ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 4, December 1979

SEMI-FREE GROUP ACTIONS

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1. Introduction

In the study of equivariant differential topology, there has been considerable recent interest in the classification of semi-free actions; i.e. differential actions of a compact Lie group with every isotropy subgroup being either the entire group or the unit subgroup. This interest stems largely from the fact that free actions are "understood" and that the next simplest case is the semi-free case to which recently developed tools apply nicely.

The question with which this paper will be concerned is: Given a compact Lie group G, which unoriented bordism classes of compact manifolds contain a representative on which G acts semi-freely?

The answer is trivial, of course, for every manifold M admits a semi-free action which is trivial; i.e. every isotropy group is the entire group, and so every class contains such a representative. This trivial case must be excluded, and one seeks those classes which contain a representative M on which G acts semi-freely and non-trivially in the sense that no component of M consists entirely of points fixed by G. This problem is somewhat less trivial.

In Section 2, the necessary general nonsense of setting up appropriate bordism groups will be accomplished. In addition, the problem will be reduced to a special class of groups—those G which admit an orthogonal representation which is free on the corresponding sphere, and which are not finite of odd order. In Section 3, Conner and Floyd's methods will be used to compute the bordism groups partially, or rather theoretically. This reduces the problem to understanding the classifying space for principal G bundles and the representations of G. In addition the case $G = Z_2$ will be thoroughly studied, since everything will map into this case.

In Section 4, the work really begins. Based on the general nonsense, and the partial calculations, one can state exactly what will be proved. This reduces the problem to verifying certain properties for each group involved, and in essence gives the plan for the remaining sections.

In Section 5, the infinite groups admitting the appropriate representations will be studied, and in Section 6 the finite groups will be studied. In Section 7, some examples of groups will be given to show that all potential images in $\mathfrak{N}_*$ actually arise, and examples of manifolds to show that the images are distinct.

Received February 24, 1978.

I am indebted to John Ewing for several conversations about representations which greatly simplified my original arguments, and to the National Science Foundation for financial support during this work.

2. General nonsense

Throughout this paper, all manifolds (and manifolds with boundary) will be compact and differentiable. G will denote a compact Lie group, and all G actions will be differentiable. If G acts on X and $x \in X$, G_x will denote the isotropy group $\{g \in G \mid gx = x\}$.

Following Conner and Floyd's ideas in [3], one may introduce bordism groups of G actions:

(a) $\mathfrak{N}_*^G(F)$, the unoriented bordism group of free G actions.

(b) $\mathfrak{N}_*^G(SF)$, the unoriented bordism group of semi-free G actions. (Note that semi-free actions do not come from a family in the sense of [4], but there is no difficulty in extending the techniques to these actions.)

(c) $\mathfrak{N}_*^G(SF, F)$, the relative bordism group of semi-free G actions on manifolds with boundary for which the action is free on the boundary.

There is an exact sequence of $\mathfrak{N}_*$ modules, where $\mathfrak{N}_*$ denotes the unoriented bordism ring, given by

$$\begin{tikzcd} \mathfrak{N}_*^G(F) \arrow[rr, "i"] & & \mathfrak{N}_*^G(SF) \arrow[dl, "j"] \\ & \mathfrak{N}_*^G(SF, F) \arrow[ul, "\partial"] & \end{tikzcd}$$

in which i considers a free action as semi-free, j considers a closed manifold as a manifold with empty boundary, and ∂ takes the class of the boundary. In each case the module structure is obtained by taking the cartesian product with a closed manifold and letting G act on the product by acting only on the "second" coordinate.

One has augmentations

$$\varepsilon : \mathfrak{N}_*^G(F) \to \mathfrak{N}_* \quad \text{and} \quad \varepsilon : \mathfrak{N}_*^G(SF) \to \mathfrak{N}_*$$

which ignore the G action, which are $\mathfrak{N}_*$ module homomorphisms.

There is an $\mathfrak{N}_*$ module homomorphism $t: \mathfrak{N}_* \to \mathfrak{N}_*^G(SF)$ which assigns to M the class of M with trivial G action, $gx = x$ for all x. Since $\varepsilon t = 1$, this is monic. Being given $\phi: G \times M \to M$ a semi-free G action, M decomposes into a disjoint union $M_1 \cup M_2$, where M_1 is the union of those components of M for which every point has isotropy group $G_x = G$, and M_2 is the union of the remaining components. Clearly M_1 and M_2 are invariant under the action of G. Further, if M is a manifold with boundary, $(\partial M)_1 = \partial(M_1)$ and $(\partial M)_2 = \partial(M_2)$, i.e. if some boundary component X belongs to $(\partial M)_1$, the entire component of M to which it belongs is fixed pointwise by G. There are then $\mathfrak{N}_*$ module homomorphisms

$$\pi_i: \mathfrak{N}_*^G(SF) \to \mathfrak{N}_*^G(SF)$$

which assign to M the class of M_i, $i=1,2$, with image $\pi_1 =$ image t, and with the image of π_2 being a complementary summand for image t.

The image of π_2 is the bordism group of non-trivial semi-free actions. As outlined in the introduction, the main problem of this paper is:

Problem. For a compact Lie group G, what is the image of

$$\varepsilon: \pi_2\mathfrak{N}_*^G(SF) \to \mathfrak{N}_*?$$

Similarly, π_i may be defined on $\mathfrak{N}_*^G(SF, F)$ and if $M = M_1 \cup M_2$ where M is a manifold with boundary, then M_1 must be closed for the action on $\partial(M_1)$ is both free and every isotropy group is G. (Except when $G=\{1\}$, a case we shall ignore, this gives $\partial(M_1)=\emptyset$.)

