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CHEVALLEY GROUPS AS STANDARD SUBGROUPS, III

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Introduction

In this paper we complete the proof of the main theorem stated in [13]. This involves proving that a certain subgroup, G_0, of a group G is, in fact, normal in G.

Throughout the paper we let G be a finite group having a standard subgroup A such that $|Z(A)|$ is odd, $\tilde{A} = A/Z(A)$ is a group of Lie type of Lie rank at least 3 and defined over a field of characteristic 2, and $C_G(A)$ has cyclic Sylow 2-subgroups. Let t be an involution in $C_G(A)$ and assume $t \notin Z^*(G)$. In [13] we showed that with the assumption of a certain hypothesis (Hypothesis $(*)$ in [13]) either $\tilde{A} \cong O^+(8,2)'$ and $G \cong \mathrm{Aut}(M(22))$, or there exists a t-invariant semisimple subgroup $G_0 \leq G$ such that $|Z(G_0)|$ is odd and $\tilde{G}_0 = G_0/Z(G_0)$ satisfies one of the following:

(i) $\tilde{G}_0 \cong \tilde{A} \times \tilde{A}$, the components interchanged by t.
(ii) $\tilde{G}_0$ is a group of Lie type defined over a field of characteristic 2 and t acts on $\tilde{G}_0$ as an outer automorphism.

Our goal in the present paper is to show $G_0 = E(G)$. We remark that our work here might be useful in the verification of Hypothesis $(*)$ of [13]. In that hypothesis we assumed results concerning standard subgroups of type $PSp(6,2)$, $PSU(6,2)$, and $O^{\pm}(8,2)'$. However, these assumptions are only needed in establishing the existence of the group G_0. Therefore, when one considers standard subgroups of type $PSp(6,2)$, $PSU(6,2)$, or $O^{\pm}(8,2)'$ it will probably be sufficient to just construct an appropriate group G_0. The methods of this paper will yield $G_0 \trianglelefteq G$ once results analogous to those in §15 are verified.

It is in this part of the proof of the main theorem that we make use of the $B(G)$-conjecture, and here it is only used for the "wreathed case" (type (i)). With extra work and signalizer arguments it may be possible to work around the $B(G)$-conjecture. In the "quasisimple case" (type (ii) we will use a recent result of Holt [10].

Throughout the paper we keep the above notation. Also, we set $\Omega = \{G_0^g\colon g \in G\}$ and write $\alpha = G_0$. Our goal is to show that $\Omega = \{\alpha\}$. Notice that $G_\alpha = N_G(G_0)$. For $X \leq G$, let $\Delta(X)$ denote the set of fixed points of X on Δ.

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The paper is organized as follows. In §13 we prove a result on automorphisms of the Sylow 2-subgroups of Chevalley groups (normal or twisted) in characteristic 2. This is based on a paper of Gibbs [6]. In §14 we prove some results about the action of G on Ω, under the assumption that $\Delta(t)=\{\alpha\}$. In §15 we verify that $\Delta(t)=\{\alpha\}$ and apply the results of §14. Then §16 and §17 handle the wreathed case and the quasisimple case, respectively.

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13. Preliminaries

In this section we reduce to the case $O(G)=1$ and prove a result concerning $\mathrm{Aut}(U)$, where $U\in Syl_2(G)$. This result is based on a result of Gibbs [3].

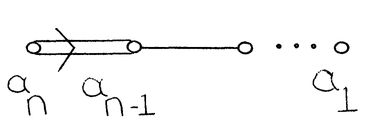
(13.1) *$L(G)=E(G)$, so we may assume $O(G)=1$.*

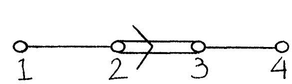
Proof. $A=L(C_G(t))\leq L(G)$. The argument of (5.1) of [13] shows that $A^G\leq C_G(O(G))$. Since $C_G(A)$ is solvable, $\langle A^G\rangle = L(G)$ and the result follows.

From now on we assume $O(G)=1$.

(13.2) *Let G be a finite group of Lie type defined over a field $\mathbf{F}_q$ of characteristic 2. Assume that $G/Z(G)\not\cong L_4(q)$ and that G has Lie rank $n\geq 3$. Let $U\in Syl_2(G)$ and $U_{\alpha_1},\ldots,U_{\alpha_n}$ the root subgroups of U, corresponding to a root system Σ of G. Let σ be an automorphism of U and write $U_0=\langle U_\alpha\colon \alpha\in\Sigma^+,\ \alpha$ a compound root$\rangle$. Then $U_0=U_0^\sigma$ and σ permutes the groups $U_{\alpha i}U_0$, $i=1,\ldots,n$, the permutation given by a graph automorphism of the Dynkin diagram of G.*

Proof. The ideas here are similar to those in (6.2)A of Gibbs [6]. First we show that U_0 is σ-invariant. By Lemma 4 of [12], $U_0=U'$ except in the cases $G\cong PSp(2n,2)$ and $G\cong F_4(2)$. In the latter cases label the Dynkin diagram as follows:





Then it is easily checked using the commutator relations that

$$U'=\langle U_{\alpha_n+\alpha_{n-1}}(1)U_{\alpha_n+2\alpha_{n-1}}(1),\quad U_\alpha\colon \alpha\in\Sigma^+ \ \text{compound},\ \ \alpha\neq\alpha_n+\alpha_{n-1},\ \alpha_n+2\alpha_{n-1}\rangle$$

or

$$\langle U_{\alpha_2+\alpha_3}(1)U_{\alpha_2+2\alpha_3}(1),\ U_\alpha\colon \alpha\in\Sigma^+ \ \text{compound},\ \ \alpha\neq\alpha_2+\alpha_3,\ \alpha_2+2\alpha_3\rangle.$$

Certainly α normalizes U'. We get $U_0^\sigma = U_0$ by checking that $U_0/\gamma_3(U) = Z(U/\gamma_3(U))$.

We consider the action of σ on U/U_0. As U/U_0 is the direct product of the groups $U_{\alpha_i}U_0/U_0 \cong U_{\alpha_i}/U'_{\alpha_i}$ (U_{α_i} may be non-abelian if $G/Z(G) \cong PSU(2n+1, q)$), we may indicate elements of U/U_0 by n-tuples, each entry an element of $\mathbf{F}_q$ (or possibly $\mathbf{F}_{q^2}$ in twisted cases). Choose $t_1, \ldots, t_n$ with $t_i \in \mathbf{F}_q^\#$ (or $\mathbf{F}_{q^2}^\#$) and consider the elements $U_{\alpha_i}(t_i)^\sigma = \prod U_{\alpha_j}(t_{ij})$ (mod U_0). We claim that the matrix $T = (t_{ij})$ is monomial with permutation given by a graph automorphism of the Dynkin diagram. Identify $U_{\alpha_i}(t_i)^\sigma$ with $(t_{il}, \ldots, t_{in})$.

Note that except for the case $\tilde{G} \cong PSU(2n+1, q)$ each U_{α_i} is abelian. In the following we exclude the case $\tilde{G} \cong PSU(2n+1, q)$, but the arguments for $PSU(2n+1, q)$ are only slightly different than those presented. It is never the case that α_i, $\alpha_j \in \Sigma^+$ are short roots and $\alpha_i + \alpha_j \in \Sigma^+$ is a long root. Consequently, if $\alpha_i, \alpha_j, \alpha_i + \alpha_j$ are all in Σ^+, then $[U_{\alpha_i}, U_{\alpha_j}]$ projects onto $U_{\alpha_i + \alpha_j}$. Using this fact we easily see that given any $i = 1, \ldots, n$, there do not exist adjacent roots α_j and α_k with $t_{ij} \neq 0 \neq t_{ik}$. For otherwise, $U_{\alpha_i}(t_i)^\sigma$ would not be an involution.

We indicate the proof of the claim by considering an example, thereby simplifying notation and case analysis. Say $\tilde{G} \cong PSp(2n, q)$ $n \geq 4$ and label the Dynkin diagram as before. We begin by letting $\hat{U}_0 = [U_0, U]$ and checking that $|[U_{\alpha_i}(t), U]\hat{U}_0/\hat{U}_0| = q$ or q^2, depending on whether α_i is an end node ($i = 1$ or n) or not. This fact, together with the adjacency condition, above, and the Chevalley commutator relations implies the following:

(a) $(t_{i1}, \ldots, t_{in})$ has at most 3 non-zero entries, equality possible only if $n = 5$ and $(t_{i1}, \ldots, t_{in}) = (t_{i1}, 0, t_{i3}, 0, t_{i5})$.

(b) If $i = 1$ or n, then $(t_{i1}, \ldots, t_{in}) = (t_{i1}, 0, \ldots, 0)$ or $(0, \ldots, 0, t_{in})$.

(c) If $(t_{i1}, \ldots, t_{in})$ has precisely two non-zero entries, then

$$(t_{i1}, \ldots, t_{in}) = (t_{i1}, 0, t_{i3}, 0, \ldots, 0),$$
$$(0, \ldots, t_{i,n-2}, 0, t_{in}) \quad \text{or} \quad (t_{i1}, 0, \ldots, 0, t_{in}).$$

Suppose $t_{11} \neq 0$, so $(t_{11}, \ldots, t_{1n}) = (t_{11}, 0, \ldots, 0)$ by (b). Since

$$[U_{\alpha_1}(t_1), U_{\alpha_2}(t_2)] \not\in \hat{U}_0,$$

we must have $t_{22} \neq 0$. As $[U_{\alpha_2}, U_{\alpha_n}] = 0$, this and (b) imply that

$$(t_{n1}, \ldots, t_{nn}) = (0, \ldots, 0, t_{nn}).$$

By (a) and (c), $t_{2j} = 0$ for $j \neq 2$ unless $n = 4$ and

$$(t_{21}, t_{22}, t_{23}, t_{24}) = (0, t_{22}, 0, t_{24}).$$

Suppose this occurs. By symmetry, $(t_{31}, \ldots, t_{34}) = (t_{31}, 0, t_{33}, 0)$ with $t_{33} \neq 0$.

Now

$$[U_{\alpha_1}(_1), U_{\alpha_2}(t_2), U_{\alpha_3}(t_3), U_{\alpha_2}(t_2)] = 1,$$

whereas the image of this commutator under σ is non-trivial. So $t_{2j} = 0$ for $j \neq 2$.

Now consider $t_{n-1,j}$. As above $t_{n-1,n-1} \neq 0$ and $t_{n-1,j} = 0$ for $j \neq n-1$, unless $n = 4$ and $j = 1$. In the latter case

$$(t_{n-1,1}, \ldots, t_{n-1,n}) = (t_{31}, 0, t_{33}, 0).$$

Computing $\gamma_5(U)$, we have $[U_{\alpha_3}(t_3), \gamma_5(U)] = 1$, but $[U_{\alpha_3}(t_3)^{\sigma}, \gamma_5(U)] \neq 1$. This is a contradiction. Thus, $t_{n-1,j} = 0$ for $j \neq n-1$.

Next, consider $(t_{31}, \ldots, t_{3n})$. As $[U_{\alpha_2}(t_2), U_{\alpha_3}(t_3)] \neq 1$, either t_{31} or t_{33} is non-zero. Suppose $t_{31} \neq 0 \neq t_{33}$. Since $[U_{\alpha_1}, U_{\alpha_4}] = [U_{\alpha_2}, U_{\alpha_4}] = 1$, we conclude that $t_{41} = t_{42} = t_{43} = 0$. If $n = 5$, then (c) yields $t_{4j} = 0$ for $j \neq 4$. If $n > 5$, then (a) and (c) imply $t_{3j} = 0$, for $j > 3$. Since $[U_{\alpha_3}(t_3), U_{\alpha_4}(t_4)] \neq 1$, we must have $t_{44} \neq 0$ in either case. From here we get a contradiction by looking at $[U_{\alpha_4}(t_4), U_{\alpha_3}(t_3), U_{\alpha_2}(t_2), U_{\alpha_3}(t_3)]$. This commutator is trivial, whereas its image under σ is non-trivial. Therefore, $t_{31} = 0$ or $t_{33} = 0$.

Suppose $t_{33} = 0$. As above, $t_{41} = t_{42} = t_{43} = 0$. Since

$$[U_{\alpha_3}(t_3), U_{\alpha_4}(t_4)] \neq 1,$$

$t_{3j} \neq 0$ for some $j > 3$. Then (c) implies that $j = n$. But then

$$[U_{\alpha_3}(t_3), U_{\alpha_{n-1}}(t_{n-1})]^{\alpha} \notin [U_0, U],$$

and we conclude $n - 1 = 4$, $n = 5$. The commutator relations imply that

$$[U_{\alpha_2}(t_2), U_{\alpha_3}(t_3), U_{\alpha_4}(t_4)] = U_{\alpha_2+\alpha_3+\alpha_4}(t_2 t_3 t_4),$$

whereas the image of the left side (under σ) is in $[U_0, U, U]$. This is a contradiction. We now have $t_{31} = 0$ and $t_{33} \neq 0$. From here we conclude $t_{3j} = 0$ for $j \neq 3$. Otherwise, (c) forces $n = 5$ and $j = 5$, and we get a contradiction by considering the commutator of length 4 of the previous paragraph.

