Im}\,e_l, \mathbf{F}_l^*).$$

Let $\{N_i\}$ represent the distinct isomorphism classes of simple $\mathbf{F}_l\Gamma$-modules. We may view the N_i as a $\mathbf{Z}$-basis of $R_{\Gamma,l}$. From (3.3), $l^M N_i \in \mathrm{Im}\,(c_l)$ for each i, and so we may view the $l^M N_i$ as projective $\mathbf{F}_l\Gamma$ modules. We set $P_i = \theta^{-1}(l^m N_i)$ and we let χ_i be the complex character associated to the $\mathbf{Q}_l\Gamma$ module $\mathbf{Q}_l \otimes_{\mathbf{Z}_l} P_i$ (i.e. $e_l(l^M N_i) = \chi_i$). We define T to be the sub-group of R_Γ generated by the χ_i. Then we have for some $M' \geq 0$ that $(\mathrm{Im}\,(e_l)\colon T) = l^{M'}$, and so we are reduced to showing

$$(3.6)\qquad r_l(\mathrm{Det}\,(\mathbf{Z}_l\Gamma^*))|_T = \mathrm{Hom}\,(T, \mathbf{F}_l^*).$$

We now use Wedderburn's theorem on the structure of semi-simple rings to prove (3.5). Let $\mathfrak{r}$ be the radical of the ring $\mathbf{F}_l\Gamma$. Then by Wedderburn's Theorem (see [3] Chapter XVII) we have an isomorphism of rings

$$(3.7)\qquad \mathbf{F}_l\Gamma/\mathfrak{r} \cong \oplus_{N_i} M_{n_i}(\mathbf{F}_{q_i})$$

where

$$\mathbf{F}_{q_i} = \mathrm{End}_{\mathbf{F}_l\Gamma}(N_i) \quad \text{and} \quad M_{n_i}(\mathbf{F}_{q_i}) = \mathrm{End}_{\mathbf{F}_{q_i}}(N_i).$$

Let $r \in \mathbf{F}_l^*$ and choose $s_i \in \mathbf{F}_{q_i}^*$ such that $N_{\mathbf{F}_{q_i}/\mathbf{F}_l}(s_i) = r$ (the norm from $\mathbf{F}_{q_i}^*$ to

$\mathbf{F}_l^*$). We choose $\bar{\alpha}_{i,r} \in \mathbf{F}_l\Gamma$ such that under the isomorphism (3.7),

$$\bar{\alpha}_{i,r} \text{ has image } \begin{cases} \begin{pmatrix} 1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & 1 \end{pmatrix} & \text{in } M_{n_j}(\mathbf{F}_{q_j}), \quad j \neq i, \\[2ex] \begin{pmatrix} s_i & & & & 0 \\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & 1 \end{pmatrix} & \text{in } M_{n_i}(\mathbf{F}_{q_i}). \end{cases}$$

We choose $\alpha_{i,r} \in \mathbf{Z}_l\Gamma$ such that $\alpha_{i,r} \to \bar{\alpha}_{i,r}$ under the surjection $\mathbf{Z}_l\Gamma \to \mathbf{F}_l\Gamma$. We consider $\mathrm{Det}_{\chi_j}(\alpha_{i,r})$:

$$\begin{aligned} \mathrm{Det}_{\chi_j}(\alpha_{i,r}) &= \text{det of } \alpha_{i,r} \text{ viewed as a } \mathbf{Z}_l \text{ endomorphism of } \theta^{-1}(l^M N_j) \\ &\equiv \text{det of } \bar{\alpha}_{i,r} \text{ viewed as an } \mathbf{F}_l \text{ endomorphism of } l^M N_j, \bmod (l). \end{aligned}$$

However, because $\bar{\alpha}_{i,r}$ commutes with $\mathbf{F}_{q_i}$ action it is easily seen that det of $\bar{\alpha}_{i,r}$ viewed as an $\mathbf{F}_l$ endomorphism of $l^M N_j$, is equal to the $l^{M\text{th}}$ power of $N_{\mathbf{F}_{q_i}/\mathbf{F}_l}$ (det of $\bar{\alpha}_{i,r}$ viewed as an $\mathbf{F}_{q_i}$ endomorphism of N_j). Thus, by our choice of $\alpha_{i,r}$,

$$\begin{aligned} \mathrm{Det}_{\chi_j}(\alpha_{i,r}) &\equiv \begin{cases} N_{\mathbf{F}_{q_i}/\mathbf{F}_l}(s_i)^{l^M}, & i = j \\ 1, & i \neq j' \end{cases} \\ &\equiv \begin{cases} r^{l^M}, & i = j, \\ 1, & i \neq j, \end{cases} \quad \text{by choice of } s_i, \\ &\equiv \begin{cases} r, & i = j, \\ 1, & i \neq j, \end{cases} \end{aligned}$$

and so (3.6) is shown.