Further $i: \mathfrak{N}_*^G(F) \to \mathfrak{N}_*^G(SF)$ has image contained in the image of π_2 (except for $G=\{1\}$), and one has an exact sequence

$$\begin{tikzcd} \mathfrak{N}_*^G(F) \arrow[rr, "i"] & & \pi_2\mathfrak{N}_*^G(SF) \arrow[dl, "j"] \\ & \pi_2\mathfrak{N}_*^G(SF, F) \arrow[ul, "\partial"] & \end{tikzcd}$$

This brings one to the first and simplest result:

PROPOSITION 2.1. *If $G \neq \{1\}$ is a compact Lie group, the image of $\varepsilon: \mathfrak{N}_*^G(F) \to \mathfrak{N}_*$ is*

(a) *all of $\mathfrak{N}_*$ if G is finite of odd order,*
(b) *zero otherwise.*

Thus, if G is finite of odd order, $\varepsilon: \pi_2\mathfrak{N}_*^G(SF) \to \mathfrak{N}_*$ is epic.

Proof. If G is finite of odd order, and M is any closed manifold, $M \times G$ has the free G action given by multiplication in the second variable, and is bordant to M.

If G is not finite of odd order, then G contains a subgroup Z_2, cyclic of order 2, with generator t, and if G acts freely on M, then M is the boundary of $M \times [-1, 1]/(m, x) \sim (tm, -x)$. ■

Having analyzed free actions, one then observes that a semi-free action on a closed manifold decomposes into 3 disjoint portions, $M = M_1 \cup M_2' \cup M_2''$, where as before M_1 is the union of those components pointwise fixed by G, M_2' is the union of those components on which every point has isotropy group equal to $\{1\}$, and M_2'' is the remainder.

Now being given a component N of M_2'', there is some point $n \in N$ with

$gn = n$ for all $g \in G$, so N is G invariant. However, there must also be a point $n' \in N$ for which $G_{n'} = \{1\}$.

Being given such a component N, the fixed set of N, F is the disjoint union of closed submanifolds F^k (the union of those components of dimension k) with $k < \dim(N) = n$. A tubular neighborhood of F^k in N may then be identified with the disc bundle $D(\nu^{n-k})$ of the normal bundle ν^{n-k} of F^k in N, with G acting linearly in the fibers of ν^{n-k}. Since the G action is semi-free, the induced representation on a fiber is a linear representation which acts *freely* on the sphere of that representation. (Note that with an invariant Riemannian metric, the representation is orthogonal, and so preserves the unit sphere.)

Groups which admit orthogonal representations which are free on the sphere have been studied extensively. A very nice reference is Wolf [6]. Summarizing, one has:

PROPOSITION 2.2. *If G does not admit an orthogonal representation which is fixed point free on the sphere, then every non-trivial semi-free action is free. Thus $i: \mathfrak{N}_*^G(F) \to \pi_2\mathfrak{N}_*^G(SF)$ is an isomorphism and the image of $\varepsilon: \pi_2\mathfrak{N}_*^G(SF) \to \mathfrak{N}_*$ is*

(a) *all of $\mathfrak{N}_*$ if G is finite of odd order,*
(b) *zero otherwise.*

Thus, one is left to consider those G which admit an orthogonal representation which is free on the sphere, and further, need consider only those G which are *not* finite of odd order.

Note. For G finite of odd order, one might try to consider actions $\phi: G \times M \to M$ for which each component of M contains points fixed by G and points with isotropy group $\{1\}$ in the hope of splitting off image (i). This does not work. Being given M as above and a connnected closed manifold N, one may form a G-connected sum $M \#_G (N \times G)$ by cutting out disc neighborhoods of an orbit Gm for $m \in M$ with $G_m \neq G$ and of $n \times G$ for some $n \in N$ and sewing along the resulting boundaries. This gives a G action of the desired type on a manifold bordant to $M \cup N$.

One final point of general nonsense is worth mentioning. If G acts semi-freely on M and $H \subset G$ is a subgroup, then H also acts semi-freely on M. This defines restriction homomorphisms, denoted ρ_H^G relating each of the groups. The elementary properties of these restrictions will play a major role in studying the bordism problem.

In particular, if G acts orthogonally on V (a representation space) and freely on the sphere, and if G is not finite of odd order, then G contains an element of order 2. Since this element acts as multiplication by -1 in V, it must be unique. Further, it is central in G. This subgroup will be denoted Z_2, and ρ will denote restriction to this subgroup.

3. Calculation of the groups

In order to understand the image of ε more thoroughly, one needs to analyze the groups of G actions. It will be assumed that G acts orthogonally on some representation space, and freely on the sphere, and that G is not finite of odd order.

First, one recalls the result of Conner and Floyd [3; (19.1)]:

PROPOSITION 3.1. *Assigning to $\phi: G\times M \to M$ the class of the map $f: M/G \to BG$ classifying the principal G bundle $\pi: M \to M/G$ defines an isomorphism $\pi: \mathfrak{N}_*^G(F) \to \mathfrak{N}_{*-\dim G}^G(BG)$*

If G is finite, one may make use of facts about the cohomology of groups to analyze free actions. One of the main results needed will be:

PROPOSITION 3.2. *If G is finite and $S \subset G$ is a Sylow 2-subgroup of G, then the restriction $\rho_s^G: \mathfrak{N}_*^G(F) \to \mathfrak{N}_*^S(F)$ is monic.*

Proof. Given $\phi: G\times M \to M$, one has the map $f: M/G \to BG$. $\rho_s^G(M, \phi)$ is represented by the map $f': M/S \to BS$ which is the induced covering from the diagram

$$\begin{array}{ccc} M/S & \to & BS \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ M/G & \to & BG. \end{array}$$