Similarly, we show $t_{n-2,j} = 0$ if $j \neq n-2$. First show that one, but not both of $t_{n-2,n-2}$ and $t_{n-2,n}$ are non-zero. If $t_{n-2,n} \neq 0$, argue, as before, that $n = 5$ and $t_{n-2,1} \neq 0$. But we have shown that $t_{n-2,1} = t_{31} = 0$. Thus, $t_{n-2,n-2} \neq 0$, $t_{n-2,n} = 0$ and arguing as above, we show $t_{n-2,j} = 0$ for $j \neq n-2$.

At this stage it is easy to show that $T = (t_{ij})$ is diagonal. The preceding arguments allow us to assume $n > 5$. If $3 < i < n-2$, then (a) implies $(t_{i1}, \ldots, t_{in})$ has at most two non-zero entries. For $i = 4$, (c) and $[U_{\alpha_3}(t_3), U_{\alpha_4}(t_4)] \neq 1$ yield $t_{4j} = 0$ for $j \neq 4$. Continue in this way to complete the argument.

If $t_{11} = 0$, then $t_{n1} \neq 0$ and we argue, as above, that T is skew-diagonal. But then

$$[[U_{\alpha_1}(t_1), U_{\alpha_2}(t_2)], [U_{\alpha_2}(t_2), U_{\alpha_3}(t_3)]] = 1,$$

whereas the image of this commutator under σ is non-trivial. We have now proved the claim for $\tilde{G} \cong PSp(2n, q)$ and $n \geq 4$.

For $\tilde{G} \cong PSp(6, q)$ we use slightly different arguments. It is easy to see that

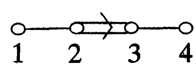
$$(t_{11}, t_{12}, t_{13}) = (t_{11}, 0, t_{13}), \qquad (t_{21}, t_{22}, t_{23}) = (0, t_{22}, 0),$$

and

$$(t_{31}, t_{32}, t_{33}) = (t_{31}, 0, t_{33}).$$

Next, check that $[U_{\alpha_1}(t_1), U_0]$ covers $\hat{U}_0/[\hat{U}_0, U]$ (a group of order q^2), while $[U_{\alpha_3}(t_3), U_0]\hat{U}_0/\hat{U}_0$ has order q. Comparing with the images of these groups under σ, we have $t_{31} = 0$. If $t_{13} \neq 0$, then $[U_{\alpha_1}(t_1), U_{\alpha_2}(t_2)]$ is an involution, while the image of this element under σ is not an involution. Therefore, $t_{13} = 0$ and the claim holds in this case.

For the other Chevalley groups use similar arguments. When the Dykin diagram has more than two end nodes, the arguments are simpler. For the cases $\tilde{G} \cong F_4(q)$ or ${}^2E_6(q)$ reason as follows. Label the Dynkin diagram



and let $n_i = |[U_{\alpha_i}(t_i), U]U_0/U_0|$. Then

$$(n_1, n_2, n_3, n_4) = (q, q^2, q^2, q) \quad \text{or} \quad (q, q^3, q^3, q^2),$$

according to $\tilde{G} \cong F_4(q)$ or ${}^2E_6(q)$. An easy check gives $t_{1j} = 0$ if $j \neq 1$, $t_{4j} = 0$ if $j \neq 4$,

$$(t_{21}, t_{22}, t_{23}, t_{24}) = (0, t_{22}, 0, t_{24})$$

and

$$(t_{31}, t_{32}, t_{33}, t_{34}) = (t_{31}, 0, t_{33}, 0).$$

To conclude that $t_{24} = 0 = t_{31}$, compare the action of $U_{\alpha_2}(t_2)$ and $U_{\alpha_3}(t_3)$ on $\hat{U}_0/[\hat{U}_0, U]$ with the action of their images under σ. The claim follows.

We can now complete the proof of (13.2). Choose $t_i' \in \mathbf{F}_q$ (or $\mathbf{F}^{\#}_{q^2}$) for $i = 1, \ldots, n$, and consider the images of $U_{\alpha_i}(t_i)$, $U_{\alpha_i}(t_i')$, and $U_{\alpha_i}(t_i + t_i')$ under σ. By the claim, each of the corresponding matrices is monomial. This forces each to have the same underlying permutation matrix, and the result follows.

(13.3) *Let G, U, σ be as in* (13.2). *Then there exist graph, diagonal, and field automorphisms g, d, f, respectively, such that σ and gdf induce the same automorphism on U/U_0.*

Proof. This follows from (13.2) as in the proof of (6.5) of [6].

14. The embedding of G_0 in G.

In this section we prove results in a more general situation than actually needed for the proof of the main theorem. This is done so that the results will be helpful in the verification of Hypothesis (*). We let A be quasisimple such that $\tilde{A}$ is a group of Lie type having Lie rank at least 2 and defined over a field of characteristic 2. Assume $\tilde{A} \not\cong L_3(2)$, $Sp(4, 2)'$ and that G contains a t-invariant semi-simple subgroup $G_0 > A$ such that either

(i) $\tilde{G}_0$ is a group of Lie type in characteristic 2, with t inducing an outer automorphism of $\tilde{G}_0$; or
(ii) $\tilde{G}_0 \cong \tilde{A} \times \tilde{A}$, with t interchanging the components of $\tilde{G}_0$.

We assume that t fixes a unique point of $\Omega = \{G_0^g : g \in G\}$, namely G_0. Note that $C_G(G_0) \leq C_G(A)$ implies that $|C_G(G_0)|$ is odd. Moreover, $|\Omega|$ is odd.

(14.1) *Assume that* (i) *holds and* $(\tilde{A}, \tilde{G}_0) \neq ({}^2F_4(q)', F_4(q))$, $(PSp(n, q), O^+(n+2, q)')$ *with* $4 \mid n+2$, *or* $(O^-(n, q)', O^+(n, q^2)')$ *with* $4 \mid n$. *Then* $N_G(G_0)$ *controls fusion of its involutions.*

Proof. To prove this it is first necessary to check that with the exception of the cases listed t centralizes a G_0 conjugate of each involution in $N_G(G_0)$. For this we use the results of [3]. First use §8 and §19 of [3] to determine the possible action of t on G_0, and then check that each class of involutions in G_0 has a representative centralized by a conjugate of t. This is easy using the results of [3], although a case analysis is required.

Now consider G acting as a transitive permutation group on Ω. Fix an involution $z \in N(G_0)$. If β is any point of Ω fixed by z, then by the previous paragraph z is centralized by a conjugate of t fixing β. It follows (by Gleason's lemma) that $C_G(z)$ is transitive on the set of fixed points of z on Ω. But now the converse of Witt's theorem gives the result. Indeed, suppose $z^g \in N(G_0)$. Then z fixes G_0, $G_0^{g^{-1}}$ and so for some $c \in C_G(z)$, $G_0^{g^{-1}} = G_0^c$. But then $cg \in N(G_0)$ and $z^{cg} = z^g$.

(14.2) *Each pair of points in* Ω *is interchanged by a conjugate of* t.

Proof. Let $\beta \neq \gamma$ be points in Ω, $t_1 \in t^G \cap G_\beta$ and $t_2 \in t^G \cap G_\gamma$. Consider $D = \langle t_1, t_2 \rangle$ and $E = O_2(D)$. We claim that $\langle t_1 \rangle E$ fixes β. Suppose $(\langle t_1 \rangle E)_\beta < \langle t_1 \rangle E$. Since t_1 fixes just β, $\langle t_1 \rangle E_\beta$ fixes just β, and hence $N((\langle t_1 \rangle E)_\beta) \cap \langle t_1 \rangle E$ must also fix β. This is a contradiction.

Now $\langle t_1 \rangle E \sim \langle t_2 \rangle E$ by an element $d \in \langle t_1 t_2 \rangle$. So $t_1 d$ is an involution interchanging β and γ. As $t_1 d$ is conjugate to t_1 or t_2 we have the result.

For the remainder of this section we assume that condition (ii) holds. Set $\alpha = G_0 \in \Omega$. Let G_1, G_2 be the components of G_0. Let $S_1 \in Syl_2(G_1)$, $S_2 = S_1^t$, $S_0 = S_1 \times S_2$, and $S_0 \leq S \in Syl_2(N(G_0))$. We assume $t \in S$. Let $\bar{S}$ be the weak closure of S_0 in S, with respect to G, and for $j=1, 2$ write $\hat{G}_j = C_{G_\alpha}(G_i)$, where $\{i, j\}=\{1, 2\}$. Let $\hat{S}_i = S \cap \hat{G}_i$.

(14.3) *$\bar{S} = S_1 \times S_2$ or $S_1\langle d_1\rangle \times S_2\langle d_2\rangle$, where for $i=1, 2$, d_i is an involution inducing a graph automorphism of G_i and centralizing $Z(S_i)$. Also, $S \in Syl_2(G)$. In particular, $\bar{S} = (\bar{S} \cap \hat{S}_1) \times (\bar{S} \cap \hat{S}_2)$.*

Proof. As t fixes just one point of Ω, $|\Omega|$ is odd and $S \in Syl_2(G)$. Suppose $\bar{S} > S_0$ and $S_0 \neq S_0^g \leq \bar{S}$. If $x \in S_1^g$, then

$$|C(x) \cap S_0^g| \geq |S_2|\,|S_1 \colon S_1'| > |C_S(y)|$$

for any $y \in S-(N(S_1) \cap N(S_2))$. So $x \in N(S_1) \cap N(S_2)$. Letting x vary and using symmetry we have $S_0^g \leq N(S_1) \cap N(S_2)$.

Regard $(N(G_1) \cap N(G_2))/Z(G_0)$ as a subgroup of $\mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$. Since $S_0^g/(S_0^g)'$ is elementary abelian, we have $S_0^g S_0/S_0$ elementary of order at most 2^4, and of order at most 2^2 unless $\tilde{G}_1 = PSL(n, q)$, $O^+(n, q)'$ or $E_6(q)$. If $\tilde{G}_i \cong PSU(n, q)$, $O^-(n, q)'$, ${}^2E_6(q)$, or ${}^2F_4(q)'$, then $Z(\bar{S}) = Z(S_0)$, as $Z(S_i)$ is centralized by all involutions in a Sylow 2-subgroup of $\mathrm{Aut}(\tilde{G}_i)$, for $i=1, 2$.

Suppose $\tilde{G}_1 \cong PSL(n, q)$ for $n \geq 4$, $O^+(n, q)'$ for $n \geq 8$, $E_6(q)$, $E_7(q)$, $E_8(q)$, $G_2(q)$, or ${}^3D_4(q)$. Then $|Z(S_1)| = q$ and we are done if $q=2$. So suppose $q>2$ and let $I = O_2(C_{G_1}(Z(S_1)))$. Then $Z(I) = Z(S_1)$, I is special, and for L maximal in $Z(I)$, I/L is extraspecial of order at least $2q^4$. These facts follow from the Chevalley commutator relations (for the groups $E_6(q)$, $E_7(q)$, and $E_8(q)$ a reference is (4.4) of [5]). If L can be chosen such that $S_0^g \cap Z(S_1) \leq L$, then $S_0^g \cap I$ maps to an abelian subgroup of I/L not containing $Z(I/L)$. This contradicts $|I \colon S_0^g \cap I| \leq 2^4$. Therefore, $S_0^g \geq Z(S_1)$. If $x \in Z(S_1)$, then $|S_0^g \colon C_{S_0^g}(x)| \leq 2$, so the commutator relations imply that $x \in Z(S_0^g)$. So $Z(S_1) \leq Z(S_0^g)$, and by symmetry we have $Z(S_0) = Z(S_0^g)$.