4. Proof of Theorem 2(ii)

Firstly we recall some facts on the representation theory of the symmetric group. Our main references for this are [2] and [4]. From page 13 of [2] we have an isomorphism of additive groups $\theta: R_{S_n} \xrightarrow{\sim} \Lambda_n$. Under this isomorphism the irreducible characters of S_n map to the Schur functions attached to the various partitions of n. If π is a partition of n, then $\theta^{-1}(\sigma_\pi)$ is the virtual character which takes the following values:

$$(4.1)\qquad \theta^{-1}(\sigma_\pi)(\gamma) = \begin{cases} n!/|\pi| & \text{if } \gamma \in \pi, \\ 0 & \text{if } \gamma \notin \pi. \end{cases}$$

θ then induces an isomorphism

$$(4.2)\qquad \operatorname{Hom}(R_{S_n}, 1+l\mathbf{Z}_l) \cong \operatorname{Hom}(\Lambda_n, 1+l\mathbf{Z}_l),$$

We define the l-adic logarithm log: $\mathbf{Z}_l^* \to l\mathbf{Z}_l^+$ via

$$\log(x) = \frac{-1}{l-1}\sum \frac{(1-x^{l-1})^n}{n} \quad \text{for } x \in \mathbf{Z}_l^*.$$

(Analogously, of course, one can define an l-adic logarithm for the units of the integers of any finite extension of $\mathbf{Q}_l$.) It is easily seen that

$$\log\colon 1+l\mathbf{Z}_l \to l\mathbf{Z}_l, \quad l \neq 2,$$
$$\log\colon 1+4\mathbf{Z}_2 \to 4\mathbf{Z}_2, \quad l = 2,$$

are isomorphisms. Hence $(1/l)\log$ for $l\neq 2$, and $(1/4)\log$ for $l=2$, induce isomorphisms

$$(4.3)\qquad \begin{aligned} \Phi_l&\colon \operatorname{Hom}(R_{S_n}, 1+l\mathbf{Z}_l) \cong \operatorname{Hom}(\Lambda_n, \mathbf{Z}_l^+) \quad \text{for} \quad l \neq 2, \\ \Phi_2&\colon \operatorname{Hom}(R_{S_n}, 1+4\mathbf{Z}_2) \cong \operatorname{Hom}(\Lambda_n, \mathbf{Z}_2^+) \quad \text{for} \quad l = 2. \end{aligned}$$

If $\{\lambda_\pi\}$ denotes the $\mathbf{Z}$ basis of Λ_n given by the Schurfunctions, then the map $f \mapsto \sum f(\lambda_\pi).\ \lambda_\pi$ yields an isomorphism

$$(4.4)\qquad K_l\colon \operatorname{Hom}(\Lambda_n, \mathbf{Z}_l^+) \xrightarrow{\sim} \Lambda_{n,l}^+.$$

By means of a classical polynomial identity (see Chapter 1.3 of [2]) we have that in $\Lambda_n \otimes_{\mathbf{Z}} \Lambda_n$,

$$(4.5)\qquad \sum_\pi \lambda_\pi \otimes \lambda_\pi = \sum \frac{|\pi|}{n!}\sigma_\pi \otimes \sigma_\pi.$$

So because K_l is given by evaluation in the left hand factors of (4.5), K_l is also given by

$$(4.6)\qquad K_l(f) = \sum_\pi \frac{|\pi|}{n!} f(\sigma_\pi)\cdot\sigma_\pi.$$

Setting $\eta_l = K_l \circ \Phi_l$ we have

$$\eta_l\colon \operatorname{Hom}(R_{S_n}, 1+l\mathbf{Z}_l) \xrightarrow{\sim} \Lambda_{n,l}, \qquad \eta_2\colon \operatorname{Hom}(R_{S_n}, 1+4\mathbf{Z}_2) \xrightarrow{\sim} \Lambda_{n,2}.$$

The main aim of this section is, of course, to calculate η_l (Ker r_l).

For $\chi \in R_{S_n}$, and for an integer m, we say $\chi \equiv 0 \bmod (m)$ if, and only if, $\chi(\gamma) \equiv 0 \bmod (m)$ for each $\gamma \in S_n$. Define $\Psi^l(\chi)$ to be the central function of S_n given by $\Psi^l(\chi)(\gamma) = \chi(\gamma^l)$ for $\gamma \in S_n$. Because $\Psi^l(\chi)$ can be expressed as a polynomial in exterior powers of χ, $\Psi^l(\chi) \in R_{S_n}$ (see [2] for details).

From [9] we have:

THEOREM 3. *Let $\chi \in R_{S_n}$ with $\chi \equiv 0 \bmod (l^r)$, then for $z \in \mathbf{Z}_l S_n^*$,*

$$\log(\mathrm{Det}_{l\chi - \Psi^l\chi}(z)) \equiv 0 \bmod (l^{r+1}).$$

Remark. In [9] the result is proved only for l-groups, but the proof extends to arbitrary finite groups without difficulty.