If $f': M/S \to BS$ bounds, then for all $x \in H^*(BS; Z_2)$ and all $\omega = (i, \ldots, i_r)$,

$$\langle w_\omega(M/S) \cup f'^*(x), [M/S]\rangle = 0.$$

Now $\pi'^* w_\omega(M/G) = w_\omega(M/S)$, so

$$\begin{aligned}\langle w_\omega(M/G) \cup f^*(y), \pi'_*[M/S]\rangle &= \langle w_\omega(M/S) \cup \pi^* f^*(y), [M/S]\rangle \\ &= \langle w_\omega(M/S) f'^* \pi^*(y), [M/S]\rangle \\ &= 0\end{aligned}$$

for all $y \in H^*(BG; Z_2)$. Since $\pi'_*[M/S] = [M/G]$, all Stiefel-Whitney numbers of $f: M/G \to BG$ are zero, and $f: M/G \to BG$ bounds. ∎

In order to analyze $\mathfrak{N}_*^G(SF, F)$, one follows Conner and Floyd [3]. Being given $\phi: C\times M^n \to M^n$ which is semi-free, and free on ∂M, one gives M^n a Riemannian metric invariant under G. The fixed set F of G is then a disjoint union of closed submanifolds F^k (the union of those components of F of dimension k) imbedded in the interior of M. Let ν_{n-k} denote the normal bundle of F^k in M; G induces, via the differential, an action by bundle maps on ν_{n-k} preserving the Riemannian metric. A tubular neighborhood of F in M may be identified equivariantly with the disjoint union of the disc bundles $D(\nu_{n-k})$ with their linear actions, and in fact (M, ϕ) is bordant to the union of these bundle actions, for ϕ is free on the complement of the tubular neighborhood.

Now examining the bundle ν_{n-k} we see that ν_{n-k} decomposes into the Whitney sum of G-invariant subbundles ν^V_{n-k}, corresponding to the irreducible orthogonal representations V of G for which G acts freely on the sphere of V, where ν^V_{n-k} is characterized by the fact that each fiber is a sum of copies of the representation V.

Now letting V be any such irreducible representation of G, let $d(V)$ denote the real dimension of V, and for each m, let $C_m(V) \subset O(md(V))$ be the subgroup of the orthogonal group of $V \oplus \cdots \oplus V$ (m times) which centralizes the action of G. The bundle ν^V_{n-k} (over a given component of F^k) has fiber dimension $md(V)$ for some m and is classified by a map into $BC_m(V)$. It is then immediate that:

PROPOSITION 3.3. *Let $V_1, \ldots, V_r$ denote the distinct irreducible orthogonal representations of G with G acting freely on the sphere of each V_i. Then the fixed point homomorphism*

$$F\colon \mathfrak{N}^G_n(SF, F) \to \oplus_{(m)} \mathfrak{N}_{n-\sum m_i d_i}(BC_{m_1}(V_1) \times \cdots \times BC_{m_r}(V_r))$$

is an isomorphism, where $d_i = d(V_i)$ and the sum is over all sequences $(m) = (m_1, \ldots, m_r)$.

Note. The fixed components of M^n of dimension $n - \sum m_i d_i = n - k$ over which each ν^{Vi}_{n-k} has the dimension $m_i d_i$ is mapped into

$$BC_{m_1}(V_1) \times \cdots \times BC_{m_r}(V_r)$$

to classify the bundles ν^{Vi}_{n-k}.

The term $(m) = (0, \ldots, 0)$ may be identified with $\mathfrak{N}_n$ and is the image of π_1. The remaining summands form the image of π_2.

The homomorphism $\partial\colon \mathfrak{N}^G_n(SF, F) \to \mathfrak{N}^G_{n-1}(F)$ may be described as follows: Given $f\colon F^k \to BC_{m_1}(V_1) \times \cdots \times BC_{m_r}(V_r)$, one forms the induced vector bundle and takes the sphere bundle, on which G acts freely.

Thus, to compute $\mathfrak{N}^G_*(F)$ and $\mathfrak{N}^G_*(SF, F)$ one must understand in some detail the classifying space BG and the appropriate representations of G.

This is, of course, all a generalization of the original work of Conner and Floyd [3] on involutions. Before proceeding, it is useful to summarize this work, since every case will be mapped into it.

Thus with $G = Z_2$, the results above become

$$\mathfrak{N}^{Z_2}_*(F) \cong \mathfrak{N}_*(BZ_2) \quad \text{and} \quad \mathfrak{N}^{Z_2}_*(SF, F) \cong \bigoplus_{k=0}^{*} \mathfrak{N}_{*-k}(BO(k))$$

with $\pi_2 \mathfrak{N}^{Z_2}_*(SF, F) \cong \oplus^*_{k=1} \mathfrak{N}_{*-k}(BO(k))$.

Now for $k = 1$, one has $O(k) = Z_2$, and in fact, the homomorphism ∂ maps the summand $\mathfrak{N}_{*-1}(BO(1))$ isomorphically onto $\mathfrak{N}^{Z_2}_*(F)$. To see this, note that if $T\colon M \to M$ is a free involution, (M, T) is the boundary of

$Mx[-1,1]/(m,x)^{\sim}(Tm,-x)$ with involution induced by $T\times 1$ or 1×-1. The fixed set of this involution is M/Z_2 and the normal bundle is the line bundle $\pi: M\to M/Z_2$.