We claim that $Z(S_0) = Z(S_0^g)$ in all cases. We are left with $\tilde{G}_1 \cong L_3(q)$, $F_4(q)$, or $PSp(n, q)$. In the first case note that each involution in S_0^g is contained in an elementary abelian subgroup of order q^4, and, recall, $q>2$ here. From $S_0^g = \Omega_1(S_0^g)$ we see that $S_0^g = S_0$. For the other cases $|S_0^g S_0/S_0| \leq 4$, and so $L_i = S_0^g \cap S_i$ is of index at most 4 in S_i, for $i=1, 2$. If $q>2$, argue as follows. From (19.5) of [3] we easily have $Z(S_0^g) \leq S_0$. Choose $a \in Z(S_0^g) - Z(S_0)$ with $a = a_1 a_2$ and $a_i \in S_i$, for $i=1, 2$. We may assume $a_1 \notin Z(S_1)$. Then $|S_1 \colon C_{S_1}(a_1)| \geq q$, while $L_1 \leq C_{S_1}(a_1)$. So $q=4$ and $L_1 = C_{S_1}(a_1) \geq Z(S_1)$. As before, $Z(S_1) \leq S_0^g$ implies $Z(S_1) \leq Z(S_0^g)$. But this forces $S_0^g \leq S_1 \hat{S}_2$ and $|S_0^g S_0/S_0| \leq 2$. In turn $|S_1 \colon L_1| \leq 2$, contradicting $L_1 = C_{S_1}(a_1)$. So we must have $q=2$. We may assume $\tilde{G}_1 \cong F_4(2)$, for otherwise $S_0^g = S_0$. Here we show, directly, that $S_0^g = S_0$. Let $U \in Syl_2(A)$, write U as a product of root subgroups U_α, for $\alpha \in \Sigma^+$, and set

$$\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2+\alpha_3, \alpha_2+2\alpha_3\}.$$

From the commutator relations $U = \langle U', U_\alpha \colon \alpha \in \Delta\rangle$. For each $\alpha \in \Delta$ a check of the commutator relations shows that $|C_U(U_\alpha)| \geq 2^{16}$. Assuming $U = C_{S_0}(t)$ and letting $S_{i\alpha}$ be the projection of U_α to S_i, we then have $|S_0^g \cap C(S_{i\alpha}^g)| \geq 2^{40}$. But, if $x \in N(G_0) - G_0$ is an involution, then Sylow 2-subgroups of $C_{G_\alpha}(x)$ have order strictly less than 2^{40} (see (19.5) of [3]). So $S_{i\alpha}^g \leq S_0$ for each $i = 1, 2$ and $\alpha \in \Delta$. This gives $S_0^g = S_0$ and we have proved the claim. Therefore, $Z(\bar{S}) = Z(S_0)$.

Regard $\bar{S} \leq \mathrm{Aut}(\tilde{G}_1) \times \mathrm{Aut}(\tilde{G}_2)$. We may think of $\bar{S} \leq S_1\langle d_1 \rangle \times S_2\langle d_2 \rangle$, where d_i is a graph automorphism of G_i, centralizing G_i^t and $Z(S_i)$. Of course it may happen that $d_i \notin G_\alpha$, although we will show that this does not occur. As $\bar{S}$ is t-invariant, $\bar{S}$ is as desired or $\bar{S} = S_0\langle d_1 d_2 \rangle$. Suppose the latter holds and set $(\alpha)g^{-1} = \beta$. Then $S_0^g \leq G_\alpha$ implies $S_0 \leq G_{\alpha\beta}$. By (14.2) there is a conjugate, t_1 of t, such that t_1 interchanges α and β and t_1 normalizes $\bar{S} \cap G_{\alpha\beta}$. If t_1 normalizes S_0, then $S_0^g = S_0^{t_1 g}$. But $t_1 g \in G_\alpha$, so $S_0^{t_1 g} \leq G_1 \times G_2$, a contradiction. Therefore $S_0^{t_1} \neq S_0$. So $\langle S_0, S_0^{t_1} \rangle = \bar{S} = S_0\langle d_1 d_2 \rangle$. However, if $x \in S_1$, then $C_{\bar{S}}(x) \geq C_{S_1}(x) \times S_2$ as a subgroup of index at most 2, while no element in $\bar{S} - S_0$ has such a centralizer. Therefore, $S_1^{t_1} \leq S_0$ and, similarly, $S_2^{t_1} \leq S_0$. But then $S_0 = S_0^{t_1}$, a contradiction.

(14.4) *If $G_i^g \leq G_\alpha$, then $G_i^g \in \{G_1, G_2\}$.*

Proof. Say $G_1^g \leq G_\alpha$. As $G_\alpha^{(\infty)} = G_0$, $G_1^g \leq G_0 = G_1 G_2$. Assume $G_1^g \notin \{G_1, G_2\}$. Then $G_1^g \cap G_1$ and $G_1^g \cap G_2$ are both contained in $Z(G_0)$.

Let $X_i = Z(S_i) \cap \bar{S}_i'$, for $i = 1, 2$. We may assume $S_1^g \leq S_1 \times S_2$. Then considering the image of G_1^g in G_0/G_i for $i = 1, 2$ we have

$$X_1^g \leq Z(S_1) \times Z(S_2).$$

However, $X_1^g \cap Z(S_1) = X_1^g \cap Z(S_2) = 1$.

By (14.3) $\bar{S}, \bar{S}^g \leq C(X_1^g)$. As $S \in Syl_2(G)$ and $\bar{S}$ is weakly closed in S, there is an element $h \in C(X_1^g)$ such that $\bar{S} = \bar{S}^{gh}$. By (14.3), $\bar{S}$ is a direct product $\bar{S} = \bar{S}_1 \times \bar{S}_2$ where $\bar{S}_i = S_i$ or $S_i\langle d_i \rangle$, for $i = 1, 2$.

Apply the Krull–Schmidt theorem to get

$$\bar{S} = \bar{S}_1 \times \bar{S}_1^{gh} \quad \text{or} \quad \bar{S} = \bar{S}_1 \times \bar{S}_2^{gh}.$$

Then

$$Z(\bar{S}_1) \times \bar{S}_2 = C_{\bar{S}}(\bar{S}_1) \geq \bar{S}_1^{gh} \quad \text{or} \quad \bar{S}_2^{gh}.$$

So $X_2 = Z(C_{\bar{S}}(\bar{S}_1)) \cap C_{\bar{S}}(\bar{S}_1)' = X_1^{gh}$ or X_2^{gh}, respectively. From the remarks in the second paragraph we conclude that $X_2 = X_2^{gh}$. Repeat this argument starting with $\bar{S} = \bar{S}_2 \times \bar{S}_1^{gh}$ or $\bar{S} = \bar{S}_2 \times \bar{S}_2^{gh}$ and conclude that $X_1 = X_1^{gh}$ or X_2^{gh}. Since $X_2^{gh} = X_2$ we must have $X_1 = X_1^{gh}$, which is a contradiction. This completes the proof of (14.4).

(14.5) *For $i = 1, 2$, Gi fixes precisely one point of Ω, namely α.*

Proof. Let $\Omega(G_i)$ be the set of fixed points of G_i on Ω, and assume that $\alpha \neq \beta \in \Omega(G_1)$. By (14.2) there is a conjugate, t_1 of t, with $\alpha^{t_1} = \beta$. Then

$\alpha \in \Omega(G_1^{t_1})$, so $G_1^{t_1} \leq G_\alpha$ and by (14.4), $G_1^{t_1} = G_1$ or $G_1^{t_1} = G_2$. If $G_1^{t_1} = G_2$, then $\Omega(G_1^{t_1}) = \Omega(G_2)$, forcing $\{\alpha, \beta\} \subseteq \Omega(G_1) \cap \Omega(G_2)$. But then G_0 fixes α and β, which is impossible. Therefore, $G_1^{t_1} = G_1$.

Now t_1 normalizes a Sylow 2-subgroup, V, of $C(G_1)$. Conjugating by an element of $N(G_1)$, we may assume $t_1 \in N(\bar{S})$ (although we may no longer have $\alpha^{t_1} = \beta$). Write $\bar{S} = \bar{S}_1 \times \bar{S}_2$, where for $i = 1, 2$ $\bar{S}_i = \bar{S} \cap \hat{S}_i = S_i$ or $S_i\langle d_i \rangle$. Also, $t_1 \in N(\bar{S} \cap G_1) = N(S_1)$.

Apply the Krull–Schmidt theorem to $\bar{S} = \bar{S}_1 \times \bar{S}_2$, and conclude that the pair $\{\bar{S}_1 Z(\bar{S}), \bar{S}_2 Z(\bar{S})\}$ is determined uniquely by the abstract structure of $\bar{S}$. Let $J = N(\bar{S}) \cap N(\bar{S}_1 Z(\bar{S}))$. Then $J\langle t \rangle = N(\bar{S})$ and $S \in Syl_2(J\langle t \rangle)$. Since $t_1 \in N(S_1)$ we must have $t_1 \in J$ and hence S contains a conjugate, t_2, of t_1, normalizing $\bar{S}_1 Z(\bar{S})$. But then t_2 fixes α, so t and t_2 are conjugate in G_α. The contradiction is that t_2 normalizing $\bar{S}_1 Z(\bar{S})$ forces $t_2 \in N(G_1)$ and $N_{G_\alpha}(G_1) = N_{G_\alpha}(G_1) \cap N_{G_\alpha}(G_2) \trianglelefteq G_\alpha$. We have now proved (14.5).

(14.6) *Suppose $N_G(\hat{S}_1) \not\leq G_\alpha$. Then there is a point $\beta \neq \alpha$ fixed by $\hat{S}_1$ and a conjugate t_1 of ι, such that t_1 interchanges α and β, and $tt_1 \in N(\hat{S}_i)$, for $i = 1, 2$.*

Proof. We have $G_2 \leq C(\hat{S}_1)$. If $G_2 \trianglelefteq C(\hat{S}_1)$, then by (14.5) $C(\hat{S}_1)$ fixes just the point α, and since $C(\hat{S}_1) \trianglelefteq N(\hat{S}_1)$ we also have $N(\hat{S}_1) \leq G_\alpha$, against the hypothesis. So $G_2 \ntrianglelefteq C(\hat{S}_1)$, and we choose $g \in C(\hat{S}_1)$ with $G_2^g \neq G_2$. Let $\beta = \alpha^g$.

Then $\hat{S}_1 \times G_2^g \leq G_\beta = N(G_1^g G_2^g)$, and so $\hat{S}_1 \leq \hat{G}_1^g$. Also, $S_1 \leq G_1$ implies $\hat{S}_1 \leq G_1$. Let t_1 be a conjugate of t with t_1 interchanging α and β. Then $G_0^{t_1} = G_0^g$, so $G_1^{t_1} = G_1^g$ or G_2^g. Say $G_1^{t_1} = G_1^g \geq S_1$. Then $\langle S_1, S_1^{t_1} \rangle \leq G_1$ and so $S_1 \in Syl_2(\langle S_1, S_1^{t_1} \rangle)$. Since t_1 normalizes $\langle S_1, S_1^{t_1} \rangle$, t_1 normalizes a Sylow 2-subgroup of $\langle S_1, S_1^{t_1} \rangle$, and we may assume $t_1 \in N(S_1)$. We then get a conjugate of $t_1 \in N(S_1) \cap N(\bar{S})$. The argument of (14.5) gives a contradiction.

Therefore, $G_1^{t_1} = G_2^g$ and $G_1 = G_2^{gt_1}$. By (14.5), $gt_1 \in G_\alpha$ and so

$$\hat{S}_1 \times \hat{S}_1^{t_1} = \hat{S}_1 \times \hat{S}_1^{gt_1} \in Syl_2(\hat{G}_1 \hat{G}_2).$$

Replace t by a G_2-conjugate of t, if necessary, so that $\hat{S}_1^{t_1} = \hat{S}_2 = \hat{S}_1^t$. Then $t_1 t \in N(\hat{S}_1)$. But also, $S_1 \leq G_2^{t_1}$ implies $S_1^{t_1} \leq G_2$ and so $S_1^{t_1 t} \leq G_1$. So $S_1^{t_1 t} \leq \hat{S}_1 \cap G_1 = S_1$ and $t_1 t \in N(S_1)$. Consequently, $tt_1 = (t_1 t)^{-1} \in N(S_1)$ and so $S_2^{t_1 t} = S_1^{tt_1 t} = S_1^t = S_2$. Similarly, $tt_1 \in N(\hat{S}_2)$, proving the result.

We now prove our main result concerning groups satisfying (ii).

(14.7) *Assume that*:

(a) $N_G(\hat{S}_i) \leq G_\alpha$, *for $i = 1, 2$.*

(b) *If $I \cong G_2$ is a standard subgroup of a proper section, W of G, with $m_2((C_W(I)) = 1$, then $E(\tilde{W})$ is a Chevalley group defined over a field of characteristic 2, $E(\tilde{W}) \cong \tilde{G}_2 \times \tilde{G}_2$, or $IO(W) \trianglelefteq W$.*

(c) *If $1 \neq X \trianglelefteq \hat{S}_1$ and $O(N(X))G_2 \trianglelefteq N(X)$, then $G_2 \trianglelefteq N(X)$.*

Then $|\Omega| = 1$; that is $G_0 \trianglelefteq G$.

Remarks. We see from (14.6) that if (a) fails then $\hat{S}=\hat{S}_1\times\hat{S}_2$ fixes more than one point of Ω. In classifying groups having a standard subgroup A with $\tilde{A}\cong\tilde{G}_2$, condition (b) will hold in a minimal counterexample. Condition (c) will be a consequence of the B-conjecture. So only condition (a) needs checking in a given problem.