We now introduce some further notation. We define a partial order relation $\geq$ (resp. $\geq_l$ for a fixed prime l) on the conjugacy classes of S_n as follows: For conjugacy classes π_1, π_2 of n, we say $\pi_1 \geq \pi_2$ (resp. $\pi_1 \geq_l \pi_2$) if, and only if, for some integer $m \geq 0$, $\pi_1^m = \pi_2$ (resp. $\pi_1^{l^m} = \pi_2$). Let G (resp. G_l) be the directed graph whose vertices are the conjugacy classes of S_n, and where there is an edge from π_1 to π_2 if, and only if, for some prime p, $\pi_1^p = \pi_2$ (resp. $\pi_1^l = \pi_2$), and $\pi_1 \neq \pi_2$. G is a connected graph (since $\pi^m = 1$ for some m). We let $\{\mathscr{L}_{l,i}\}_i$ denote the connected components of G_l. We set $\tau_\pi = \theta^{-1}(\sigma_\pi)$, for each partition π. For a fixed prime l we set

$$\rho_\pi \ (=\rho_\pi(l)) = l\,|\pi|\,\tau_\pi - \Psi^l(|\pi|\,\tau_\pi).$$

From 4.1,

$$\Psi^l(|\pi|\,\tau_\pi) = \sum_{\pi_i \mid \pi_i^l = \pi} |\pi_i|\,\tau_{\pi_i}. \tag{4.7}$$

We define $\Delta_n(l) = \sum_\pi \mathbf{Z}\rho_\pi$,

LEMMA 1. $\mathbf{Q} \otimes_{\mathbf{Z}} \Delta_n(l) = \mathbf{Q} \otimes_{\mathbf{Z}} R_{S_n}$.

Proof. Let $\mathscr{S}_n = \sum_\pi \mathbf{Z}\sigma_\pi$; it is well known that $(\Lambda_n : \mathscr{S}_n) < \infty$; so it is sufficient to show that for each π there exists a non-zero integer m_π such that

$$m_\pi \tau_\pi \in \Delta_n(l). \tag{4.8}$$

Suppose π lies in the connected component $\mathscr{L}_{l,i}$ of G_l. We suppose initially that π is maximal in G_l (i.e. $\pi' \geq \pi \Rightarrow \pi' = \pi$). Then we have $\Psi^l(\tau_\pi) = 0$, or, τ_π, according as $\pi^l \neq \pi$, or, $\pi^l = \pi$; and so $\rho_\pi = l\,|\pi|\,\tau_\pi$, or, $(l-1)\,|\pi|\,\tau_\pi$, respectively.

So now we assume that π is not maximal. By careful choices of π it is easily seen that inductively we may assume that (4.8) holds for all π' such that $\pi' >_l \pi$. Now the sum $\sum_{\pi_i \mid \pi_i^l = \pi}$ clearly contains only π_i such that $\pi_i \geq_l \pi$. So if $\pi_i \neq \pi$, by induction $m_{\pi_i}\tau_{\pi_i} \in \Delta_n(l)$ for some non-zero $m_{\pi_i} \in \mathbf{Z}$. Thus

$$\rho_\pi = l\,|\pi|\,\tau_\pi - \sum_{\pi_i \mid \pi_i^l = \pi, \pi_i \neq \pi} |\pi_i|\,\tau_{\pi_i} - \binom{0}{1}|\pi|\,\tau_\pi \tag{4.9}$$

where the value 0, or, 1 is taken according as $\pi^l \neq \pi$, or, $\pi^l = \pi$. (4.8) now follows.

We define ν_l to be the homomorphism $\nu_l\colon \mathrm{Hom}\,(\Lambda_n, \mathbf{Z}_l^+) \to \prod_\pi \mathbf{Z}_l^+$ given by

$$\nu_l(f) = \prod f(\theta(\rho_\pi)) \quad \text{for } f \in \mathrm{Hom}\,(\Lambda_n, \mathbf{Z}_l),$$

where the product is taken over distinct partitions π of n.

(4.10) By Lemma 1, $(R_{S_n}\colon \Delta_n(l)) < \infty$, so because $\mathbf{Z}_l^+$ is torsion free, ν_l is injective.

We now suppose that $l \neq 2$.

PROPOSITION 1. $\nu_l \circ \Phi_l(\mathrm{Ker}\, r_l) = \prod_\pi n!\, \mathbf{Z}_l$.

Proof. Because $|\pi|\, \tau_\pi \equiv 0 \bmod n!$ (from (4.1)), by Theorem 3 we have, for $z \in \mathbf{Z}_l S_n^*$, $\log\,(\mathrm{Det}_{\rho_\pi}(z)) \equiv 0 \bmod ln!\, \mathbf{Z}_l$. Therefore

$$\nu_l \circ \Phi_l(\mathrm{Ker}\, r_l) \subseteq \prod_\pi n!\, \mathbf{Z}_l,$$

so now we must show $\prod_\pi n!\, \mathbf{Z}_l \subseteq \nu_l \circ \Phi_l(\mathrm{Ker}\, r_l)$. To do this we exhibit various $\mathrm{Det}\,(y_\pi) \in \mathrm{Ker}\,(r_l)$ such that

$$\log\,(\mathrm{Det}_{\rho_{\pi'}}(y_\pi)) \begin{aligned} &= 0 && \text{if } \pi \not\geq \pi', \\ &= u\, ln! && \text{if } \pi' = \pi, \\ &= v_{\pi'}\, ln! && \text{if } \pi > \pi', \end{aligned} \tag{4.11}$$

where $u \in \mathbf{Z}_l^*$, $v_{\pi'} \in \mathbf{Z}_l$. Firstly we need a lemma on the logarithms of determinants. For any finite group Γ, we denote the radical of the ring $\mathbf{Z}_l\Gamma$, by $\mathfrak{r}\,(\mathbf{Z}_l\Gamma)$.