This gives [3, 28.1], and its analog, that the sequences

$$0\to \mathfrak{N}_*^{Z_2} \xrightarrow{j} \mathfrak{N}_*^{Z_2}(SF,F) \xrightarrow{\partial} \mathfrak{N}_*^{Z_2}(F)\to 0$$

and

$$0\to \pi_2\mathfrak{N}_*^{Z_2}(SF) \xrightarrow{j} \pi_2\mathfrak{N}_*^{Z_2}(SF,F) \xrightarrow{\partial} \mathfrak{N}_*^{Z_2}(F)\to 0$$

are split and short exact. (Note that every Z_2 action is automatically semi-free.) Corresponding to this splitting one has a left inverse to j given by

$$P: \mathfrak{N}_*^{Z_2}(SF,F)\to \mathfrak{N}_*^{Z_2} \quad \text{or} \quad P: \pi_2\mathfrak{N}_*^{Z_2}(SF,F)\to \pi_2\mathfrak{N}_*^{Z_2}(SF)$$

which assigns to the bundle ξ over N, the manifold

$$D(\xi)/\{x^{\sim}-x/x\in S(\xi)\}$$

obtained from the disc bundle of ξ, $D(\xi)$, by identifying antipodal points of the sphere bundle, $S(\xi)$, with the involution induced by multiplication by -1 in the fibers of ξ. This may be alternately described as $RP(\xi\oplus 1)$, the projective space bundle of lines in the fibers of the Whitney sum of ξ and a trivial line bundle 1, with the involution induced by multiplication by -1×1 or 1×-1 in the fibers of $\xi\oplus 1$.

In particular, $P: \oplus_{k=2}^{*} \mathfrak{N}_{*-k}(BO(k))\to \pi_2\mathfrak{N}_*^{Z_2}(SF)$ is an isomorphism, and $\varepsilon: \pi_2\mathfrak{N}_*^{Z_2}(SF)\to \mathfrak{N}_*$ has image the ideal generated by the $RP(\xi\oplus 1)$ with ξ a vector bundle of fiber dimension at least 2. One may then compute this ideal, as has been done many times to show

PROPOSITION 3.4. *The ideal in $\mathfrak{N}_*$ consisting of classes which admit a representative with non-trivial semi-free Z_2 action; i.e.*

$$\text{image}\,\{\varepsilon: \pi_2\mathfrak{N}_*^{Z_2}(SF)\to \mathfrak{N}_*\}$$

is precisely the augmentation ideal of positive dimensional classes.

4. The plan

Again, it will be assumed that G acts orthogonally on some representation space and freely on the sphere, and that G is not finite of odd order.

For any positive integer d, one introduces an ideal $I_d\subset \mathfrak{N}_*$ as follows:

(1) Let $A_d=\oplus_{k=1}^{\infty} \mathfrak{N}_{*-kd}(BO(k))$, and let

$$f_d: A_d\to \bigoplus_{j=1}^{\infty} \mathfrak{N}_{*-j}(BO(j))\cong \pi_2\mathfrak{N}_*^{Z_2}(SF,F)$$

be the homomorphism assigning to the bundle $\xi\to N$ in $\mathfrak{N}_{*-kd}(BO(k))$ the bundle $d\xi=(\xi\oplus\cdots\oplus\xi)\to N$, the d-fold Whitney sum, in $\mathfrak{N}_{*-kd}(BO(kd))$.

(2) Let $K_d\subset A_d$ be the kernel of the composite

$$A_d \xrightarrow{f_d} \pi_2\mathfrak{N}_*^{Z_2}(SF,F) \xrightarrow{\partial} \mathfrak{N}_{*-1}^{Z_2}(F).$$

(3) Let $I_d \subset \mathfrak{N}_*$ be the image of K_d under $\varepsilon P f_d$, where

$$\varepsilon P\colon \pi_2\mathfrak{N}_*^{Z_2}(SF, F) \to \mathfrak{N}_*$$

assigns to $\xi \to N$ the bordism class of $RP(\xi \oplus 1)$.

The main result of this paper will be:

PROPOSITION 4.1. *If G acts orthogonally on some representation space and freely on the sphere, and if G is not finite of odd order, then there exists an integer $d = \dim(V)$, the dimension of each irreducible representation V of G for which G acts freely on the sphere, so that the image of $\varepsilon\colon \pi_2\mathfrak{N}_*^G(SF) \to \mathfrak{N}_*$ is precisely I_d.*

Actually, much more will be shown. Specifically, it will be verified that:

Fact 1. The image of the restriction $\rho\colon \pi_2\mathfrak{N}_*^G(SF, F) \to \pi_2\mathfrak{N}_*^{Z_2}(SF, F)$ is precisely the image of f_d.

Fact 2. In the commutative diagram

$$\begin{tikzcd}
\pi_2\mathfrak{N}_*^G(SF, F) \arrow[r, "\partial"] \arrow[d, "\rho"] & \mathfrak{N}_*^G(F) \arrow[d, "\rho'"] \\
\pi_2\mathfrak{N}_*^{Z_2}(SF, F) \arrow[r, "\partial'"] & \mathfrak{N}_*^{Z_2}(F),
\end{tikzcd}$$

ρ' is monic on image ∂.

LEMMA 4.2. *These facts imply Proposition* 4.1.

Proof. First note that $\varepsilon\colon \pi_2\mathfrak{N}_*^G(SF) \to \mathfrak{N}_*$ vanishes on the image of $i\colon \mathfrak{N}_*^G(F) \to \pi_2\mathfrak{N}_*^G(SF)$ and further factors through the restriction to Z_2; i.e. one has the commutative diagram

$$\begin{tikzcd}
 & \pi_2\mathfrak{N}_*^G(SF) \arrow[d, "\theta"] \arrow[dr, "j"] & & \\
0 \arrow[r] & \pi_2\mathfrak{N}_*^G(SF)/imi \arrow[r, "j'"] \arrow[d, "\rho''"] & \pi_2\mathfrak{N}_*^G(SF, F) \arrow[r, "\partial'"] \arrow[d, "\rho"] & \mathfrak{N}_*^G(F) \arrow[d, "\rho'"] \\
0 \arrow[r] & \pi_2\mathfrak{N}_*^{Z_2} \arrow[r, "j''"] \arrow[d, "\varepsilon"] & \pi_2\mathfrak{N}_*^{Z_2}(SF, F) \arrow[r, "\partial'"] \arrow[dl, "\varepsilon P"] & \mathfrak{N}_*^{Z_2}(F) \\
 & \mathfrak{N}_* & &
\end{tikzcd}$$

in which the composite $\varepsilon\rho''\theta = \varepsilon P\rho j'\theta$ is $\varepsilon\colon \pi_2\mathfrak{N}_*^G(SF) \to \mathfrak{N}_*$.