Proof. Let $1<X\leq\hat{S}_1$. We claim that $N_G(X)\leq N_G(G_2)$. If $X=\hat{S}_1$, this is (a). Assuming the claim false choose X as a maximal counterexample. Let $Y=N_{\hat{S}_1}(X)$ and consider the group $\bar{N}=N_G(X)/X$.

By our choice of X, $\bar{Y}\in Syl_2(C_{\bar{N}}(\bar{G}_2))$. Using the maximality of X and the fact that G_2 fixes just α, we see that $\bar{G}_2$ is a standard subgroup of $\bar{N}$. So by (b), the main theorem in [4], and Corollary II in [2], the structure of $N_G(X)$ is known. Let $P/X=E(\bar{N})$ and $P_0=\langle G_2^P\rangle$. Then $P_0\leq C_G(X)$ and P_0 is semi-simple. By (c) we may assume $P_0>G_2$. Let $X_0=X\cap P_0$.

Assume that $m(Y/X)=1$. Then from (b) we see that $|Y/X|=2$ and that the involution in Y/X induces an outer automorphism on P_0. Also $\hat{S}_2\cap P_0=S_2$. Choose $z\in N_{P_0}(\hat{S}_2X_0)-\hat{S}_2X_0$ and such that $z^2\in\hat{S}_2X_0$. Then z normalizes $V=\hat{S}_2\cap\hat{S}_2^z$. From (3.1) and the fact that $Z(G_2)$ has odd order we conclude that either $|X_0|\leq 4$ or that $\tilde{P}_0\cong L_3(4)\times L_3(4)$ and $|X_0|\leq 16$. We claim that $|V|\geq\frac{1}{4}|\hat{S}_2|$. This is clear except in the latter situation. So suppose $\tilde{P}_0\cong L_3(4)\times L_3(4)$. Then $\hat{S}_2=\Omega_1(\hat{S}_2)\leq C(X_0)$ and X_0 is an image of $Z_4\times Z_4$ (the Sylow 2-subgroup of the multiplier of $L_3(4)$). For j an involution in $\hat{S}_2$ we have j^z an involution in $\hat{S}_2X_0=\hat{S}_2\times X_0$, so $j^z\in\hat{S}_2\times\Omega_1(X_0)$. Thus $\hat{S}_2^z\leq\hat{S}_2\times\Omega_1(X_0)$ and the claim follows.

If $|V|>|X|$, then by the maximality of X and symmetry we have $G_1\trianglelefteq N(V)$. But then (14.5) implies that $z\in N(G_2)$, which is not the case. Therefore $|V|\leq|X|$. Let r be the order of a Sylow 2-subgroup of $\tilde{P}_0$. A Sylow 2-subgroup, I, of $P_0\hat{S}_2Y$ has order at least

$$|Y|\,|\hat{S}_2/S_2|\,r=2\,|X|\,|\hat{S}_2/S_2|\,r\geq\tfrac{1}{2}\,|\hat{S}_2|\,|\hat{S}_2/S_2|\,r=\tfrac{1}{2}\,|S_2|\,|\hat{S}/S|\,r.$$

However, it is easy to check that then $|I|>|S|$, whereas $S\in Syl_2(G)$. This is a contradiction and so $m_2(Y/X)>1$.

By the main theorem in [4] the only possibilities for $\tilde{A}$ are $L_3(4)$ and $G_2(4)$. Also Y/X is a Klein group. If $\tilde{A}\cong\tilde{G}_2=L_3(4)$, then $\tilde{P}_0\cong Sz$. By Theorem 6 of [15], $|X_0|\leq 2$, and from here we can argue as in the preceding paragraph. Suppose $\tilde{A}\cong G_2(4)$ and set $Z/X=Z(S_2/X)$. Then Z/X is elementary of order 16 and contains 4 conjugates of Y/X (see (3.5) of [4]). Also, $N_{P_0}(Z)$ is 2-transitive on these conjugates of Y/X. Choose $g\in N_{P_0}(Z)$ with $Y^g\neq Y$. By (18.2)(i) of [3] we have $I/X=(C(Z/X)\cap P_0/X)^{(\infty)}$ contained in G_2X/X and $S_2X/X\in Syl_2(I/X)$. As $G_2X=G_2\times X$ we have $S_2\in Syl_2(I')$ and we may choose $g\in N(S_2)$. We now argue as in the above paragraph to get $|X|\geq|S_2|$. However, this is impossible as $Y\leq\hat{S}_1$, $|\hat{S}_1|\leq 2\,|S_1|$ and $|Y/X|=4$. We have now proved the claim.

In particular, if x, x^g are 2-elements centralizing G_i and G_j, respectively, then $G_i \trianglelefteq C_G(x)$ and $G_j \trianglelefteq C_G(x^g)$. By (14.5), $C_G(x)$ and $C_G(x^g)$ each fix α, and α is the unique point of Ω fixed by $C_G(x)$. It follows that $g \in G_\alpha$. Now apply Theorem 5 of Aschbacher [1] and conclude $G_0 \trianglelefteq G$.

15. Ω

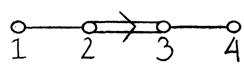
In this section we consider the action of G on Ω. Notation will be as in [13]. In particular, G_0 is the group constructed in Sections 8–12 of [13]. We will show, among other things, that t fixes a unique point of Ω. This will put us in a position to apply the results of §14.

Suppose that $\tilde{A}$ is defined over the field $\mathbf{F}_q$.

(15.1) *$N_G(A) \leq N_G(G_0) = G_\alpha$. In particular, $C_G(t) \leq G_\alpha$.*

Proof. Let $g \in N_G(A)$. Assume, for the moment, that $\tilde{A} \not\cong O^{\pm}(8, q)'$. Then $G_0 = \langle A, E\rangle$, where $E = E(C_G(\bar{J}_r))$ (see (7.3) of [14] for the definition of $\bar{J}_r$). Consider the group $\bar{J}_r^g$. If $\bar{J}_r^g = \bar{J}_r^a$ for some $a \in A$, then $G_0^g = \langle A, E\rangle^a = G_0^a = G_0$. So, suppose $\bar{J}_r^g \neq \bar{J}_r^a$ for any $a \in A$. Then $\tilde{A} \cong F_4(q)$ and the automorphism induced by g on A is not contained in the subgroup of Aut (A) generated by inner automorphisms and field automorphisms.

Let



be the labeling of the Dynkin diagram of A. Then

$$r = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4.$$

Let

$$s = \alpha_2 + 2a_3 + 2\alpha_4, \quad \alpha = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 \quad \text{and} \quad \beta = \alpha_1 + \alpha_2 + \alpha_3.$$

Then $\langle J_r, J_s\rangle = J_r \times J_s$ and $\langle J_r, J_s\rangle^g = \langle J_\alpha, J_\beta\rangle^x$ for some $x \in A$. Choose I to be a $(q+1)$-Hall subgroup of $J_r \times J_s$. Using (7.12) of [14] we see that there is an element $y \in A$ with $I^{gy} = I$. So gy normalizes $O^{2'}(C_G(I)) = O^{2'}(C_E(J_s))$ (see (7.11) of [14]). One now checks that $G_0 = \langle A, O^{2'}(C_G(I))\rangle$, so gy, and hence g, is in $N_G(G_0) = G_\alpha$.

Finally, assume $\tilde{A} \cong O^{\pm}(8, q)'$. Recall that $q > 2$ here. Then for some $x \in A$, gx normalizes J_r. Therefore, gx normalizes $O^{2'}(C_G(J_r))$ and $C_A(J_r)$. Now

$$O^{2'}(C_A(J_r)) \cong L_2(q) \times L_2(q) \times L_2(q) \quad \text{or} \quad L_2(q^2) \times L_2(q),$$

depending on whether $\tilde{A} \cong O^+(8, q)'$ or $O^-(8, q)'$. Also, $O^{2'}(G_0)$ is isomorphic to one of the groups $\tilde{A} \times \tilde{A}$ or $O^+(8, q^2)'$. In any case it follows that the Sylow 2-subgroups of $O^{2'}(C_{G_0}(J_r))$ are elementary abelian of order q^6. Let $t \in X \in Syl_2(C_{G_\alpha}(J_r))$, $X_0 = X \cap G_0$, and

$$X \leq Y \in Syl_2(C_G(J_r)).$$

Then X_0 is elementary of order q^6 and a free $\mathbf{F}_2[\langle t\rangle]$-module. So $t^G \cap X_0 t = t^{X_0}$. It is easy to see that X_0 is weakly closed in X. This, and consideration of the action of X on X_0 shows that $N_Y(X) = X(N_Y(X) \cap C(t)) = X$. So $X \in Syl_2(C_G(J_r))$. Now, as X_0 is weakly closed in X we can apply Corollary 4 of Goldschmidt [8] to get X_0 strongly closed in X (with respect to $C_G(J_r)$). The main theorem of [8] shows that $X_0 \in Syl_2(C)$, where $C = \langle X_0^{C_G(J_r)}\rangle$, so we know the structure of $C/O(C)$. As $O^{2'}(C_C(t)) \leq C_A(J_r)$ we must have $C = O(C)O^{2'}(C_{G_0}(J_r))$.

Let y be an involution in $C_{X_0}(t)$. Then there exists $v \in X_0$ with $t^v = ty$. Write $O = O(C)$ and $O = C_O(t)C_O(ty)C_O(y)$. Now $C_O(t) \leq N(A)$, so

$$[C_O(t), C_A(J_r)] \leq C_O(t) \cap C_A(J_r) = 1 \quad \text{and} \quad C_O(t) \leq O(C_G(A)).$$

Similarly, $C_O(ty) \leq O(C_G(A^v))$. Therefore, $C_O(t)$, $C_O(ty) \leq C_O(y)$ and $[y, 0] = 1$. As $q > 2$, if we let y range over $C_{X_0}(t)$ we conclude that $O \leq Z(C)$ and, finally, that

$$C = O^{2'}(C_{G_0}(J_r)) = O^{2'}(C_G(J_r)).$$

Therefore, $gx \in N(O^{2'}(C_{G_0}(J_r)))$ and hence gx normalizes

$$\langle A, O^{2'}(C_{G_0}(J_r))\rangle = G_0.$$

As $x \in A \leq G_0$, $g \in N(G_0)$. This completes the proof of (15.1).

(15.2) *α is the unique point in Ω fixed by t. In particular, $|\Omega|$ is odd and $G_\alpha = N_G(G_0)$ contains a Sylow 2-subgroup of G.*

Proof. Suppose t fixes $\beta \neq \alpha$ and write $\beta = G_0^g = G_1$. Then $C_{G_1}(t) \leq N_G(A)$, so $C_{G_1}(t)^{(\infty)} \leq A$ and $|C_{G_1}(t)^{(\infty)}|$ divides $|A|$. If $E(C_{G_1}(t)) \cong \tilde{A}$, then $A = E(C_{G_1}(t))$ and $t \sim t^g$ by an element of G_1 (see §19 of [3]). Say $t^{gy} = t$, with $y \in G_1$. Then $gy \in C(t) \leq G_\alpha$ (by (15.1)), and so $G_1 = G_0^{gy} = G_0$, a contradiction. Therefore $E(C_{G_1}(t)) \not\cong \tilde{A}$. In case $\tilde{G}_1 = \tilde{A} \times \tilde{A}$ we write $G_1 = G_{11}G_{12}$, where G_{11} and G_{12} are the components of G_1.

First assume that $C_{G_1}(t)$ is not 2-constrained. If G_1 is quasisimple, then we may choose g so that $[t, t^g] = 1$ (§19 of [3]). Consider $A_1 = E(C_{G_1}(t))$ and the groups $A_1 \cap A^g < A_1 < A$. Since t^g acts on each of these groups we contradict (5.3) of [14]. Say $G_1 = G_{11}G_{12}$. Then t stabilizes G_{11} and G_{12}, so write $X_i = C_{G_{1i}}(t)^{(\infty)}$. We may assume that $X_1 = E(X_1)$. Now $X_1X_2 \leq A$. Let $P \in Syl_2(X_2)$ and consider $C_A(P)$. Using the results of [3] we see that in no case is $\tilde{X}_1$ a section of $C_A(P)$. Therefore, we have a contradiction here as well.

For the remainder of the proof we assume that $C_{G_1}(t)$ is 2-constrained. We have $t \notin N(A)^{(\infty)} = A$, so $t \notin C_{G_1}(t)^{(\infty)}$. Also $|C_{G_1}(t)|_2$ must divide $|N(A)|_2$. Using the results of [3] it is fairly easy to show that either

$$(\tilde{A}, \tilde{G}_0) = (Sp(n, q), O^{\pm}(n+2, q)');$$

or $t \notin G_1$ and

$$(\tilde{A}, \tilde{G}_0) = (Sp(n, q), PSL(n, q)), (Sp(n, q), PSU(n, q)), (F_4(q), E_6(q)), \quad \text{or} \quad (F_4(q), {}^2E_6(q)).$$

(For the exceptional groups the number $|C_{G_1}(t)|_2$ is available from the work in §§13–17 of [3]. For the classical groups, this number is not presented explicitly, although it is fairly easy to calculate it. One way is to embed the centralizer in a parabolic subgroup of G_1 and look at the 2-part of the index).