LEMMA 2. *Let χ be a virtual character of a finite group Γ. Let $\alpha \in \mathfrak{r}\,(\mathbf{Z}_l\Gamma)$. Then*

$$\log\,(\mathrm{Det}_\chi\,(1-\alpha)) = -\sum_{n=1}^{\infty} \frac{1}{n}\chi(\alpha^n)$$

Proof. By additivity it is clearly sufficient to prove the Lemma when χ is afforded by a representation, T_χ say. Let $\{a_i\}$ be the roots of the polynomial $\det\,(T_\chi(X-\alpha))$. Because $\alpha \in \mathfrak{r}\,(\mathbf{Z}_l\Gamma)$, the a_i all lie in the maximal ideal of the field to which they belong. We have

$$\mathrm{Det}_\chi\,(1-\alpha) = \prod (1-a_i),$$

and so

$$\log\,(\mathrm{Det}_\chi\,(1-\alpha)) = \sum_i \log\,(1-a_i) = -\sum_{m=1}^{\infty}\sum_i \frac{1}{m} a_i^m = -\sum_{m=1}^{\infty}\frac{1}{m}\chi(\alpha^m).$$

We now return to the proof of Proposition 1. Let γ_π be an element of the conjugacy class π. We can write γ_π uniquely in the form $\gamma_\pi = \gamma'_\pi\gamma''_\pi$ where γ'_π has order prime to l, γ''_π has l-power order and where γ'_π and γ''_π commute. We set

$$y_\pi = \begin{matrix} 1 - l\gamma_\pi & \text{if } \pi^l = \pi, \\ 1 - \gamma'_\pi(1-\gamma''_\pi)^{l-1} & \text{if } \pi^l \neq \pi. \end{matrix}$$

(Clearly $\gamma''_\pi = 1$ if, and only if, $\pi^l = \pi$.) We observe that $y_\pi - 1 \in \imath(\mathbf{Z}_l\langle\gamma_\pi\rangle)$ in both cases, and thus $\operatorname{Det}(y_\pi) \in \operatorname{Ker}(r_l)$. We now show that $\operatorname{Det}(y_\pi)$ satisfies (4.11).

Case 1. $\pi^l = \pi$ (i.e. π is minimal in G_l). Then

$$\rho_\pi = |\pi|\,(l-1)\tau_\pi + \sum_{\pi_i \mid \pi_i{}^l = \pi, \pi_i \neq \pi} |\pi_i|\,\tau_{\pi_i}$$

However, from (4.1), if $\pi_i > \pi$, then we have $\tau_{\pi_i}|_{\langle\gamma_\pi\rangle} = 0$ (zero character). Hence,

$$\log(\operatorname{Det}_{\rho_\pi}(y_\pi)) = (l-1)\,|\pi|\log(\operatorname{Det}_{\tau_\pi}(1 - l\gamma_\pi))$$

and, by Lemma 2,

$$\begin{aligned}\log(\operatorname{Det}_{\tau_\pi}(1-l\gamma_\pi)) &\equiv -l\tau_\pi(\gamma_\pi) \bmod \mathbf{Z}_l l^2 n!\,|\pi|^{-1} \\ &\equiv -ln!\,|\pi|^{-1} \bmod \mathbf{Z}_l l^2 n!\,|\pi|^{-1}.\end{aligned}$$

Thus $\log(\operatorname{Det}_{\rho_\pi}(y_\pi)) = uln!$ for $u \in \mathbf{Z}_l^*$. If $\pi > \pi'$, then, from Theorem 3,

$$\log(\operatorname{Det}_{\rho_{\pi'}}(y_\pi)) \equiv 0 \bmod \mathbf{Z}_l ln!.$$

On the other hand if $\pi \not\geq \pi'$, then by (4.1), $\tau_{\pi'}|_{\langle\gamma_\pi\rangle} = 0$, and further for each π'_i such that $(\pi'_i)^l = \pi'$, $\tau_{\pi'_i} \mid \langle\gamma_\pi\rangle = 0$. Thus $\rho_{\pi'} \mid \langle\gamma_\pi\rangle = 0$, and so

$$\log(\operatorname{Det}_{\rho_{\pi'}}(y_\pi)) = 0.$$

Case 2. $\pi^l \neq \pi$ (i.e. π not minimal in G_l). Arguing as in Case 1 we see that if $\pi > \pi'$, or if, $\pi \not\geq \pi'$ then (4.10) holds. So now we suppose that $\pi' = \pi$, and we are required to show

$$\log(\operatorname{Det}_{\rho_\pi}(y_\pi)) = uln! \quad \text{for some} \quad u \in \mathbf{Z}_l^*. \tag{4.12}$$