For any $\alpha \in \pi_2\mathfrak{N}_*^G(SF)$, $\varepsilon\alpha = \varepsilon P\rho j'\theta(\alpha)$, and

$$j'\theta(\alpha) \in \ker \rho'\partial = \ker \partial'\rho,$$

so $\rho j'\theta(\alpha) \in \ker \partial'$, and $j'\theta(\alpha) \in f_d(A_d)$ by Fact 1. Thus

$$\rho j'\theta(\alpha) \in f_d(K_d) = f_d(A_d) \cap \ker \partial' \quad \text{and} \quad \varepsilon P\rho j'\theta(\alpha) \in \varepsilon Pf_d(K_d) = I_d.$$

On the other hand, if $\beta \in I_d$, $\beta = \varepsilon Pf_d(K)$ for some $K \in K_d$, and $f_d(K) = \rho(\gamma)$ for some $\gamma \in \pi_2\mathfrak{N}_*^G(SF, F)$. Now $\partial'\rho(\gamma) = \partial' f_d(K) = 0$ so $\rho'\,\partial(\gamma) = 0$ but by Fact 2), ρ' is monic on im ∂, so $\partial(\gamma) = 0$ and $\gamma = j'(\delta)$ for some $\delta \in \pi_2\mathfrak{N}_*^G(SF)/imi$. Since θ is epic, there is an $\alpha \in \pi_2\mathfrak{N}_*^G(SF)$ with $\theta(\alpha) = \delta$. Then

$$\varepsilon(\alpha) = \varepsilon P\rho j'\theta(\alpha) = \varepsilon P\rho j'(\delta) = \varepsilon P\rho(\gamma) = \varepsilon Pf_d(K) = \beta.$$

Thus imagine $\{\varepsilon\colon \pi_2\mathfrak{N}_*^G(SF) \to \mathfrak{N}_*\} = I_d$. ■

Note. Actually, this is not quite precise. One must know that dim (V) is independent of V. This needs to be verified for each group.

There are a couple of minor results which will be useful in the sequel.

LEMMA 4.3. *If V is an irreducible representation of G with G acting freely on the sphere, and if $d = \dim V$, then the image of*

$$\rho\colon \pi_2\mathfrak{N}_*^G(SF, F) \to \pi_2\mathfrak{N}_*^{Z_2}(SF, F)$$

contains image (f_d).

Proof. If $\xi \to N$ represents a class in $\mathfrak{N}_{n-kd}(BO(k))$, then $d\xi \cong \xi \oplus_{\mathbf{R}} V$ and admits an action of G with each fiber being a sum of k copies of V. Thus the class of $d\xi \to N$ belongs to the image of ρ. ■

LEMMA 4.4. *If G is finite, Fact 2 is true for G if it is true for the Sylow 2 subgroup of G.*

Proof. Let S be a Sylow 2 subgroup of G and consider

$$\begin{tikzcd}
\pi_2\mathfrak{N}_*^G(SF, F) \arrow[r, "\partial"] \arrow[d, "\rho_s^{G'}"] & \mathfrak{N}_*^G(F) \arrow[d, "\rho_s^G"] \\
\pi_2\mathfrak{N}_*^S(SF, F) \arrow[r, "\partial'"] \arrow[d, "\rho'"] & \mathfrak{N}_*^S(F) \arrow[d, "\rho"] \\
\pi_2\mathfrak{N}_*^{Z_2}(SF, F) \arrow[r, "\partial''"] & \mathfrak{N}_*^{Z_2}(F)
\end{tikzcd}$$

If Fact 2 holds for S, $\rho\ \partial'(x) = 0$ implies $\partial'(x) = 0$. Then for

$$\alpha \in \pi_2\mathfrak{N}_*^G(SF, F),$$

$0=\rho\rho_S^G\,\partial(\alpha)=\rho\partial'\rho_S^{G'}(\alpha)$ implies $0=\partial'\rho_S^{G'}(\alpha)=\rho_S^G\,\partial(\alpha)$, but by Proposition 3.2, ρ_S^G is monic, so $\partial(\alpha)=0$, giving Fact 2 for G.

Finally, one should note:

PROPOSITION 4.5. *Facts* 1 *and* 2 *hold for* $G=Z_2$.

Proof. Clearly $A_1=\pi_2\mathfrak{N}_*^{Z_2}(SF,F)$ and f_1 is the identity, giving Fact 1. Since ρ' is the identity, Fact 2 is obvious. Note also that the only irreducible representation of Z_2 for which the action on the sphere is free is one-dimensional. ■

5. The infinite groups

The infinite groups admitting appropriate representations are classified in Bredon [1, Theorem 8.5]. One has:

PROPOSITION 5.1. *If G is an infinite compact Lie group admitting an orthogonal representation which is free on the sphere, then G is one of the following groups:*

(1) *the circle group* $S^1=U(1)$,
(2) *the group of quaternions of unit norm,* $S^3=Sp(1)$,
(3) *the subgroup A of the group of unit norm quaternions generated by* $S^1=\{\exp(2\pi it)\}$ *and* j.