Suppose $t \notin G_1$ and $(\tilde{A}, \tilde{G}_0) = (Sp(n, q), PSL(n, q))$, $(Sp(n, q), PSU(n, q))$, $(F_4(q), E_6(q))$, or $(F_4(q), {}^2E_6(q))$. By (19.9) of [3], $t \in G_1 t^g$ and $G_1 t^g$ contains two G_1-classes of involutions. In fact, setting $x = tt^g$ we have

$$O^{2'}(C_{G_1}(t)) = O^{2'}(C_{G_1}(t^g) \cap C_{G_1}(x)) = O^{2'}(C_{A^g}(x))$$

and x is in a long root subgroup of A^g. We may choose g so that g normalizes $\langle t, t^g \rangle = \langle x \rangle \times \langle t \rangle$. Then g normalizes

$$O^{2'}(C_G(\langle t, t^g \rangle)) = O^{2'}(C_A(x)) = C.$$

Let X be a $(q+1)$-Hall subgroup of J, as defined in (4.1) of [13]. We may assume $X \leq C$ (otherwise, replace X by a conjugate), and since $C = C^g$ we can assume that $X = X^g$. But

$$G_0 = \langle A, E(C_G(X)) \rangle = \langle C, E(C_G(X)) \rangle = G_0^g = G_1.$$

This is a contradiction.

Now we assume $(\tilde{A}, \tilde{G}_0) = (Sp(n, q), O^{\pm}(n+2, q)')$. If $t \in G_1$, then t is of type a_l or c_l (in the notation of §8 of [3]), and we claim $t \in C_{G_1}(t)'$. If t is of type c_l, then this was checked in the middle of Case 2, in the proof of (3.5) of [13]. If t is of type a_l, then argue by direct computation using the results of §8 of [3], or pass to the Lie notation and use the Chevalley commutator relations within a parabolic subgroup of A. Since $C_{G_1}(t) \leq N(A)$ and $t \notin N(A)'$, we conclude that $t \notin G_1$. So t induces an involution on G_1 of type b_l, where $l > 1$.

Assume $n+2 > 8$ and $2l \leq n-4$. We may assume that $[t, J^g] = 1$, for otherwise replace g by gg_1 for some $g_1 \in G_1$. Then $[t, X^g] = 1$, where X is a $(q+1)$-Hall subgroup of J (as in (4.1) of [13]). Now t normalizes $E^g = E(C_G(X^g))$ and $E^g \leq G_0^g = G_1$. Also, $\tilde{E}^g \cong O^{\pm}(n, q)'$. Since $t \in E^g \langle t^g \rangle$, either $C_{E^g}(t)$ is 2-constrained or $C_{E^g}(t) \cong Sp(n-2, q)$. However, considering $C_{G_1}(t)$ we see that the latter is impossible. Therefore $C_{E^g}(t)$ contains a proper normal 2-subgroup, as does $C_G(X^g) \cap C_G(t)$. View $C_G(X^g) \cap C_G(t)$ as a subgroup of $N(A)$. Since $X^g \leq J^g \leq C(t) \leq N(A)$, we have $X^g \leq AO(C(A))$. Arguments with classical groups or (B, N)-pairs show that $C_A(X^g)$ does not contain a proper normal 2-subgroup. This is a contradiction. Therefore, t induces, on G_1, an involution of type b_l for $l = \frac{1}{2}(n+2)$ or $\frac{1}{2}n$, depending on whether $4 \mid n$ or $4 \nmid n$.

If $n+2=8$, then t induces, on G_1, an involution of type b_3, and we can argue, as before, that $G_0=G_1$. So suppose $n+2>8$. Write $t=xy$, where $x \in G_1 t$ is a transvection, $y \in G_1$ is of type a_{l-1}, and $[x, y]=1$. Conjugating, within G_1, if necessary, we may assume $x=t^g$. Then $t^g \in N_G(G_0)$ and, by symmetry, t^g induces, on G_0, an involution of type b_l. We have $y \in C(t)$, and it follows that $y \in A$ is an a_{l-1} involution in A. Let $Q=O_2(C_A(y))= O_2(C_A(t^g))$ and $Q_0=Q\langle t\rangle$. Then $t^G \cap Q_0 \subseteq Qt$ and $N_A(Q_0)$ is transitive on $(t^G \cap Q_0)-\{t\}$. Considering $N_{A^g}(Q_0)$ and $Y=\langle N_{A^g}(Q_0), N_A(Q_0)\rangle$, we get a contradiction (as in Case 1 of (3.5) of [13]) by using (2.1) of [13].

(15.3) *Suppose G_0 is quasisimple and*

$$(\tilde{A}, \tilde{G}_0) \not\cong (O^-(n, q)', O^+(n, q^2)') \quad \text{or} \quad (PSp(n-2, q), O^+(n, q)'),$$

with $4 \mid n$. Then G_α controls G-fusion of its involutions.

Proof. This follows from (15.2) and (14.1).

(15.4) *Suppose*

$$(\tilde{A}, \tilde{G}_0) \cong (O^-(n, q)', O^+(n, q^2)') \quad \textit{or} \quad (Sp(n-2, q), O^+(n, q)'),$$

with $4 \mid n$. Let $c \in G_0$ be a root involution in G_0. Then $c^G \cap G_0 = c^{G_0}$.

Proof. Since $c^{G_0}=c^{G_\alpha}$ it suffices to show that $c^G \cap G_\alpha = c^{G_\alpha}$. Consider the argument used to prove (14.1). As mentioned in the proof of (8.12) of [3], t centralizes a representative of every class of involutions of $\mathrm{Aut}(G_\alpha)$, with the one exception of involutions of type $a_{n/2}$. Say $c \in G_\beta$. If c does not correspond to an involution of type $a_{n/2}$ in $E(G_\beta)$, then $C_G(c)$ contains a conjugate of t, fixing just β. So, if this is true for all $\beta \in \Delta(c)$, then Gleason's lemma [7] implies that $C_G(c)$ is transitive on $\Delta(c)$, and Witt's theorem yields the result.

Suppose, now, that $\Delta(c)=\Delta_1 \cup \Delta_2$, where $\Delta_2 \neq \varnothing$ consists of those $\beta \in \Delta(c)$ such that $c \in E(G_\beta)$ is of type $a_{n/2}$ and $\Delta_1=\Delta(c)-\Delta_2$. Then $C_G(c)$ acts on Δ_1 and the above argument shows that $C_G(c)$ is transitive on Δ_1. It follows that if $c^g \in G_\alpha$, and if it is not the case that $c^g \in G_\alpha$ is of type $a_{n/2}$, then $c^g \in c^{G_\alpha}$, and hence c^g is of type a_2.

Also, we are assuming that for some $g \in G$, $c^g \in G_0$ and is of type $a_{n/2}$. It now follows that $c^G \cap G_\alpha$ consists of all involutions in G_0 of type a_2 or $a_{n/2}$. If $d \in G_\alpha$ is not of type a_2 or $a_{n/2}$, then $d \notin c^G$ and the argument of the first paragraph implies that $d^G \cap G_\alpha = d^{G_\alpha}$.

Choose $\alpha^g=\beta \in \Delta(c)$ such that $c \in G_\beta$ is of type $a_{n/2}$. Let $G_1=G_0^g\langle t^g\rangle$ and consider G_1 acting as $O^+(n, q)$ on its usual $\mathbf{F}_q$-module, M. There is a decomposition $M=M_1 \perp \cdots \perp M_{n/4}$ with each M_i a non-degenerate 4-space of index 2. Then $O^+(n, q)$ contains the direct product $X_1 \times \cdots \times X_{n/4}$, where $X_i \cong O^+(4, q)$, for $i=1, \ldots, n/4$, and X_i is trivial on M_j if $i \neq j$. We may assume that c stabilizes each M_i and so we write $c=x_1 \cdots x_{n/4}$ with $x_i \in M_i$ an involution of type a_2, for $i=1, \ldots, n/4$. For each $i=1, \ldots, n/4$, choose

$g_i \in X_i$ such that $\langle x_i, g_i\rangle \cong D_8$ and $z_i = x_i x_i^{g_i}$ is of type c_2. Let

$$I = \langle x_i, z_i\colon i = 1, \ldots, n/4\rangle.$$

Since c is 2-central in G_α and $|G\colon G_\alpha|$ is odd, we may choose $b \in C_G(c)$ such that $I^b \leq G_\alpha$.

We need an observation about orthogonal groups. Suppose j is an a_2 involution, k is an a_2 (respectively c_2) involution and $[j, k] = 1$. Let W be the natural orthogonal space and suppose $\dim([W, jk]) \geq 4$. Setting $W_0 = [W, j] + [W, k]$ we have $\langle j, k\rangle$ centralizing W/W_0, so $W_0 = [W, jk]$. Now $(W_0, W_0) = 0$ (see §8 of [1]), hence $W_0 = [W, j] \perp [W, k]$, and W_0 is totally singular if and only if $[W, k]$ is. We conclude that jk has type a_4 (respectively c_4).

Now c is of type a_2 in G_0 and $(cz_1)^b = c^{g_1 b}$ is of type a_2 or $a_{n/2}$ in G_0. But $c(cz_1)^b = z_1^b$ is of type c_2. Writing this as $cz_1^b = (cz_1)^b$ and using the above we see that $(cz_1)^b$ must be of type a_2. This can be repeated. Using $(cz_1)^b$ of type a_2 in G_0 and $cz_1z_2 = c^{g_1g_2}$, we conclude that $(cz_1z_2)^b$ is of type a_2 in G_0. Eventually, we obtain $(cz_1 \cdots z_{n/4})^b$ of type a_2. But then $c(cz_1 \cdots z_{n/4})^b = (z_1 \cdots z_{n/4})^b$ has type $c_{n/2}$, a contradiction. This completes the proof of (15.4).

(15.5) *Suppose G_0 is quasisimple and $c \in G_0$ is a root involution in G_0. Then $C_G(c)$ is transitive on $\Delta(c)$.*

Proof. This is immediate from (15.4), (15.5), Gleason's lemma [7], and Witt's theorem.

16. The wreathed case

Throughout this section we assume that $\tilde{G}_0 \cong \tilde{A} \times \tilde{A}$. Write $G_0 = G_1G_2$ with $[G_1, G_2] = 1$, $G_2 = G_1^t$, G_1 quasisimple, and $\tilde{G}_1 \cong \tilde{G}_2 \cong \tilde{A}$. Notation follows that in §14. Choose $S_1 \in Syl_2(G_1)$ and set $S_2 = S_1^t$. For $i = 1, 2$ we define subgroups $\bar{S}$ and $\hat{S}_i$ as in the paragraph preceding (14.3).

Recall that $\Omega = \{G_0^x\colon x \in G\}$, $\alpha = G_0$, and $O(G) = 1$. We will be assuming the following case of the $B(G)$-conjecture.

Hypothesis (16.1). If $1 \neq X \leq \hat{S}_1$ and $O(N_G(X))G_2 \trianglelefteq N_G(X)$, then $G_2 \trianglelefteq N_G(X)$.

(16.2) *Assume that Hypothesis* (16.1) *holds. Then $G_0 \trianglelefteq G$.*

Proof. Suppose the result false and apply (14.7). Inductively, we have $N_G(\hat{S}_i) \not\leq G_\alpha$, $j = 1, 2$. Then (14.6) implies that there is a point $\alpha \neq \beta \in \Omega$ fixed by $\hat{S}_1$ and a conjugate, t_1 of t, such that t_1 interchanges α and β and $g = tt_1$ is in $N_G(S_i) \cap N_G(\hat{S}_i)$ for $i = 1, 2$. As in the proof of (14.2), $O_2(\langle g\rangle) \leq G_\alpha \cap G_\beta$. Replacing g by a generator of $O(\langle g\rangle)$ we may assume that $|g|$ is odd.

Consider $X = \langle g\rangle C_G(S_1)S_1/S_1$. This group contains G_2S_1/S_1 and using

(14.3) we check that $\hat{S}_2S_1/S_1 \in Syl_2(X)$. Let H_2 be a 2-complement in $N_{G_2}(S_2)$, $U = S_2S_1/S_1$, and $Y = N_X(U)$. The Sylow 2-subgroups of Y/U are abelian of rank at most 2. We claim that Y/U has a normal 2-complement. Otherwise, a Sylow 2-subgroup, I, of Y/U has rank 2 and $\Omega_1(I)^{\#}$ is fused in $N(I)$. But the involutions in I correspond to involutions in $\hat{S}_2S_1 - S_2S_1$, and we get a contradiction by using §19 of [3] and checking centralizers. So the claim holds, and we let Z/U be the normal 2-complement of Y. Since $H_2S_1/S_1 \leq X$ we have H_2S_1/S_1 and $\langle g\rangle S_1/S_1$ contained in Z.