Because $\pi^l \neq \pi$, $\rho_\pi = l\,|\pi|\,\tau_\pi - \sum_{\pi_i \mid \pi_i{}^l = \pi} |\pi_i|\,\tau_{\pi_i}$ with $\pi_i \neq \pi$ for all i. Thus by (4.1), for each i, $\log(\operatorname{Det}_{\tau_{\pi_i}}(y_\pi)) = 0$. So it is sufficient to show $\log(\operatorname{Det}_{\tau_\pi}(y_\pi)) = u'n!\,|\pi|^{-1}$ for some $u' \in \mathbf{Z}_l^*$. By Lemma 2,

$$\begin{aligned}\log(\operatorname{Det}_{\tau_\pi}(y_\pi)) &= -\sum_{a=1}^{\infty} \frac{1}{a}\,\tau_\pi((\gamma'_\pi(1-\gamma''_\pi)^{l-1})^a) \\ &= -\sum_{a=1}^{\infty} \frac{1}{a}\,\tau_\pi(\gamma'^a_\pi(1-\gamma''_\pi)^{a(l-1)}).\end{aligned}$$

We set $T_a = \gamma'^a_\pi(1-\gamma''_\pi)^{a(l-1)}$. By the Binomial theorem

$$T_1 = \gamma'_\pi \sum_{r=0}^{l-1} (-1)^r \binom{l-1}{r} \gamma''^r_\pi.$$

But for all $1 \leq r \leq l-1$, $\gamma'_\pi \gamma''^r_\pi$ generate the same subgroup of S_n, viz $\langle \gamma_\pi \rangle$; whence they are S_n conjugate to γ_π. Thus we have

$$\sum_{r=1}^{l-1} \tau_\pi(\gamma'_\pi \gamma''^r_\pi)\binom{l-1}{r}(-1)^r = -\tau_\pi(\gamma_\pi),$$

and since $\tau_\pi(\gamma'_\pi) = 0$, $\tau_\pi(T_1) = -n!\,|\pi|^{-1}$.

We now consider $\tau_\pi(T_a)$ for $a > 1$. Define m_a to be the integer such that $lm_a \leq a(l-1) < l(m_a+1)$. Because $a > 1$, we have $m_a \geq 1$. Again by the Binomial theorem $(1-\gamma''_\pi)^l = 1 - \gamma''^l_\pi + l\alpha$ for some $\alpha \in \mathbf{Z}_l\langle \gamma''_\pi \rangle$, and more generally

(4.13) $$(1-\gamma''_\pi)^{lm_a} = \alpha_1(1-\gamma''^l_\pi) + l^{m_a}\alpha_2 \quad \text{for } \alpha_1, \alpha_2 \in \mathbf{Z}_l\langle \gamma''_\pi \rangle.$$

If $(a, \operatorname{ord}(\gamma'_\pi)) \neq 1$ then the element $\gamma'^a_\pi(1-\gamma''_\pi)^{a(l-1)}$ belongs to the group ring of a proper sub-group of $\langle \gamma_\pi \rangle$. So by (4.1) we have

(4.14) $$\tau_\pi(T_a) = 0.$$

So now we assume that $(a, \operatorname{ord}(\gamma'_\pi)) = 1$. From (4.12),

(4.15) $$\tau_\pi(T_a) \equiv \tau_\pi(\gamma'^a_\pi(1-\gamma''^l_\pi)\alpha_3) \bmod \mathbf{Z}_l l^{m_a} n!\,|\pi|^{-1}$$

for some $\alpha_3 \in \mathbf{Z}_l\langle \gamma''_\pi \rangle$. We now consider $\tau_\pi(\gamma'^a_\pi(1-\gamma''^l_\pi)\gamma''^r_\pi)$ for various integers r.

If $(r, l) \neq 1$ then $\langle \gamma'^a_\pi \gamma''^{l+r}_\pi \rangle \subsetneq \langle \gamma_\pi \rangle$ and $\langle \gamma'^a_\pi \gamma''^l_\pi \rangle \subsetneq \langle \gamma_\pi \rangle$. So by (4.1),

$$\tau_\pi(\gamma'^a_\pi(1-\gamma''^l_\pi)\gamma''^r_\pi) = 0.$$

If $(r, l) = 1$, then $\langle \gamma'^a_\pi \gamma''^r_\pi \rangle = \langle \gamma'^a_\pi \gamma''^{l+r}_\pi \rangle = \langle \gamma_\pi \rangle$. So both $\gamma'^a_\pi \gamma''^r_\pi$ and $\gamma'^a_\pi \gamma''^{l+r}_\pi$ are S_n conjugate to γ_π and thus again

$$\tau_\pi(\gamma'^a_\pi(1-\gamma''^l_\pi)\gamma''^r_\pi) = 0.$$

So from (4.14) and (4.15) we have shown that for all $a > 1$,

$$\tau_\pi(T_a) \equiv 0 \bmod l^{m_a} n!\,|\pi|^{-1}\,\mathbf{Z}_l.$$

It is clear that for $a > 1$, $m_a \geq v_l(a) + 1$ where v_l is the usual l-valuation. So for $a > 1$, we have shown that

$$-\frac{1}{a}\tau_\pi(T_a) \equiv 0 \bmod ln!\,|\pi|^{-1}\,\mathbf{Z}_l$$

and hence

$$\log(\operatorname{Det}_{\tau_\pi}(y_\pi)) \equiv n!\,|\pi|^{-1} \bmod ln!\,|\pi|^{-1} \cdot \mathbf{Z}_l$$

as was required.

