ILLINOIS JOURNAL OF MATHEMATICS
Volume 23, Number 4, December 1979

CHEVALLEY GROUPS AS STANDARD SUBGROUPS, II

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Introduction

This paper continues the work that was begun in [13]. Our situation is that A is a standard subgroup of a finite group G and $\tilde{A} = A/Z(A)$ is a group of Lie type having Lie rank at least 3 and defined over a field of characteristic 2. Our goal, in this paper, is to show that under the hypotheses of the main theorem of [13], either (a), (d), or (e) of that theroem holds, or there is an involution $t \in C_G(A)$ and a t-invariant subgroup, $G_0 \leq G$, such that G_0 satisfies (b) or (c) of the main theorem. Once we prove the existence of such a group G_0, all that will remain in the proof of the main theorem is the verification that $G_0 = E(G)$. That verification will occur in part three of the series.

Our construction of the group G_0 is as follows. Using the results of §4 of [13] we find a subgroup $X \leq A$ so that $O^{2'}(C_A(X))$ is a standard subgroup of $C_G(X)$ and $t \notin Z^*(C_G(X))$. By induction, Hypothesis (*), or by appealing to the literature, we have the structure of $E = E(C_G(X))$. The group G_0 will be $\langle E, E^w \rangle$, where w is a suitable element of the Weyl group of A. The structure of G_0 is obtained by developing sufficient commutator information in order to apply the work of Curtis [5]. However, there are some difficulties in obtaining the necessary commutator relations. This is due, in part, to the fact that root subgroups of A may be properly contained in root subgroups of G_0, and in some cases not even contained in root subgroups of G_0. Another difficulty occurs when X is taken as an abelian Hall subgroup of a group, J, generated by two opposite root subgroups of A, and we find that J does not centralize $E(C_G(X))$.

Throughout the paper we operate under the following assumptions: $|Z(A)|$ is odd, $K = C_G(A)$ has cyclic Sylow 2-subgroups, and $\tilde{A} \not\cong Sp(6,2)$, $U_6(2)$, $O^{\pm}(8,2)'$, or $L_n(2^a)$. The omission of $\tilde{A} \cong L_n(2^a)$ is justified by the corollary in [14]. Let $R \in Syl_2(K)$ and $\langle t \rangle = \Omega_1(R)$.

5. Preliminaries

If X is any subgroup of G we set $X_A = \langle (O^{2'}(A \cap X))^X \rangle$. So $X_A \trianglelefteq X$. We will need a slight generalization of (1.3) of [14].

(5.1) *Let X be a finite group, P a standard subgroup of X with $C_X(P)$ of*

Received July 6, 1976.
[1] Supported in part by a National Science Foundation grant.

2-rank 1 and $|Z(P)|$ odd. Let $S \in Syl_2(N(P))$ and let t be the involution in $C_S(P)$. Suppose that there is an element $g \in N(S) - S$ with $g^2 \in S$ and $t^g \in PC_X(P)$. Then $[P, O(X)] = 1$. So if L is a t-invariant 2-component of X with $P \leq L$, then L is quasisimple.

Proof. This is just (1.3) of [14] with slightly weaker hypotheses. These hypotheses are precisely what was needed to prove that result.

(5.2) *Let $X < Y < Z$ be finite groups of Lie type defined over a field of characteristic 2, and each generated by its root subgroups. Suppose that σ is an involutory automorphism of Z and of Y and $X = E(C_Z(\sigma))$. Then there is an even integer n and $q = 2^a$, such that $(\tilde{X}, \tilde{Y}, \tilde{Z})$ is either*

$$(PSp(n, q), PSU(n, q), PSU(n+1, q)) \quad \text{or} \quad (PSp(n, q), PSL(n, q), PSL(n+1, q)).$$

Proof. First note that by the Borel–Tits Theorem ((3.9) of [3]) σ must induce an outer automorphism of Z. Checking centralizers of outer automorphisms (see §19 of [1]) we obtain the result.

Next, we discuss national conventions. Let X be a group of Lie type defined over a field of characteristic 2 and having root system Σ. Then $|Z(A)|$ is odd. Let $\{\alpha_1, \ldots, \alpha_n\}$ be a fundamental system of roots for Σ. Once we have chosen a Borel subgroup, B_1, of X and fundamental reflections $s_1, \ldots, s_n$ of the Weyl group of X we often write $X = \langle K_{\alpha_1}, \ldots, K_{\alpha_n} \rangle$ where each K_{α_i} is generated by the root subgroups corresponding to the roots $\pm\alpha_i$. Let B_1^0 be the opposite Borel subgroup.