Let $U_0 = (S_2)_0S_1/S_1$, where $(S_2)_0$ is as in (13.2). Then by (13.3) we see that Y acts on U/U_0, inducing a subgroup of the group of automorphisms induced by graph, diagonal and field automorphisms of G_2. In particular, if $C/U = C_{Y/U}(U/U_0)$, then $H_2C/U \trianglelefteq Y$. Notice that as $\hat{S}_2S_1/S_1 \in Syl_2(X)$, C/U has odd order. Therefore, replacing H_2 by an S_2-conjugate of H_2, if necessary, we may assume that H_2 and g are in a common 2-complement, say L, of $H_2C\langle g\rangle$.

Now U is generated by root subgroups, U_α, each normalized by H_2. We show:

(16.3) *If $\tilde{G}_2 \not\cong O^+(8, q)'$, then, for suitable choice of Σ, g normalizes each root subgroup of U.*

Proof. Suppose $\tilde{G}_2 \cong PSp(n, 2)$ or $F_4(2)$. Then $[U_0: U'] = 2$. Also, each root group has order 2, so g centralizes U/U_0 and U_0/U'. As $|g|$ is odd, g centralizes U/U' and hence $g \in C(U)$. So we may assume $\tilde{G}_2 \not\cong PSp(n, 2)$ or $F_4(2)$. Similarly, (16.3) holds if $\tilde{G}_2$ is an untwisted group defined over the field of 2 elements.

As noted in the proof of (13.2), $U_0 = U'$ in the remaining cases. Consequently, $L_0 = C_L(U/U_0) = C_L(U)$. Also, $L_0H_2 \trianglelefteq L$. So $(H_2L_0)^g = H_2L_0$. If $\alpha \in \Sigma$ and U_α a root subgroup of U, then H_2 normalizes U_α^g. If $q \geq 8$, then we can apply Lemma 3 of [9] and conclude that U_α^g is also a root group of U. Since we are dealing with groups of rank at least 3 we can use the proof of Lemma 3 of [12] and conclude U_α^g is a root group, even if $q = 4$. What is needed in that proof is that H_2 acts irreducibly on $U_\alpha/\Phi(U_\alpha)$ for each $\alpha \in \Sigma^+$ and that if $\alpha \neq \beta$ are in Σ^+, then $C_{H_2}(U_\alpha) \neq C_{H_2}(U_\beta)$. But these can be checked directly. Therefore g permutes the root subgroups. Since we have excluded the case $\tilde{A} \cong O^+(8, q)'$, g must induce the trivial permutation.

To prove (16.3) we may now assume that $q = 2$ and that $\tilde{G}_2 \cong {}^2E_6(2)$, $O^-(n, 2)'$, or $PSU(n, 2)$. Moreover, in the latter case we may assume n is even, for otherwise argue as above. So for $\alpha \in \Sigma$, $U_\alpha \cong Z_2$ or $Z_2 \times Z_2$. But then L induces only diagonal automorphisms on U. In the case of $\tilde{G}_2 \cong O^-(n, 2)'$ this implies that $L = H_2L_0$ and the claim is immediate. In the other cases $|L: H_2L_0| \leq 3$. Suppose equality holds. By (2.2) $\mathrm{Aut}(U)$ contains a normal 2-subgroup I (the centralizer of $U/U' = U/U_0$, in this case) such that $\mathrm{Aut}(U) = II_1$, where I_1 is the group induced by diagonal, field, and graph automorphisms of the corresponding Chevalley group. Therefore, L induces a group of automorphisms conjugate in $\mathrm{Aut}(U)$ to the group of diagonal automorphisms. It follows that $C_U(H_2)=C_U(L)$, and this is the product of all root subgroups, U_α, for $\alpha\in\Sigma^+$ and α short.

If $\tilde{G}_2\cong PSU(n,2)$ for $n>6$, the argument in Lemma 3 of [12] can be modified to show that any H_2-invariant subgroup Q *of* U is generated by certain root subgroups U_α for $\alpha\in\Sigma^+$, together with $C_Q(H_2)$. It follows that $\alpha\in\Sigma^+$ implies U_α^g is a root subgroup.

If $\tilde{G}_2\cong PSU(6,2)$ or ${}^2E_6(2)$ we use the following method. It will suffice to show that for each α_i, $U_{\alpha_i}^g=U_{\alpha_i}$. For once we have this it will follow from the commutator relations that $U_\alpha^g=U_\alpha$ for all $\alpha\in\Sigma^+$. If $|U_{\alpha_i}|=2$, then $U_{\alpha_i}\le C_U(H_2)=C_U(L)$, so we need only consider the case where $U_{\alpha_i}\cong Z_2\times Z_2$. Let U_i be the subgroup of U generated by root subgroups U_α for $\alpha\in\Sigma^+$ of height at least i. Then $U_0=U_2$, and for each i, U_i is the ith term in the lower central series of U. We know that $U_{\alpha_i}^g\le U_{\alpha_i}U_2$ for each i. The idea is to show that in fact $U_{\alpha_1}^g\le U_{\alpha_i}U_j$ for $j=2,3,\ldots$. Choosing j with $U_j=1$ we have the result.

Let $V=U_2/U_3$. We will indicate the proof that $U_{\alpha_i}^g\le U_{\alpha_i}U_3$. The rest is similar. Take the more difficult case of ${}^2E_6(2)$. Here

$$V=(U_{\alpha_1+\alpha_2}U_3/U_3)\times(U_{\alpha_2+\alpha_3}U_3/U_3)\times(U_{\alpha_3+\alpha_4}U_3/U_3)=V_1\times V_2\times V_3.$$

We know that g centralizes U_{α_1} and U_{α_2} so consider $U_{\alpha_3}^g$. The group H_2 acts on the abelian group $V(U_{\alpha_3}U_3/U_3)$ and

$$U_{\alpha_3}U_3/U_3\le[V(U_{\alpha_3}U_3/U_3),H_2]=U_{\alpha_2+\alpha_3}U_{\alpha_3+\alpha_4}U_{\alpha_3}U_3/U_3.$$

So $U_{\alpha_3}^g\le U_{\alpha_2+\alpha_3}U_{\alpha_3+\alpha_4}U_{\alpha_3}U_3$ and projects onto $U_{\alpha_3}U_3/U_3$. Since $[U_{\alpha_1},U_{\alpha_3}]=1$, $U_{\alpha_3}^g$ must project trivially to $U_{\alpha_2+\alpha_3}$. Using (14.2) and induction we see that if $\alpha\in\Sigma^+$ has height i, then $U_\alpha^gU_{i+1}=U_\alpha U_{i+1}$. Therefore, $U_{\alpha_2+\alpha_3}^g\le U_{\alpha_2+\alpha_3}U_3$. Suppose that $U_{\alpha_3}^gU_3\ne U_{\alpha_3}U_3$. Then $U_{\alpha_3}^g$ projects onto $U_{\alpha_3+\alpha_4}$. Let $\delta=\alpha_1+2\alpha_2+3\alpha_3+\alpha_4$. Then $U_\delta^g\le U_\delta U_8$. Suppose $U_{\alpha_3}^gU_3\ne U_{\alpha_3}U_3$. Then $U_{\alpha_3}^g$ projects onto $U_{\alpha_3+\alpha_4}$. Now $[U_{\alpha_3},U_\delta]=1$, but consideration of $[U_{\alpha_3}^g,U_\delta^g]$ shows that this commutator is non-trivial. This is impossible, so $U_{\alpha_3}^g\le U_{\alpha_3}U_3$. Similarly, using the fact that $[U_{\alpha_1},U_{\alpha_4}]=[U_{\alpha_2},U_{\alpha_4}]=1$, we easily see that $U_{\alpha_3}^g\le U_{\alpha_3}U_4$ and $U_{\alpha_4}^g\le U_{\alpha_4}U_4$. Continuing we get the result.

(16.4) $\tilde{A}\cong O^+(8,q)'$.

Proof. Suppose $\tilde{A}\not\cong O^+(8,q)'$. By (16.3), g normalizes each root subgroup of $U=S_2S_1/S_1$, and we know that $g\in N(S_2)$. So with appropriate choice of root subgroups we may assume that g normalizes each root subgroup of S_2. Since $\langle g\rangle^t=\langle g\rangle$, we have g normalizing the corresponding root subgroups in S_1.

Let r be the root of highest height in Σ^+. For $\alpha\in\Sigma^+$, Let $Z_\alpha=\Omega_1(\hat{Z}_\alpha)$, where $\tilde{Z}_\alpha$ is the corresponding root subgroup of S_1. Similarly, we have the groups $\hat{Y}_\alpha$ and Y_α of S_2. If $\tilde{A}$ is an orthogonal group, let $\bar{Z}_r=Z_r\times Z_{\alpha_1}$ and $\bar{Y}_r=Y_r\times Y_{\alpha_1}$. Otherwise, set $\bar{Z}_r=Z_r$, $\bar{Y}_r=Y_r$. Let $Q_1=O_2(C_{G_1}(\bar{Z}_r))$. If $\tilde{A}$ is not an orthogonal group, then $Q_1 = O_2(I_1)$, for I_1 a maximal parabolic subgroup of G_1. If $\tilde{A}$ is an orthogonal group, then $I_1 = N_{G_1}(Q_1)$ is a parabolic subgroup of G_1. In either case, set $\tilde{Q}_1 = O_2(I_1)$. Similarly, we have subgroups Q_2, $\tilde{Q}_2$ of G_2. Set $Q = \tilde{Q}_1\tilde{Q}_2$. Let L_1 be a Levi factor of I_1 such that $L_1 \cap S_1 \in Syl_2(L_1)$ and $L_1 \cap S_1$ is the product of root subgroups, $\tilde{X}_\alpha$, of S_1. Set $L_2 = L_1^t$ and $L = C(t) \cap L_1L_2$.

Consider $D = N_G(Q)$ and let bars denote images in D/Q. Then $\bar{L}$ is a standard subgroup of $\bar{D}$ and $\langle \bar{t} \rangle \in Syl_2(C_{\bar{D}}(\bar{L}))$. It follows by induction that $\bar{L}_1 \times \bar{L}_2 \trianglelefteq \bar{D}$. In particular, g normalizes $\bar{L}_1 \times \bar{L}_2$. Unless $\tilde{A} \cong PSp(6, 2)$, $\bar{L}_i = \bar{L}_i^{(\infty)}$ for $i = 1, 2$, and so g normalizes $(L_iQ)^{(\infty)} = \tilde{Q}_iL_i$ for $i = 1, 2$. If $\tilde{A} \cong PSp(6, 2)$, then, by (16.3), g is trivial on $\bar{L}_1$ and on $\bar{L}_2$. Since g must also centralize $\tilde{Q}_1$ and $\tilde{Q}_2$, $g \in C_G(QL_1L_2)$.

Assume $\tilde{A} \not\cong PSp(6, 2)$. By the above, g normalizes $\tilde{Q}_iL_i$, for $i = 1, 2$. Since g normalizes each root subgroup of S_1 and S_2 we conclude that the action of g on $L_i\tilde{Q}_i/\tilde{Q}_i \cong L_i$ is given by the product of a diagonal and field automorphism of L_i. For $C \leq A$ and $i = 1$ or 2, let C_i denote the projection of C to G_i. Assume that $\tilde{A} \not\cong O^-(8, 2)'$.

Let s be the root of highest height in the root system, Δ, of $L \cong L_1 \cong L_2$. Let $\bar{J}_s = J_s = \langle U_{\pm s} \rangle$ if L is not an orthogonal group and $\bar{J}_s = J_s \times J_{\alpha_3}$, otherwise. Then

$$\bar{J}_{s1} \cong \bar{J}_{s2} \cong SL(2, q)$$

(or $SL(2, q) \times SL(2, q)$ if $\tilde{A}$ is an orthogonal group). Also, g normalizes $\tilde{Q}_i\bar{J}_{si}$ for $i = 1, 2$. There are elements $h_i \in \bar{J}_{s1}$ such that gh_1h_2 induces a field automorphism of odd order on each of $\bar{J}_{si}\tilde{Q}_i/\tilde{Q}_i$. Also, gh_1h_2 has odd order in $QL_1L_2\langle g \rangle$. There is a $(q+1)$-Hall subgroup $X_s \leq \bar{J}_{s1}\bar{J}_{s2}$ such that gh_1h_2 normalizes X_sQ. Conjugating by an element, $q \in Q$, we have $(gh_1h_2)^q \in N_G(X_s)$. Note that by previous remarks this also holds of $\tilde{A} \cong PSp(6, 2)$.