It is not difficult to see that the only irreducible orthogonal representations which are free on the sphere are the standard ones with S^1 acting as multiplication on $C=R^2$ and $A\subset S^3$ both acting as multiplication on $H=R^4$. Thus, one has:

(a) *Every irreducible representation of S^1 which is free on the sphere is of dimension* 2. *The group $C_m(V)$ is the unitary group $U(m)$, and* $\pi_2\mathfrak{N}_*^{S^1}(SF,F)\cong\oplus_{k=1}^{*}\mathfrak{N}_{*-2k}(BU(k))$.

(b) *Every irreducible representation of A or S^3 which is free on the sphere is of dimension* 4. *The group $C_m(V)$ is the symplectic group $Sp(m)$, and*

$$\pi_2\mathfrak{N}_*^{S^3}(SF,F)\cong\pi_2\mathfrak{N}_*^{A}(SF,F)\cong\bigoplus_{k=1}^{*}\mathfrak{N}_{*-4k}(BSp(k)).$$

One now needs the structure of the bordism groups $\oplus\mathfrak{N}_{*-k}(BO(k))$ and similarly for the unitary and symplectic groups. The requisite argument may be found in [2, Lemma 2.2], and the result is:

If $\xi\to M$ and $\eta\to N$ are two vector bundles, their product is represented by $\xi\oplus\eta\to N\times M$ defining products in

$$\oplus\mathfrak{N}_{*-k}(BO(k))\quad(\text{or}\quad\oplus\mathfrak{N}_{*-2k}(BU(k)),\quad\text{or}\quad\mathfrak{N}_{*-4k}(BSp(k))).$$

With this product, the ring is the polynomial algebra over $\mathfrak{N}_*$ on the classes of $\lambda\to RP(n)$, $n\geq 0$, where λ is the usual line bundle over projective space (or $\lambda\to CP(n)$ or $\lambda\to HP(n)$). In particular, $\mathfrak{N}_{*-k}(BO(k))$ has a base given by the monomials

$$\lambda_1 \oplus \cdots \oplus \lambda_k \to RP(n_1) \times \cdots \times RP(n_k), \quad n_1 \le \cdots \le n_k,$$

(and similarly in the other cases).

To obtain the result one wants, it is only necessary to note that the bundles $\lambda \otimes C = 2\lambda \to RP(2n)$ and $\lambda \otimes H = 4\lambda \to RP(4n)$ are also suitable polynomial generators for

$$\oplus \mathfrak{N}_{*-2k}(BU(k)) \quad \text{and} \quad \oplus \mathfrak{N}_{*-4k}(BSp(k)).$$

Thus one sees that for S^1, $d = 2$ and A or S^3, $d = 4$, the image of $\rho: \pi_2 \mathfrak{N}_*^G(SF, F) \to \pi_2 \mathfrak{N}_*^{Z_2}(SF, F)$ is precisely $f_d(A_d)$.

Now turning to the free bordism groups, one knows that:

(a) $\mathfrak{N}_*(RP(\infty)) = \mathfrak{N}_*(BZ_2)$ is the free $\mathfrak{N}_*$ module on the classes of the inclusions $RP(i) \subset RP(\infty)$,

(b) $\mathfrak{N}_*(CP(\infty)) = \mathfrak{N}_*(BS^1)$ is the free $\mathfrak{N}_*$ module on the classes of the inclusions $CP(i) \subset CP(\infty)$, and

(c) $\mathfrak{N}_*(HP(\infty)) = \mathfrak{N}_*(BS^3)$ is the free $\mathfrak{N}_*$ module on the classes of the inclusions $HP(i) \subset HP(\infty)$.

Taking induced bundles, one has that $\mathfrak{N}_*^{Z_2}(F)$, $\mathfrak{N}_*^{S^1}(F)$, and $\mathfrak{N}_*^{S^3}(F)$ are the free $\mathfrak{N}_*$ modules on the classes of the $[S^i, a]$, a the antipodal map, or S^{2i-1} or S^{4i-1} with standard multiplication action of S^1 or S^3. Restricting from S^1 or S^3 to Z_2 gives the antipodal map, so

$$\rho: \mathfrak{N}_*^{S^1}(F) \to \mathfrak{N}_*^{Z_2}(F) \quad \text{and} \quad \rho: \mathfrak{N}_*^{S^3}(F) \to \mathfrak{N}_*^{Z_2}(F)$$

are monic.

For A, one has the commutative diagram

$$\begin{tikzcd}
\pi_2 \mathfrak{N}_*^{S^3}(SF, F) \arrow[r, "\partial"] \arrow[d, "\rho"] & \mathfrak{N}_*^{S^3}(F) \arrow[d, "\rho'"] \\
\pi_2 \mathfrak{N}_*^{A}(SF, F) \arrow[r, "\partial"] \arrow[d, "\rho''"] & \mathfrak{N}_*^{A}(F) \arrow[d, "\rho_0"] \\
\pi_2 \mathfrak{N}_*^{Z_2}(SF, F) \arrow[r, "\partial"] & \mathfrak{N}_*^{Z_2}(F)
\end{tikzcd}$$

and ρ is an isomorphism. Thus $\rho_0 \partial'(x) = 0$ implies $x = \rho(y)$ for some y, so $0 = \rho_0 \partial' \rho(y) = \rho_0 \rho' \partial(y)$ and $\partial(y) = 0$ but then $\partial'(x) = \partial' \rho(y) = \rho' \partial(y) = 0$.

Thus one has shown:

PROPOSITION 5.2. *Every infinite compact Lie group admitting an appropriate representation satisfies Facts* 1 *and* 2 *and Proposition* 4.1.