Proposition 2. $\nu_l \circ K_l^{-1}(\Xi_{n,l}) = \prod_\pi n!\,\mathbf{Z}_l$

Proof. Let $f \in \operatorname{Hom}(\Lambda_n, \mathbf{Z}_l)$. From the definition of $\Xi_{n,l}$ we have $K_l(f) \in \Xi_{n,l}$ if, and only if, $K_l(f) = \sum_\pi a_\pi \sigma_\pi$ where, for each π,

$$la_\pi \equiv \sum_{\pi_i \mid \pi_i^l = \pi} a_{\pi_i} \bmod \mathbf{Z}_l.$$

From (4.6), $a_\pi = |\pi|\, n!^{-1} f(\sigma_\pi)$. So $K_l(f) \in \Xi_{n,l}$, if, and only if, for each π,

$$|\pi|\, n!^{-1} f(l\sigma_\pi) \equiv \sum_{\pi_i \mid \pi_i^l = \pi} n!^{-1} |\pi_i|\, f(\sigma_{\pi_i}) \bmod \mathbf{Z}_{l'},$$

i.e. if, and only if, for each π, $f(\theta(\rho_\pi)) \equiv 0 \bmod n!\, \mathbf{Z}_l$. Thus $K_l^{-1}(\Xi_{n,l})$ consists of those $f \in \operatorname{Hom}(\Lambda_n, \mathbf{Z}_l)$ such that $\nu_l(f) \in \prod_\pi n!\, \mathbf{Z}_l$. However, from Proposition 1, $\operatorname{Im}(\nu_l) \supseteq \prod_\pi n!\, \mathbf{Z}_l$; so that

$$\nu_l \circ \mathrm{K}_l^{-1}(\Xi_{n,l}) = \prod_\pi n!\, \mathbf{Z}_l.$$

We now prove Theorem 2(ii) for the case $l \neq 2$. By (4.10) ν_l is injective, so from Proposition 1 and Proposition 2, $\Phi_l(\operatorname{Ker}(r_l)) = K_l^{-1}(\Xi_{n,l})$. Because K_l is an isomorphism with $\eta_l = K_l \circ \Phi_l$, we have $\Xi_{n,l} = \eta_l(\operatorname{Ker}(r_l))$. So we have shown that η_l yields an isomorphism

$$C_l^{(2)} = \frac{\operatorname{Hom}(R_{S_n}, 1 + l\mathbf{Z}_l)}{\operatorname{Ker}(r_l)} \xrightarrow{\sim} \frac{\Lambda_{n,l}}{\Xi_{n,l}},$$

as was required.

We now outline the proof of Theorem 2(ii) when $l = 2$.

PROPOSITION 1′. $\prod_\pi \frac{1}{2}n!\, \mathbf{Z}_2 \supseteq \nu_2 \circ \Phi_2(\operatorname{Ker} r_2)$.

Proof. As in Proposition 1, Theorem 3 implies that

$$\nu_2 \circ \Phi_2(\operatorname{Ker} r_2) \subseteq \prod_\pi \tfrac{1}{2}n!\, \mathbf{Z}_2.$$

Remark. It would be of great interest to know whether

$$\nu_2 \circ \Phi_2(\operatorname{Ker}(r_2)) \supseteq \prod_\pi \tfrac{1}{2}n!\, \mathbf{Z}_2.$$

I cannot show this as I have not been able to find elements of $\mathbf{Z}_2 S_n^*$ which do the job of the y_π in the case $l \neq 2$.

PROPOSITION 2′. *Let* $f \in \operatorname{Hom}(\Lambda_n, \mathbf{Z}_2)$. *Then* $K_2(f) \in \Xi_{n,2}$ *if and only if,* $\nu_2(f) \in \prod_\pi \frac{1}{2}n!\, \mathbf{Z}_2$.

(Proof essentially that of Proposition 2)

By Proposition 1′, $\nu_2 \circ \Phi_2(\operatorname{Ker}(r_2)) \subset \prod \frac{1}{2}n!\, \mathbf{Z}_2$, so from Proposition 2′ we deduce that $\mathrm{K}_2 \circ \Phi_2(\operatorname{Ker}(r_2)) \subset \Xi_{n,2}$, i.e. $\eta_2(\operatorname{Ker}(r_2)) \subset \Xi_{n,2}$. The group

$$H = \frac{\Xi_{n,2}}{\eta_2(\operatorname{Ker}(r_2))}$$

is a finite abelian two group of rank less than or equal to the number of conjugacy classes of S_n. (This follows from the fact that $\mathrm{Det}(\mathbf{Z}_2S_n^*)$ is of finite index in $\mathrm{Hom}(R_{S_n}, \mathbf{Z}_2^*)$.) Thus we have an exact sequence

$$1 \to H \to \frac{\Lambda_{n,2}}{\eta_2(\mathrm{Ker}(r_2))} \to \frac{\Lambda_{n,2}}{\Xi_{n,2}} \to 1.$$

But η_2 induces an isomorphism from $C_2^{(2)}$ to $\Lambda_{n,2}/\eta_2(\mathrm{Ker}(r_2))$, so that Theorem 2(ii) is shown when $l=2$.