Therefore, $(gh_1h_2)^q \in N_G(E(C_G(X_s)))$. In the notation of (4.1) of [13], X_s contains a G_0-conjugate of X, say X^b, for $b \in G_0$. Also, from the structure of G_0 we have $E(C_G(X^b)) = E(C_G(X_s))$, so

$$E(C_G(X_s)) = E(C_G(X))^{b^{-1}} = E^{b-1}.$$

Now $G_0 = \langle E, E^w \rangle$ for some $w \in G$, so $E(C_G(X_s)) \leq G_0$. Moreover,

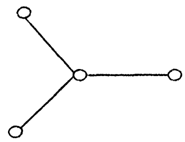
$$G_0 = \langle E(C_G(X_s)), QL_1L_2 \rangle$$

and it follows that $(gh_1h_2)^q \in N(G_0)$. So $g \in N(G_0)$, whereas $\alpha^g = \beta \neq \alpha$. This is a contradiction.

Suppose $\tilde{A} \cong O^-(8, 2)$. With s as above, $J_s \cong SL(2, q^2)$ and let X_s be a $(q-1)$-Hall subgroup with gh_1h_2 normalizing X_sQ. Argue as above, only use $X_s \geq I_I^b$, where I_1 is defined in the paragraph just before (8.4) of [13].

(16.5) $\tilde{A} \not\cong O^+(8, q)'$.

Proof. Suppose $\tilde{A} \cong O^+(8, q)'$. Label the Dynkin diagram of A



For $\alpha \in \Sigma$, let Z_α, Y_α be the corresponding root subgroups of G_1, G_2, respectively. Then $S_1' = \langle Z_\alpha : \alpha \in \Sigma^+,\ \alpha \text{ a compound root}\rangle$ and it is easy to check from the commutator relations that

$$Z_{\alpha_2} S_1' = C_{S_1}(S_1'/\gamma_4(S_1)).$$

Set $Q_1 = Z_{\alpha_2} S_1'$. Similarly, set $Q_2 = Y_{\alpha_2} S_2'$, and $Q = Q_1 Q_2$. Then g normalizes Q_1, Q_2, and Q.

Let $D = N_G(Q_1) \cap N_G(Q_2)$. Then

$$O^{2'}(D \cap G_0/Q) = Y/Q,$$

where $Y = Q(L_1 \times L_2)$, $L_i = J_{i\alpha_1} \times J_{i\alpha_3} \times J_{i\alpha_4}$ for $i = 1, 2$, and where $J_{1,\alpha_j} = \langle Z_{\alpha_j}, Z_{-\alpha_j}\rangle$, and $J_{2,\alpha_j} = \langle Y_{\alpha_j}, Y_{-\alpha_j}\rangle$. We claim that g normalizes Y/Q. Suppose $q > 2$. First we note that S_1S_2/Q is elementary of order q^6. Since $|G : N(G_0)|$ is odd (see (15.2)), we apply Corollary 4 of Goldschmidt [8] and conclude that S_1S_2/Q is strongly closed in a Sylow 2-subgroup of D. Using the main theorem of [8] we conclude that if $I/Q = O(D/Q)$, then $YI/Q \trianglelefteq D/Q$. For each $a \in (V_{\alpha_1} V_{\alpha_3} V_{\alpha_4})^\#$ there is an element $xQ \in S_1S_2/Q$ with $t^x = ta$. Writing

$$I/Q = C_{I/Q}(t) C_{I/Q}(ta) C_{I/Q}(a),$$

noting that $C_{I/Q}(t) = C_I(t)Q/Q$, and arguing as in (5.1) of [14] we see that a centralizes I/Q. As a was arbitrary,

$$Y/Q \leq \langle (V_{\alpha_1} V_{\alpha_3} V_{\alpha_4})^D Q\rangle/Q \leq C(I/Q),$$

proving that $Y/Q = E(D/Q)$. So the claim holds in this case.

Suppose $q = 2$. Then for $i = 1, 2$ g must centralize $\gamma_5(S_i)$ and $\gamma_4(S_i)$, since these groups have order 2 and 4, respectively. So consider $g \in D_0 = C_D(\gamma_4(S_1)\gamma_4(S_2)) \cap N_D(S_1'S_2')$. For $i = 1, 2$, the group D_0 acts on the elementary abelian groups $S_i'/\gamma_4(S_i)$ as a subgroup of $GL(6, 2)$. Let $i = 1$. Then L_1 acts on $S_1'/\gamma_4(S_1)$ as $S_3 \times S_3 \times S_3$, corresponding to a decomposition of $S_1'/\gamma_4(S_1)$ into a sum of 3 2-dimensional subspaces. Let $Y_i/Q = O_3(L_i)Q/Q$. It will suffice to show that g normalizes Y_1 and Y_2.

Let $i = 1$ or 2. By (13.2) g acts on S_i/S_i' as the graph automorphism of order 3 or as the identity. Then consideration of commutators leads to the fact that g acts on $S_i'/\gamma_4(S_i)$ as does the graph automorphism of order 3 or as the identity. In either case g normalizes Y_i/Q in the action on $S_i'/\gamma_4(S_i)$. Therefore,

$$g \in N_G(QY_1Y_2) C_{D_0}(S_1'S_2'/\gamma_4(S_1S_2)).$$

Now $S_1'S_2'$ is a Sylow group $C_{D_0}(S_1'S_2'/\gamma_4(S_1S_2))$, and since D_0 centralizes $\gamma_4(S_1S_2)$ we conclude that $C_{D_0}(S_1'S_2'/\gamma_4(S_1S_2))$ contains a normal 2-complement, P. Argue as in the previous paragraph to conclude that $V_{\alpha_1}\times V_{\alpha_3}\times V_{\alpha_4}\leq C(P)$. Since P centralizes Q_i/S_i' for $i=1,2$ we conclude that $P\leq C(Q)$ and $C(Q)$ covers Y/Q. It follows that $C_{D_0}(S_1'S_2'/\gamma_4(S_1S_2))\leq N(QY_1Y_2)$ and the claim holds in this case also.

We have g normalizing $Y=QL_1L_2$. Say $q>2$. Since g normalizes S_1S_2/Q we conclude that g normalizes a $(q-1)$-Hall subgroup I/Q of QL_1L_2. Then conjugating by an element of Q we may assume that g normalizes a $(q-1)$-Hall subgroup, I_0, of $L_1\times L_2$. Now we check that J_r is standard in $C_G(I_0)$ and

$$\langle t\rangle\in Syl_2(C_G(I_0)\cap C(J_r)).$$

Using the main theorem of [10] together with the fact that

$$E(C_{G_0}(I_0))\cong L_2(q)\times L_2(q),$$

we conclude that $E(C_{G_0}(I_0))=E(C_G(I_0))$. But then g normalizes

$$\langle Y, E(C_{G_0}(I_0))\rangle=G_0,$$

a contradiction.

Finally, consider the case $q=2$. Then g normalizes $Y_0/Q=O_3(Y/Q)$, so conjugating by an element of Q we may assume that g normalizes $O_3(L_1\times L_2)=I_0$. Then g normalizes $C_G(I_0)$. Now $C_A(I_0)=J_r$ and $S=C_{G_0}(I_0)=\langle Z_r, Z_{-r}\rangle\times\langle Y_r, Y_{-r}\rangle\cong S_3\times S_3$. Also, t normalizes $C_{G_0}(I_0)$, although $t\notin C_{G_0}(I_0)$. Moreover, $t^G\cap G_0=\varnothing$, since $t^g\in G_\alpha$ implies that t^g fixes just α and $g\in G_\alpha$. By the Thompson transfer lemma we conclude that $t\notin G'$. Using this we conclude that

$$Z_rY_r\langle t\rangle\in Syl_2(C_G(I_0)\langle t\rangle).$$

Let $O=O(C_G(I_0))$, $\langle a\rangle=V_r$, and $\langle x\rangle=Z_r$. Then $t^x=ta$ and $O=C_O(t)C_O(t^a)C_O(a)$. Now $J_r=O^{2'}(C_A(I_0))$, so $C_O(t)$ normalizes J_r. In particular $[C_O(t),\langle a\rangle]\leq O_3(J_r)$, so that

$$[C_O(t),\langle a\rangle]=1 \quad\text{or}\quad [C_O(t),\langle a\rangle]=O_3(J_r)\leq 0.$$

Similarly,

$$[C_O(ta),\langle a\rangle]=1 \quad\text{or}\quad [C_O(ta),\langle a\rangle]=O_3(J_r^x)\leq 0.$$

Suppose $O_3(S)\not\leq 0$. Since $O_3(S)=O_3(J_r)\times O_3(J_r^x)$ the above implies that $a\in C(O)$. In this situation $C_G(I_0)$ cannot be solvable. Considering the structure of $C_{G_0}(I_0)'$ (which has klein groups as Sylow 2-subgroups) we see that $E(C_G(I_0))\cong L_2(q)$, some q. But $E(C_G(I_0))$ cannot then contain S. This is a contradiction. Therefore $O_3(S)\leq 0$. Also, considering $Z_rY_r\langle t\rangle$-invariant Sylow subgroups of O for primes other than 3 we see that a centralizes each $3'$-chief factor of $C_G(I_0)$. Consequently $\langle a^{C_G(I_0)}\rangle$ centralizes each such factor.

In particular $O_3(S)$ centralizes each 3′-chief factor of $C_G(I_0)$ contained in O. This implies that

$$O_3(S) \leq O_3(O) = O_3(C_G(I_0)).$$

Consider the group $P = [O_3(O), \langle a \rangle]$. This group is stabilized by $Z_r Y_r \langle g \rangle \langle t \rangle$, and P contains $O_3(S)$. In view of this and previous remarks it follows that $P/\Phi(P) \cong O_3(S)$ and $P = O_3(S)\Phi(P)$. But then $P = O_3(S)$ and $g \in N_G(O_3(S))$. Consequently, $g \in N_G(S)$. But then $g \in N_G(\langle S, Y \rangle) = G_0$, a contradiction.

This completes the proof of (16.5) and hence the proof of (16.2).

17. The quasisimple case

In view of the results in §16 we may now assume that G_0 is quasisimple. Let $\hat{C}$ be a t-invariant root subgroup of G_0 and set $C = \Omega_1(\hat{C})$. Choose $\hat{C}$ such that $C^\#$ consists of root involutions. The only ambiguity in this choice is when $G_0 \cong PSp(2n, q_1)$ for some q_1. Here the Dynkin diagram has type C_n, and we take C to correspond to a long root. Let $|C| = q_1$ and fix $1 \neq c \in C$.

Except for the case $(\tilde{A}, \tilde{G}_0) \cong (PSp(n-2, q), O^\pm(n, q)')$ we may assume that $V_r = C_C(t)$. By (15.3), G_α controls fusion of its involutions, except, possibly, when

$$(\tilde{A}, \tilde{G}_0) = (PSp(n-2, q), O^+(n, q)') \quad \text{or} \quad (O^-(n, q)', O^+(n, q^2)'),$$

with $4 \mid n$. However, by (15.4) we have $c^G \cap G_0 = c^{G_0}$ in all cases.

(17.1) *Assume* $(\tilde{A}, \tilde{G}_0) \not\cong (PSp(6, q), O^+(8, q)')$. *If* $c^g \in C_{G_\alpha}(c)$, *then there exists a conjugate of* t *contained in* $C_{G_\alpha}(c)$ *and centralizing* c^g.

Proof. As G_α controls G-fusion of $c^G \cap G_\alpha$, we may take $g \in G_\alpha$. Suppose that it is not the case that $(\tilde{A}, \tilde{G}_0) \neq (F_4(q), F_4(q^2))$. Then $G_\alpha \leq G_0 C_{G_\alpha}(c)$ and we may assume $g \in G_0$. The action of G_0 on the conjugates of C is described in (12.1) of [3]. G_0 acts as a permutation group of low rank on C^{G_0}, the stabilizer of C being the unique parabolic subgroup of G_0 containing $C_{G_0}(c)$.

Now $\langle C, C^g \rangle$ is abelian, isomorphic to $SL(2, q_1)$, or isomorphic to the Sylow 2-subgroups of $SL(3, q_1)$. So $c^g \in C_{G_\alpha}(c)$ forces $[C, C^g] = 1$ and then $C^g \leq C_{G_0}(c)$. At this point we check the orbits of $C_{G_0}(c)$ on $C^{G_0} \cap C_{G_0}(c)$ to see that except for the case $\tilde{A} \cong PSp(6, q)$ and $\tilde{G}_0 \cong O^+(8, q)'$ there exists $x \in C_{G_0}(c)$ with C^{gx} t-invariant. As $(C^{gx})^\#$ is fused we have the result.

Suppose $(\tilde{A}, \tilde{G}_0) = (F_4(q), F_4(q^2))$. The above argument works unless there is an element $v \in G_\alpha$ with v inducing a graph automorphism on G_0. Here, an easy argument shows that we may assume $c^{gv} \in C(c)$, and we then apply the above argument to get the result.