Now suppose that t is an involutory field, graph, or graph-field automorphism of X defined with respect to the root system Σ. So

$$K_{\alpha_i}^t \in \{K_{\alpha_1}, \ldots, K_{\alpha_n}\} \quad \text{for each} \quad i = 1, \ldots, n.$$

Then $O^{2'}(C_X(t)) = Y$ is a Chevalley group with root system determined by Σ and we write $Y = \langle J_{\beta_1}, \ldots, J_{\beta_{m}} \rangle$ where

$$\{J_{\beta_1}, \ldots, J_{\beta_{m}}\} = \{O^{2'}(C(t) \cap \langle K_{\alpha_i}, K_{\alpha_i}^t \rangle) : i = 1, \ldots, n\}.$$

(See Theorem 33 of [15].) Note that $C_{B_1}(t)$ and $C_{B_1^0}(t)$ are opposite Borel subgroups in $C(t)$.

We will have occasion to use the fact that the set $\{J_{\beta_1}, \ldots, J_{\beta_{m}}\}$ in some sense determines $\{K_{\alpha_1}, \ldots, K_{\alpha_n}\}$.

(5.3) *Let $X = \langle K_{\alpha_1}, \ldots, K_{\alpha_n} \rangle$ and $Y = \langle J_{\beta_1}, \ldots, J_{\beta_{m}} \rangle$ be as above. C_1, C_1^0 be t-invariant opposite Borel subgroups of G for which t permutes the corresponding root subgroups. Let $L_{\alpha_1}, \ldots, L_{\alpha_n}$ be the associated subgroups, corresponding to $K_{\alpha_1}, \ldots, K_{\alpha_n}$. Assume that $C_{B_1}(t) = C_{C_1}(t)$, $C_{B_1^0}(t) = C_{C_1^0}(t)$, and, for $i = 1, \ldots, n$,*

$$O^{2'}(C(t) \cap \langle K_{\alpha_i}, K_{\alpha_i}^t \rangle) = O^{2'}(C(t) \cap \langle L_{\alpha_i}, L_{\alpha_i}^t \rangle).$$

Then $\{K_{\alpha_1}, \ldots, K_{\alpha_n}\} = \{L_{\alpha_1}, \ldots, L_{\alpha_n}\}$.

Proof. Let bars denote images in $X/Z(X)$. For each $\alpha \in \Sigma$ there is a root subgroup $\bar{U}_\alpha$ of $\bar{X}$, with $\bar{U}_\alpha \le \bar{B}_1$ if $\alpha \in \Sigma^+$ and $\bar{U}_\alpha \le \overline{B_1^0}$ if $\alpha \notin \Sigma^+$. Use Theorem (1.4) of [4] to construct a group Y such that $Y/Z(Y) \cong \bar{X}$, and Y is a group generated by isomorphic copies of the group $\bar{U}_\alpha$ and having a presentation that involves only the commutator relations that exist among these root subgroups. Then t can be regarded as an automorphism of Y. Now, if we start from root subgroups that are in $\overline{C_1} \cup \overline{C_1^0}$, then with suitable labeling of the elements, the same commutator relations exist and we are led to the same group Y. We conclude that there is an automorphism, σ, of $\bar{X}$ such that the following hold: $\sigma t = t\sigma$ (viewing $t \in \mathrm{Aut}(\bar{X})$), $\bar{B}_1^\sigma = \bar{C}_1$, $\overline{B_1^0}^\sigma = \overline{C_1^0}$, and $\bar{K}_{\alpha_i}^\sigma = \bar{L}_{\alpha_i}$, for $i = 1, \ldots, n$. Then, for $j = 1, \ldots, m$, we have

$$\bar{J}_{\beta_j}^\sigma = \bar{J}_{\beta_j}, \quad (\overline{B_1 \cap J_{\beta_j}})^\sigma = \overline{C_1 \cap J_{\beta_j}} \quad \text{and} \quad (\overline{B_1^0 \cap J_{\beta_j}})^\sigma = \overline{C_1^0 \cap J_{\beta_j}}.$$