5. Induced Swan modules

We denote the quotient group $\Lambda_{n,l}/\Xi_{n,l}$ by $\Upsilon_{n,l}$. In this section we explain how to calculate the orders of certain elementary symmetric polynomials in $\Upsilon_{n,l}$. Then we interpret $\Upsilon_{n,l}$ in terms of induced Swan modules.

We define the rth homogeneous power sum $h_r \in \Lambda_r$ by

$$h_r = \sum x_{i_1} \cdots x_{i_r}, \quad i_1 \leq i_2 \leq \cdots \leq i_r.$$

Let $\mathcal{P}_n$ denote the set of partitions of n, and let $\pi \in \mathcal{P}_n$ be the partition $n = r_1 + \cdots + r_k$. Then we define $h_\pi \in \Lambda_n$ by $h_\pi = \prod_{j=1}^k h_{r_j}$. The $\{h_\pi\}$, $\pi \in \mathcal{P}_n$, form a $\mathbf{Z}$ basis of Λ_n, and further the characters $\theta^{-1}(h_\pi) \in R_{S_n}$ have a particularly nice description. (See Chapter 3 of [2] for details of what follows.)

By splitting the numbers $1, 2, \ldots, n$ into disjoint sets of order $\{r_i\}_{i=1}^k$, and by considering the symmetric group S_{r_i} on each set of r_i numbers we may view $\bigoplus_{i=1}^k S_{r_i}$ ($=S_\pi$, say) as a sub-group of S_n. If ε_π denotes the identity character of S_π, then $\theta^{-1}(h_\pi) = \mathrm{Ind}_{S_\pi}^{S_n}(\varepsilon_\pi)$. Further, it is clear that $\theta^{-1}(h_\pi)$ does not depend on the particular choice of sets of numbers, as the various resulting S_π are all S_n-conjugate.

Let π, $\pi' \in \mathcal{P}_n$. We define $N_\pi(\pi')$ to be the number of elements of S_π which lie in the conjugacy class of π'.

PROPOSITION 3. $|S_\pi|\, h_\pi = \sum_{\pi' \in \mathcal{P}_n} N_\pi(\pi')\sigma_{\pi'}$.

Proof. From (4.1), $|\pi'|\,(n!)^{-1}\,\tau_{\pi'}$ is the characteristic function of the conjugacy class π'. Thus

$$\theta^{-1}(h_\pi) = \sum_{\pi' \in \mathcal{P}_n} \mathrm{Ind}_{S_\pi}^{S_n}(\varepsilon_\pi)(\gamma_{\pi'})\,|\pi'|\,(n!)^{-1}\,\tau_{\pi'}$$

where $\gamma_{\pi'}$ denotes an arbitrary element of the conjugacy class π'.

It is immediate that for any $\chi \in R_{S_n}$, $\chi(\gamma_{\pi'}) = (\chi, \tau_{\pi'})$ (where (,) is the standard inner product of R_{S_n}); so by Frobenius reciprocity

$$\mathrm{Ind}_{S_\pi}^{S_n}(\varepsilon_\pi)(\gamma_{\pi'}) = (\varepsilon_\pi, \tau_{\pi'} \mid S_\pi) = n!\,|\pi'|^{-1}\,N_\pi(\pi')\,|S_\pi|^{-1}.$$

COROLLARY. *Let A_π be the $\mathbf{Z}_l$ ideal generated by the numbers*

$$|S_\pi|^{-1}\left(lN_\pi(\pi') - \sum_{\pi_i \mid \pi_i^l = \pi'} N_\pi(\pi_i)\right)$$

for all $\pi' \in \mathcal{P}_n$. Then the image of h_π in $Y_{n,l}$ has order

$$\operatorname{Card}\left(\frac{\mathbf{Z}_l + A_\pi}{\mathbf{Z}_l}\right) \quad \text{if } l \neq 2 \quad \left(\text{resp.} \operatorname{Card}\left(\frac{\mathbf{Z}_2 + 2A_\pi}{\mathbf{Z}_2}\right) \text{ if } l = 2\right).$$

In particular consider the trivial partition $n = n$. Then $\theta^{-1}(h_n) = \varepsilon$ (the identity character of S_n), and $n!\, h_n = \sum_{\pi'} |\pi'|\, \sigma_{\pi'}$. Later we will show that the image of h_n in $Y_{n,l}$ corresponds to the class of a certain Swan module.

Example. The image of h_{2l} in $Y_{2l,l}$ is non-trivial when $l \neq 2$.

Proof. Let π' be the partition $2l = 1 + \cdots + 1 + l$ (l 1's). Then $|\pi'| = (2l!)(l!\, l)^{-1}$, $\pi'^l \neq \pi'$ and π' is not the lth power of any other conjugacy class. If we let $h_{2l} = \sum a_\pi \sigma_\pi$ then

$$la_{\pi'} - \sum_{\pi_i \mid \pi_i^l = \pi'} a_{\pi_i} = 1/(l!).$$

Since $l!$ is divisible by l, $A_{2l} \supset l^{-1}\mathbf{Z}_l$, and we are done by the corollary above.
The reader may wish to verify that $lh_{2l} \in \Xi_{2l,l}$.