Further, one sees that $\varepsilon \pi_2 \mathfrak{N}_*^{S^1}(SF)$ is the ideal generated by the manifolds $CP(\lambda_1 \oplus \cdots \oplus \lambda_k \oplus 1)$ over $CP(n_1) \times \cdots \times CP(n_k)$, $k \ge 2$ and $\varepsilon \pi_2 \mathfrak{N}_*^{S^3}(SF)$ by the similar quaternionic projective space bundles over products of quatemionic projective spaces. The complex and quaternionic projective space bundles are cobordant to the square and fourth power respectively of the corresponding real projective space bundles over products of real projective spaces. Hence one has:

PROPOSITION 5.3. *$I_2 = \varepsilon\pi_2\mathfrak{N}_*^{S^1}(SF)$ is the ideal in $\mathfrak{N}_*$ generated by the squares, and $I_4 = \varepsilon\pi_2\mathfrak{N}_*^{S^3}(SF)$ is the ideal in $\mathfrak{N}_*$ generated by the fourth powers.*

Finally, it should be noted that semi-free actions of S^1 and S^3 were analyzed with bordism methods by Uchida [5]. While stated in the oriented case, there is no essential difference in the unoriented situation.

6. The finite groups

Now turning to the case in which G is finite, one recalls that the Sylow 2 subgroup of G is either cyclic or generalized quaternion.

If G is cyclic of order 2^s, $s > 1$, then the only irreducible representation for which G acts freely on the sphere is the usual multiplication by 2^s-th roots of unity on $C = R^2$. Thus one has

$$\begin{array}{ccc}
\pi_2\mathfrak{N}_*^{S^1}(SF, F) & \xrightarrow{\partial} & \mathfrak{N}_*^{S^1}(F) \\
\cong \big\downarrow \rho & & \big\downarrow \rho' \\
\pi_2\mathfrak{N}_*^{Z_{2^s}}(SF, F) & \xrightarrow{\partial} & \mathfrak{N}_*^{Z_{2^s}}(F) \\
\big\downarrow \rho'' & & \big\downarrow \rho_0 \\
\pi_2\mathfrak{N}_*^{Z_2}(SF, F) & \xrightarrow{\partial} & \mathfrak{N}_*^{Z_2}(F)
\end{array}$$

with ρ an isomorphism. Since $\rho_0\rho'$ is monic, ρ_0 is monic on image ∂.

If G is generalized quaternion, then the only appropriate representation is the standard action on $H = R^4$ by quaternionic multiplication. Thus one has

$$\begin{array}{ccc}
\pi_2\mathfrak{N}_*^{S^3}(SF, F) & \xrightarrow{\partial} & \mathfrak{N}_*^{S^3}(F) \\
\cong \big\downarrow \rho & & \big\downarrow \rho' \\
\pi_2\mathfrak{N}_*^{G}(SF, F) & \xrightarrow{\partial} & \mathfrak{N}_*^{G}(F) \\
\big\downarrow \rho'' & & \big\downarrow \rho_0 \\
\pi_2\mathfrak{N}_*^{Z_2}(SF, F) & \xrightarrow{\partial} & \mathfrak{N}_*^{Z_2}(F)
\end{array}$$

with ρ an isomorphism. Since $\rho_0\rho'$ is monic, ρ_0 is monic on image ∂.

Combining these observations with Lemma 4.4 and Proposition 4.5, gives:

PROPOSITION 6.1. *If G is finite, Fact 2 is true for G.*

It now remains only to verify Fact 1 for the finite groups. By Proposition 3.3, one has

$$\mathfrak{N}_n^G(SF, F) \cong \bigoplus_{(m)} \mathfrak{N}_{n-\sum m_i d_i}(BC_{m_1}(V_1) \times \cdots \times BC_{m_r}(V_r)).$$

From Wolf's calculations (Theorem 7.2.18) all irreducible complex representations of G which are free on the sphere are of the same dimension and except for $G = Z_2$ none are "real" representations (Note that I am using quotation marks to indicate the group theoretic use of real in referring to representations) and so the irreducible real representations all have the same dimension, which will be denoted d.

There are three distinct possibilities for the group $C_m(V)$. It can be the orthogonal group O_m, the unitary group U_m, or the symplectic group Sp_m. Given

$$f: M \to BC_{m_1}(V_1) \times \cdots \times BC_{m_r}(V_r)$$

with $\xi_i \to M$ the associated m_i-dimensional R, C, or H bundle, the corresponding bundle over M is the Whitney sum of d copies of ξ_i if the group is O_m, $(d/2)$ copies if the group is U_m, and $(d/4)$ copies if the group is Sp_m, the sum being over all i, when considered in $\mathfrak{N}_*(BO_{\sum m_i d_i}) \subset \mathfrak{N}_*^{Z_2}(SF, F)$. Because, as noted for the infinite groups, the homomorphisms

$$\otimes C: \mathfrak{N}_*(BO_m) \to \mathfrak{N}_*(BU_m) \quad \text{and} \quad \otimes H: \mathfrak{N}_*(BO_m) \to \mathfrak{N}_*(BSp_m)$$

are epic, it is clear that the image of $\pi_2\mathfrak{N}_*^G(SF, F)$ in $\pi_2\mathfrak{N}_*^{Z_2}(SF, F)$ is contained in the image of f_d.

Thus one has:

PROPOSITION 6.2. *If G is finite, Fact 1 is true for G.*

Special Note. While it is not essential for the argument, it is curious that in Wolf's tables all of the irreducible fixed point free representations of G are of the same type; i.e. self conjugate, or "real" or not self conjugate.

Note. Another way to phrase the above is that a G bundle over a G fixed space has the form $\oplus M \otimes_{D_m} \mathrm{Hom}^G(M, \xi) \cong \xi$ where M runs through the irreducible real representations of G and $D_M = \mathrm{Hom}^G(M, M)$ is the field of the representation. Up to bordism, $\mathrm{Hom}^G(M, \xi)$ is equivalent to a bundle $D_M \otimes_R \eta_M$ and hence ξ to a sum $\oplus M \otimes_R \eta_M$.

7. Some examples

A very obvious question in connection with these results is:

Question 1. Which integers d can actually occur?