But we have assumed that $C_{B_1}(t) = C_{C_1}(t)$ and $C_{B_1^0}(t) = C_{C_1^0}(t)$. It follows that σ normalizes

$$\overline{B_1 \cap J_{\beta_j}} \quad \text{and} \quad \overline{B_1^0 \cap J_{\beta_j}} \quad \text{for} \quad j = 1, \ldots, m.$$

Let $\hat{X}$ be the subgroup of $\mathrm{Aut}(\bar{X})$ generated by $\bar{X}$ together with all diagonal automorphisms of $\bar{X}$. We can write $\sigma = \sigma_1\sigma_2$, where $\sigma_2 \in \hat{X}$ and σ_1 is the product of a field and a graph automorphism of $\bar{X}$, defined with respect to the Borel subgroups $\bar{B}_1$ and $\overline{B_1^0}$ of $\bar{X}$, and centralizing t. Then $\sigma_2 t = t\sigma_2$ (an equation in $\mathrm{Aut}(\bar{X})$) and σ_1 stabilizes the set $\{K_{\alpha_1}, \ldots, K_{\alpha_n}\}$, inducing a graph automorphism (possibly the identity). Now σ_2 acts on $\bar{J} = O^{2'}(C_{\bar{X}}(t))$, and from the choice of σ, we see that σ_2 normalizes each of

$$\bar{J}_{\beta_i}, \overline{B_1 \cap J_{\beta_i}}, \quad \text{and} \quad \overline{B_1^0 \cap J_{\beta_i}},$$

for $i = 1, \ldots, m$. So σ_2 induces a diagonal automorphism of $\bar{J}$ (with respect to the Borel subgroups $\overline{B_1 \cap J}, \overline{B_1^0 \cap J}$), and since $\sigma_2 \in C_{\hat{X}}(t)$, we use the Bruhat decomposition to see that σ_2 is in the Cartan subgroup of $\hat{X}$ that normalizes each of the root subgroups, $\bar{U}_\alpha$, for $\alpha \in \Sigma$. Then $\{K_{\alpha_1}, \ldots, K_{\alpha_n}\}^\sigma = \{K_{\alpha_1}, \ldots, K_{\alpha_n}\}$, proving the lemma.

(5.4) *Let* $Y = PSL(4,2)$, $PSL(5,2)$, $PSU(4,2)$, $PSU(5,2)$, $PSp(4,4)$ *or* $PSp(4,2) \times PSp(4,2)$. *Let* σ *be an involutory automorphism of* Y *with* $C_Y(\sigma) \cong PSp(4,2)$. *If* X *is a* σ*-invariant subgroup of* Y *with* $C_Y(\sigma) < X < Y$ *and* $C_Y(\sigma) \ntrianglelefteq X \ntrianglelefteq Y$, *then* $Y \cong PSU(5,2)$ *or* $PSL(5,2)$ *and* $X' \cong PSU(4,2)$ *or* $PSL(4,2)$, *respectively. We omit the details.*

Proof. If $Y \cong PSp(4,2) \times PSp(4,2)$, then this is easy. In the other cases the result follows from Sylow's theorem together with an analysis of the action of X on the underlying vector space defining Y. We omit the details.

(5.5) *Let* $\tilde{A} \cong O^{\pm}(n,2)'$, $I \le A$, *and let* $P < A$ *satisfy*

$$PZ(A)/Z(A) \cong PSO^+(8,2).$$

Suppose that $P = E(C_A(I))$ *is a standard subgroup of* $C_G(I)$ *and that*

$$R \in Syl_2(C_G(P) \cap C_G(I)).$$

Finally assume that when A *is regarded as acting on the subspace of the usual* $\mathbf{F}_2$*-module,* V*, of* $O^{\pm}(n, 2)$ *we may write* $V = V_1 \perp V_2$*, with* $\dim(V_1) = 8$*,* P *fixes each 1-space of* V_2*, and* V_1 *is* P*-invariant. Then* $C_G(I)^{\sim} \not\cong M(22)$.

Proof. Suppose otherwise. Then $C(t) \cap E(C_G(I)) \cong \mathrm{Aut}(O^+(8, 2)')$ (see Table 1, p. 441 in [2]). Let x be a 3-element centralizing t and acting as a graph automorphism of order 3 on P. We know that $x \in C(t) \leq N(A)$. However from the embedding of P in A we see that this is impossible.