Remark. The existence of a non-trivial element of $Y_{n,l}$ may also be deduced from Theorem 1 and the following result.

THEOREM (S. Ullom). *An odd prime l divides $|\mathrm{Cl}(\mathbf{Z}S_n)|$ if, and only if, $l \leq n/2$.*

(For proof see (3.9)(ii) of [12] and (3.8) of [11].)

Let m be an integer prime to $n!$ and let $\pi \in \mathcal{P}_n$. Following R. Swan in [8] we let $[m, \Sigma_\pi]$ denote the class of the locally free, right $\mathbf{Z}S_\pi$-module

$$m\mathbf{Z}S_\pi + \left(\sum_{\gamma \in S_\pi} \gamma\right)\mathbf{Z}S_\pi.$$

(5.1) From (6.1) of [8] we see that the class $[m, \Sigma_\pi]$ depends only on the class of $m \bmod |S_\pi|$.

We now consider integers m such that

$$\text{(5.2)} \qquad \begin{aligned} m &\equiv 1 \bmod n!\,\mathbf{Z}_{l'} \quad \text{for primes } l' \neq l,\ l' \leq n,\\ m &\equiv 1 \bmod (l) \quad \text{if } l \neq 2,\\ &\equiv 1 \bmod (4) \quad \text{if } l = 2. \end{aligned}$$

We let $T_l(S_\pi)$ be the group given by the classes of these modules. From (2.4) of [11] using (5.1), we have that the class $[m, \Sigma_\pi]$ is represented under isomorphism (2.1) of [10] by the homomorphism from R_{S_π} to $\prod_{l \leq n} \mathbf{Z}_l^*$ given by

$$\begin{aligned}(5.3)\qquad \phi &\to m^{(\phi,\, \varepsilon_\pi)} \quad \text{at} \quad l,\\ &\to 1 \qquad\quad \text{at primes} \quad l' \neq l,\end{aligned}$$

for $\phi \in R_{S_\pi}$.

We denote the group of classes in $\mathrm{Cl}(\mathbf{Z}S_n)$ obtained by induction from $T_l(S_\pi)$, by $\mathrm{Ind}_{S_\pi}^{S_n}(T_l(S_\pi))$.

THEOREM 3. *η_l induces an epimorphism $\sum_{\pi\in\mathscr{P}_n} \mathrm{Ind}_{S_\pi}^{S_n}(T_l(S_\pi)) \to \Upsilon_{n,l}$ and further, for $l \neq 2$, this map is an isomorphism.*

Proof. From (5.3) and from Appendix VII of [1], the class of the induced module $[m, \Sigma_\pi]\otimes_{\mathbf{Z}S_\pi} \mathbf{Z}S_n$ is represented under (2.1) by the homomorphism

$$\begin{aligned}(5.4)\qquad \chi &\to m^{(\chi|S_\pi,\varepsilon_\pi)} \quad \text{at} \quad l,\\ &\to 1 \qquad\qquad \text{at} \quad l' \neq l,\end{aligned}$$

for $\chi \in R_{S_n}$. From (5.4) and (5.2) it is immediate that this class is represented by an element of $C_l^{(2)}$. We recall that for $l \neq 2$ (resp. $l = 2$), η_l induces an isomorphism (resp. an epimorphism) $\eta_l\colon C_l^{(2)} \to \Upsilon_{n,l}$. Now by Frobenius reciprocity

$$(\chi|_{S_\pi}, \varepsilon_\pi) = (\chi, \mathrm{Ind}_{S_\pi}^{S_n}(\varepsilon_\pi)) = (\chi, \theta^{-1}(h_\pi)).$$

Let $C_{m,\pi}$ denote the projection into $\mathrm{Hom}(R_{S_n}, 1 + l\mathbf{Z}_l)$ if $l \neq 2$ (resp. $\mathrm{Hom}(R_{S_n}, 1 + 4\mathbf{Z}_2)$ if $l = 2$) of the homomorphism given in (5.4). Then

$$\begin{aligned}\eta_l(C_{m,\pi}) &= \sum_\chi \frac{1}{l} \log(m)(\chi, \theta^{-1}(h_\pi))\theta(\chi) \quad \text{if} \quad l \neq 2,\\ &= \sum_\chi \tfrac{1}{4} \log(m)(\chi, \theta^{-1}(h_\pi))\theta(\chi) \quad \text{if} \quad l = 2,\end{aligned}$$

where the sums are taken over the irreducible characters χ of S_n. Thus we have

$$\begin{aligned}\eta_l(C_{m,\pi}) &= \frac{1}{l} \log(m) h_\pi \quad \text{if} \quad l \neq 2,\\ &= \tfrac{1}{4} \log(m) h_\pi \quad \text{if} \quad l = 2.\end{aligned}$$

However, if $m \not\equiv 1 \bmod (l^2)$ when $l \neq 2$ (resp. $m \not\equiv 1 \bmod (8)$ when $l = 2$), then $(1/l)\log(m)$ (resp. $\frac{1}{4}\log(m)$) is an l-adic unit, and so we are done because $\Lambda_{n,l}$ is generated over $\mathbf{Z}_l$ by the $\{h_\pi\}_\pi \in \mathscr{P}_n$.

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