(17.2) *Assume*

$$(\tilde{A}, \tilde{G}_0) \neq (PSp(6, q), O^+(8, q)').$$

Let $c_1 \in c^{G_0} \cap C_{G_0}(c)$. *Then*

$$c_1^{C_G(c)} \cap C_{G_\alpha}(c) = c^{C_{G_\alpha}(c)}.$$

Proof. The proof is the same as that used to prove G_α controls fusion of its involutions. By (17.1) and Gleason's lemma [7], $C_G(c) \cap C_G(c_1)$ is transitive on $\Delta(c) \cap \Delta(c_1)$, so the result follows from the converse of Witt's theorem.

(17.3) *Assume* $(\tilde{A}, \tilde{G}_0) \not\cong (PSp(6, q), O^+(8, q)')$. *Then* $\Delta(c) = \Delta(C)$ *and* $[O(C_G(c)), O_2(C_{G_0}(c))] = 1$.

Proof. Let $O = O(C_G(c))$ and $Q = O_2(C_{G_0}(c))$. Let $L = O^{2'}(L)$ be a Levi factor of $C_{G_0}(c)$ and choose x such that x is an involution in L and $t \sim tx$ (in G_0). Write $O = C_O(t)C_O(tx)C_O(x)$. Then by (15.1), $C_O(t)$ and $C_O(tx)$ are in G_α. It follows that $C_O(t)$, $C_O(tx) \leq C(G_0) \leq C(x)$. Therefore, $x \in C(O) \cap C_{G_0}(c) \trianglelefteq C_{G_0}(c)$. It follows that

$$Q \leq \langle x^{C_{G_0}(c)} \rangle \leq C(O) \cap C_{G_0}(c),$$

so $[O, Q] = 1$, as asserted.

Next, use (17.2) and the fact that $C = Z(C_{G_0}(c))$ to conclude that C is strongly closed in a Sylow 2-subgroup of $C_G(c)$ (with respect to $C_G(c)$). Then by Goldschmidt [8] and the above, we have $C \leq C_G(c)$. By (15.5), $C_G(c)$ is transitive on $\Delta(c)$. Since $|\Delta(c)|$ is odd and $C \trianglelefteq C_G(c)$, we have $\Delta(c) \subseteq \Delta(C)$, which proves $\Delta(c) = \Delta(C)$.

(17.4) *Assume that* $(\tilde{A}, \tilde{G}_0) \neq (Sp(6, q), O^+(8, q)')$ *and* $\tilde{G}_0 \not\cong PSp(n, q)$ *or* $PSU(n, q)$. *Then* $Q \leq C_G(c)$, *where* $Q = O_2(C_{G_0}(c))$.

Proof. By (17.3), $\Delta(c) = \Delta(C)$, so if K is the kernel of the action of $C_G(c)$ on $\Delta(c)$, then $C \leq K \leq G_\alpha$. By (17.2) we have $C \trianglelefteq C_G(c)$. Let $L = C_G(C)$. Suppose, for the moment, that $\tilde{G}_0 \not\cong F_4(q)$. Using the results in §3 and §4 of [5] we know that $C_{G_0}(c)$ acts irreducibly on Q/C. Let $d \in c^{G_\alpha}$ be an involution in the root group of G_0 for the positive long root of next highest height to the root giving rise to C. Then as $\tilde{G}_0 \not\cong PSp(n, q)$ or $PSU(n, q)$, $d \in Q$. Also, one checks from the commutator relations that dC is 2-central in a Sylow 2-subgroup of $C_G(c)/C$. Since Q/C is abelian and $\overline{d}^{C_{G_\alpha}(c)} \subseteq \overline{Q}$, we can apply (17.2) and Corollary 2 of [8] to conclude $\overline{Q/C} \trianglelefteq \overline{C_G(c)/C}$, where bars denote images in $C_G(c)/C$ modulo its core.

Now $O(L/C) = O(L)C/C$ and by (17.3) $O(L) \leq O(C_G(c)) \leq C(Q)$. Therefore, $Q \leq O_2(L)$. Clearly, $O_2(K) \leq O_2(C_{G_\alpha}(c)) = Q$, so

$$Q = O_2(L) \cap K \trianglelefteq C_G(c),$$

giving the desired result.

Suppose $\tilde{G}_0 \cong F_4(q)$. Here the argument is only slightly different. As above Q/C is abelian and we have the involution d. However, $C_{G_0}(c)$ does not act irreducibly on Q/C. Nevertheless we can use the results in (4.5) of [5] to see that

$$\langle d^{C_{G_0}(c)} \rangle = Q.$$

Now argue as before.

(17.5) *Suppose* $(\tilde{A}, \tilde{G}_0) \neq (Sp(6, q), O^+(8, q)')$ *and* $\tilde{G}_0 \not\cong PSp(n, q)$ *or* $PSU(n, q)$. *Then* $G_0 \trianglelefteq G$. *That is,* $|\Omega| = 1$.

Proof. By (17.4), $Q \trianglelefteq C_G(c)$. As $C_G(c)$ is transitive on $\Delta(c)$ we conclude that $\Delta(Q) = \Delta(c)$. Choose $c_1, c_2 \in c^{G_0}$ such $\langle c_2, c\rangle \cong D_8$, $\langle c_1 \rangle = Z(\langle c_2, c\rangle)$ and c_2, c are in $O_2(C_{G_0}(c_1))$. That this is possible is evident by symmetry and the structure of Q. Then $\Delta(c) = \Delta(c_1) = \Delta(c_2)$ (again by symmetry). But consider

$$I = \langle c_2^g \colon g \in C_{G_0}(c)\rangle.$$

Since $C_{G_0}(c)$ stabilizes $\Delta(c)$, $\Delta(I) = \Delta(c)$, and so $\langle C_{G_0}(c), I\rangle = G_0$ stabilizes $\Delta(c)$. However, G_0 is quasisimple and c is trivial on $\Delta(c)$, forcing $\Delta(c) = \Delta(G_0) = \{\alpha\}$. At this point we can apply a result of Holt [11] to get the result (in the case $\tilde{G}_0 \cong F_4(q)$, first apply the Thompson transfer lemma in order to get a group in which c is 2-central).

(17.6) *Suppose* $\tilde{G}_0 \cong PSp(n, q)$ *or* $PSU(n, q)$. *Then* $G_0 \trianglelefteq G$.

Proof. We first claim that $Q \trianglelefteq C_G(c)$, Let L be the Levi factor of $C_{G_0}(c)$ that is generated by root subgroups of the root system of G_0. Then L acts irreducibly on Q/C. Let K be the kernel of $C_G(c)$ on $\Delta(c)$. By (17.3), $O_2(K) = C$ or Q. So suppose $O_2(K) = C$.

Either $\tilde{A} \cong PSp(n, \sqrt{q})$, or $\tilde{A} \cong PSp(n, q)$ or $PSp(n-1, q)$, depending on whether $\tilde{G}_0 \cong PSp(n, q)$ or $PSU(n, q)$ (and whether or not n is even). The action of L on V is described in §3 of [5]. Namely, L acts on V as on the natural module for $L \cong Sp(n-2, q)$ or $SU(n-2, q)$, and isotropic 1-spaces of V are images of root subgroups of G_0, for short roots (we view the root system of G_0 to be of type C_l, for $l = [n/2]$). Let $d \in V^\#$ be isotropic. Then d is a 2-central involution in $C_G(c)/C$. Say $x \in C_G(c)/C$ and $d^x \in C_{G_\alpha}(c)/C$.

Since we already know that G_α controls G-fusion of its involutions (see (15.3)), we have $d^x \in C_{G_0}(c)/C$ and hence $d^x \in V(LC/C)$. Suppose $d^x \in V$. As G_α controls G-fusion of its involutions, d^x is isotropic. Therefore, t centralizes a $C_{G_0}(c)/C$ conjugate of d^x. Next suppose that $d^x \notin V$. Write $d^x = y_1 y_2$ with $y_1 \in V$ and $y_2 \in (LC/C)^\#$. Choose a basis for V so that y_2 is in Suzuki form (see §6 and §7 of [3]). Since d^x has Jordan rank 2 on the usual module for $Sp(n, q)$ or $SU(n, q)$, y_2 is necessarily of type j_1, j_2, a_2, b_1, or c_2. As t centralizes a conjugate of y_2 in LC/C (again by [3]) we conjugate, if necessary, so that $y_2^t = y_2$. As d^x and y_2 are involutions, y_1 is in $C_V(y_2)$. Now $C_V(y_2) = V_1 \perp V_2$, with $V_1 = [V, y_2]$ isotropic and V_2 non-degenerate of dimension $\dim(V)-2\dim(V_1)$. Also $V=(V_1\oplus \hat{V}_1)\perp V_2$ where $[\hat{V}_1, y_2]=V_1$.

Conjugating y_2 by an element of $\hat{V}_1$ we have $d^x \sim y_2v_2$, with $v_2 \in V_2$. If v_2 is isotropic then since $C_L(y_2)$ contains a subgroup transitive on the non-zero isotropic vectors in V_2, we easily check that d^x is centralized by a $C_{G_0}(c)$-conjugate of t. So suppose v_2 is anisotropic. Then $\tilde{G}_0$ is a unitary group and v_2 corresponds to an element of order 4 in $C_{G_0}(c)$. Taking preimages of y_2 and v_2 we see that $y_2=aC$ and $v_2=bC$ for a an involution, ab an involution, and $|b|=4$. So a inverts b. However, another check shows that $C_V(aC)=C_Q(a)C/C$. This is a contradiction. So in all cases d^x is centralized by a $C_{G_\alpha}(c)/C$ conjugate of t.

As in the proof of (14.1) we have $C_G(c)$ transitive on $\Delta(c)$. The usual arguments now show $(C_{G_\alpha}(c)/C)\cap C(d)$ is transitive on $\Delta(c)\cap\Delta(d)$ and then that $C_{G_\alpha}(c)/C$ controls $(C_G(c)/C)$-fusion of conjugates of d. Therefore, Corollary 4 of Goldschmidt [8] gives $Q\trianglelefteq C_G(c)$, as in (17.4). This proves the claim.

To complete the proof, consider the group $C_G(c)/Q$. As c is 2-central in G it is easy to see that $(C_G(c)\cap C_G(t))Q/Q$ is standard in $C_Q(c)/Q$ and that the hypotheses of the Main Theorem in [13] hold. By that theorem and the minimality of $|G_0|$ we have

$$O^{2'}(C_{G_0}(c))/C\trianglelefteq C_G(c)/C.$$

Then $\Delta(c)=\Delta(C_{G_0}(c))$. Let $P=\langle C, C^g\rangle$ be such that $P\cong SL(2, q_1)$. Then $C_{G_0}(c)$ contains conjugates of P, so that $\Delta(c)=\Delta(P)$. Now $C_{G_0}(C)$ is transitive on the set of all such C^g. Therefore, $\Delta(I)=\Delta(c)$, where $I=\langle C^g: \langle C, C^g\rangle \cong SL(2, q_1)\rangle$. Since $G_0=\langle C_{G_0}(c), I\rangle$, G_0 fixes $\Delta(c)$ and so $\Delta(c)=\{\alpha\}$. Once again we use Holt [11] to obtain the result.

We now consider the case $(\tilde{A}, \tilde{G}_0)\cong(PSp(6, q), O^+(8, q)')$.

(17.7) *Suppose* $(\tilde{A}, \tilde{G}_0)\cong(PSp(6, q), O^+(8, q)')$. *Then* $G_0\trianglelefteq G$.

Proof. Let $c\neq d\in Q$ with $d\sim c$. We first show that if $x\in C_G(c)$ and $d^x\in d^x\in C_{G_\alpha}(c)$, then $d^x\in Q$. Suppose false. We know $d^x\in d^{G_0}$. If $d^x=d^g$, with $g\in G_0$, then as in (17.1) $C^g\leq C_{G_\alpha}(c)$. Conjugating by an element of $C_{G_0}(c)$ we may assume that $C^g\leq L$, where L is the Levi factor generated by root subgroups in the root system of $\tilde{G}_0$. Consider the action of $\tilde{G}_0$ on its usual 8-dimensional vector space. Then d is an involution of type a_2 (in the notation of §8 of [3]). Also, from the structure of Q (again see §8 of [1]) $d\sim dc$. However, this implies that $d\sim(dc)^x=d^xc=d^gc$, whereas a straight forward check shows that either $[V, d^gc]$ has dimension 4 or d^gc is of type c_2. This is a contradiction.

Given the above we can argue that if $d\in Q$, $d\sim c$, and $d^x\in C_G(c)$, then d^x is centralized by a $C_G(c)$-conjugate of t. Now Gleason's lemma allows us to conclude that $C_{G_\alpha}(c)$ controls $C_G(c)$-fusion of conjugates of d. Now argue as in (17.3) to get $\Delta(c)=\Delta(C)$ and $O(C_G(c))\leq C(Q)$. From here the proof of (17.4) gives $Q \trianglelefteq C_G(c)$. To complete the proof use the argument of (17.5).

We have now proved the Main Theorem.

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