6. Notation and the subgroup E

Write $A = \langle U_{\pm\alpha_1}, \ldots, U_{\pm\alpha_l} \rangle$, where for $\alpha \in \Sigma$ (the root system of A), U_α is the corresponding root subgroup. Set $V_\alpha = \Omega_1(U_\alpha)$ and $J_\alpha = \langle V_{\pm\alpha} \rangle$. Then for each $\alpha \in \Sigma$, $J_\alpha \cong SL(2, q)$ for some $q = 2^a$. For $i = 1, \ldots, l$ we may choose the fundamental reflection $s_i \in J_{\alpha_i}$. Choose $r \in \Sigma^+$ such that r is long and $V_r \leq Z(U)$ and set $J = J_r$. We set $\hat{J}_\alpha = \langle U_\alpha, U_{-\alpha} \rangle$.

At this point we assume that Hypothesis $(*)$ holds and that the theorem is true for all pairs (A_1, G_1) with $|A_1| < |A|$. By [14] we may assume that $\tilde{A} \not\cong PSL(n, q)$. Also we have $\tilde{A}$ of Lie rank at least 3, but $\tilde{A} \not\cong PSp(6, 2)$, $PSU(6, 2)$, $PSO^{\pm}(8, 2)$. We adopt the notation of [13].

Choose $X \leq A$ and $D = E(C_A(X))$ as in (4.1) of [13]. Set $E = E(C_G(X))$.

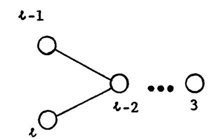
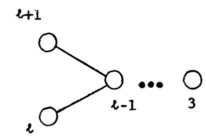
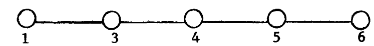
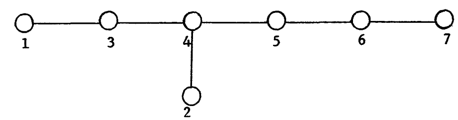
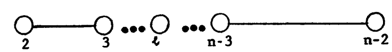
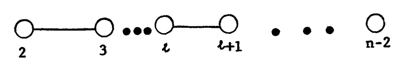
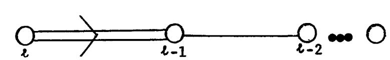
(6.1) *The pair* $(\tilde{D}, \tilde{A})$ *is one of the following (up to isomorphism):*
(i) $(O^{\pm}(n-4, q)', O^+(n, q)')$, *n even,*
(ii) $(L_6(q), E_6(q))$,
(iii) $(O^+(12, q)', E_7(q))$,
(iv) $(E_7(q), E_8(q))$,
(v) $(PSp(6, q), F_4(q))$,
(vi) $(PSU(6, q), {}^2E_6(q))$,
(vii) $(PSp(n-2, q), PSp(n, q))$, *n even,*
(viii) $(PSU(n-2, q), PSU(n, q))$.

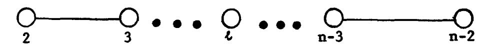
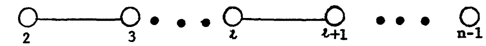
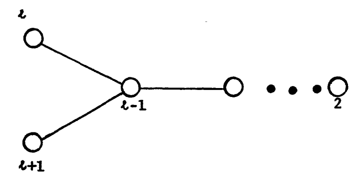
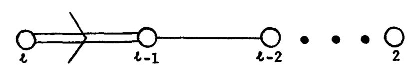
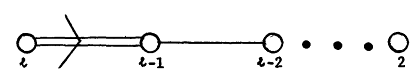
Proof. This follows from (4.1) and (4.3) of [13].

(6.2) $R = \langle t \rangle$ *and one of the following holds:*
(i) $\tilde{E} \cong \tilde{D} \times \tilde{D}$*, with* t *interchanging the factors.*
(ii) $\tilde{E}$ *is a finite group of Lie type defined over a field of characteristic* 2*, and* t *induces an outer automorphism of* $\tilde{E}$ *(a field, graph, or graph-field automorphism).*

Proof. The structure of $\tilde{E}$ is given by induction, Hypothesis $(*)$, or by application of the theorems in [11], [12], [14], and [20]. In addition, we use (5.5) in case $\tilde{D} \cong O^+(8, 2)'$. To see that $R = \langle t \rangle$ use (3.2) of [16].