Among the groups discussed in the theoretical treatment to this point, one has only encountered the cases $d = 1, 2$, or 4. If these were the only possibilities, one should know it.

If $d = d(G)$ is odd, then a d-dimensional irreducible real representation for G is not the restriction of a complex representation and so its complexification is irreducible. Thus G would have an irreducible complex representation for which the action on the sphere would be free and which is "real" in the sense used by Wolf. From Wolf's tables this situation only occurs for $G = Z_2$ when $d = 1$.

Thus one must have $d = 1$ and $G = Z_2$ or else d is even.

Going to Wolf's tables, one finds groups G, of his type I, generated by elements A and B with relations $A^m = B^n = 1$, $BAB^{-1} = A^r$ provided $m \geq 1$, $n \geq 1$, $(n(r-1), m) = 1$, $r^n \equiv 1\,(m)$ such that if k is the order of r in the multiplicative group of residues modulo m, then n/k is divisible by every prime divisor of k. For such a group, $d = 2k$.

Claim. Given an integer $k > 1$, one can find m, n and r for which the group of type I has $d = 2k$.

Given $k > 1$, there are an infinite number of primes p in the sequence $\{ak+1\}$. Choose $p = ak+1$ which is an odd prime. Then $p-1 = ak$ is the order of the multiplicative group of residues modulo p, and so there is an integer r with $1 < r < p$ having order k in this group. Taking $m = p$, $n = 2k^2$, and r as chosen you have $r^k \equiv 1\,(m)$ so $r^n \equiv 1\,(m)$, you have $n/k = 2k$ divisible by every prime divisor of k, and you have $(n(r-1), m) = 1$ for m is the prime p which is odd so prime to 2, and prime to $(r-1)$ for $1 < r < p$, and prime to k because $1 = p - ak$. For this group, one then has $d = 2k$.

Then it is essential to ask:

Question 2. Do the ideals in $\mathfrak{N}_*$ associated to the integers d actually depend on d?

First, it is obvious that the ideal associated with d can contain no non-zero elements of dimension less than d, since the dimension of the normal bundle to the fixed set of G is divisible by d.

Next, letting k and j be odd, consider $M = CP(kj-1)$ with the involution

$$t[x_1, \ldots, x_{kj}] = [-x_1, \ldots, -x_k, x_{k+1}, \ldots, x_{kj}]$$

fixing $CP(k-1)$ with normal bundle $(j-1)k\xi$ and $CP((j-1)k-1)$ with normal bundle $k\xi$. This class is then in the ideal I_{2k}. However, $CP(kj-1)$ is bordant to $RP(kj-1) \times RP(kj-1)$ and hence is the square of an indecomposable in $\mathfrak{N}_*$.

These facts show that one obtains an infinite number of distinct ideals in $\mathfrak{N}_*$. Further, they eliminate all of the plausible conjectures I have as to the structure of the ideals. It appears very difficult to determine the ideal associated with a given d.

The only general facts I know are the triviality:

Observation. If d divides d', then $I_{d'}$ is contained in I_d;
and the not at all obvious:

Observation. I_{2^s} contains the ideal generated by the 2^s-th powers.

To see this, let $\alpha \in \mathfrak{N}_*$ be a positive dimensional class and let $\alpha = [M^n]$ where M has a non-trivial involution t, with fixed data (F^k, ν^{n-k}) for various $k < n$. Then consider the involution

$$s = t \times t \times \cdots \times t \ (2^s \text{ factors})$$

On $M \times \cdots \times M = N$ (2^s factors). The fixed set of s is the union of the sets $F^{k_1} \times \cdots \times F^{k_{2^s}}$ with the normal bundle the Whitney sum. There are an even number of copies of any such factor for which the k's are not all the same, and so the fixed data of s consists of the classes

$$(F^k \times \cdots \times F^k, \nu^{n-k} \times \cdots \times \nu^{n-k})$$

in $\mathfrak{N}_{2^s k}(BO_{2^s(n-k)})$. Now

$$w_i(\nu \times \cdots \times \nu) = \sum w_i(\nu) \otimes \cdots \otimes w_{i_{2^s}}(\nu)$$

summed over all partitions of i and by symmetry the terms in the Stiefel Whitney numbers will all cancel out except for

$$w_j(\nu) \otimes \cdots \otimes w_j(\nu)$$

in the $w_{2^s j}(\nu \times \cdots \times \nu)$. Thus if $f\colon BO_{n-k} \to BO_{2^s(n-k)}$ is the map classifying $2^s\gamma$, every class in the kernel of

$$f^*\colon H^*(BO_{2^s(n-k)}; Z_2) \to H^*(BO_{n-k}; Z_2)$$

gives zero in the Stiefel Whitney numbers of

$$(F^k \times \cdots \times F^k, \nu^{n-k} \times \cdots \times \nu^{n-k}).$$

That is sufficient to guarantee that this class lies in the image of f_* on bordism, and so α^{2^s}, the class of N, is in I_{2^s}.

References

1. G. E. Bredon, *Introduction to compact transformation groups*, Academic Press, New York, 1972.
2. P. E. Conner, *The bordism class of a bundle space*, Michigan Math. J., vol. 14 (1967), pp. 289–303.
3. P. E. Conner and E. E. Floyd, *Differentiable periodic maps*, Springer Verlag, Berlin, 1964.
4. R. E. Stong, *Unoriented bordism and actions of finite groups*, Mem. Amer. Math. Soc., No. 103, Amer. Math. Soc., 1970.
5. F. Uchida, *Cobordism groups of semifree S^1 and S^3 actions*, Osaka J. Math., vol. 7 (1970), pp. 345–351.
6. J. A. Wolf, *Spaces of constant curvature*, Publish or Perish, Inc., Berkeley, 1977.

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