TABLE 2

$\tilde{D}$	$\tilde{E}$	diagram	t
(1) $O^+(n-4, q)' = \langle J_{\alpha_l}, \ldots, J_{\alpha_3}\rangle$, $l = n/2$	$O^+(n-4, q^2)'$		field
(2) $O^-(n-4, q)' = \langle \hat{J}_{\alpha_l}, \ldots, J_{\alpha_3}\rangle$, $l = (n-2)/2$	$O^+(n-4, q^2)'$		graph-field
(3) $L_6(q) = \langle J_{\alpha_1}, J_{\alpha_3}, J_{\alpha_4}, J_{\alpha_5}, J_{\alpha_6}\rangle$	$L_6(q^2)$		field
(4) $E_7(q) = \langle J_{\alpha_1}, \ldots, J_{\alpha_7}\rangle$	$E_7(q^2)$		field
(5) $PSU(n-2, q) = \langle J_{\alpha_l}, J_{\alpha_{l_1}}, \ldots, J_{\alpha_2}\rangle$, $l = n/2$	$PSL(n-2, q^2)$		graph-field
(6) $PSU(n-2, q) = \langle \hat{J}_{\alpha_l}, J_{\alpha_{l-1}}, \ldots, J_{\alpha_2}\rangle$, $l = (n-1)/2$	$PSL(n-2, q^2)$		graph-field
(7) $PSp(n-2, q) = \langle J_{\alpha_l}, \ldots, J_{\alpha_2}\rangle$, $l = n/2$	$PSp(n-2, q^2)$		field

(8) $PSp(n-2, q) = J_l, \ldots, J_2\rangle$, $l = n/2$	$PSL(n-2, q)$		graph
(9) $PSp(n-2, q) = \langle J_l, \ldots, J_2\rangle$, $l = n/2$	$PSL(n-1, q)$		graph
(10) $PSp(n-2, q) = \langle J_l, \ldots, J_2\rangle$, $l = n/2$	$O^+(n, q)'$		graph
(11) $PSp(n-2, q) = \langle J_l, \ldots, J_2\rangle$, $l = n/2$	$O^-(n, q)'$		graph (field)
(12) $PSp(n-2, q) = \langle J_l, \ldots, J_2\rangle$, $l = n/2$	$PSU(n-2, q)$		graph (field)
(13) $PSp(n-2, q) = \langle J_l, \ldots, J_2\rangle$, $l = n/2$	$PSU(n-1, q)$		graph (field)

The group D is generated by certain of the groups $\hat{J}_{\alpha_i}$, $i = 1, \ldots, l$. Indeed, for all cases except (6.1)(i), D is generated by all but one of the groups $\hat{J}_{\alpha_i}$. There is a unique root $s \in \Sigma^+$ such that $V_s \le Z(U \cap D)$ and $V_s^{\#}$ consists of root involutions in E. However, there are cases where root subgroups of A contained in D are not contained in root subgroups of E. This can occur if t induces a graph automorphism of the Dynkin diagram of E. In the accompanying table we list the possible configurations that occur in (6.2)(ii). Indicated are the groups $\tilde{D}$, $\tilde{E}$, the Dynkin diagram of $\tilde{E}$, and the type of automorphism that t induces on $\tilde{E}$.

We remark that except for cases (10) and (11) above we always have $s \sim r$ in W, so $J_s \sim J_r$ in A. When we discuss the pair $(\tilde{D}, \tilde{E})$ we will always refer to one of the entries in the preceding table with the given embedding of root systems. So, for example, we distinguish between $(PSp(4, q), PSU(4, q))$ and $(PSp(4, q), PSO^-(6, q))$, even though $PSU(4, q) \cong PSO^-(6, q)$.

(6.3) *Assume that the root system, $\Sigma_1 \subseteq \Sigma$, of D is not of type C_2, B_2, B_3, A_3, B_4, or D_4, and also assume $r \sim s$ in W. There is an involution $w \in A$ such that $\bar{J}_r^w = \bar{J}_s$ (see (4.1) for the definition of $\bar{J}_r$ and $\bar{J}_s$). If $J_{\alpha_i} \le C(\bar{J}_r)$, then there is a root $\alpha \in \Sigma$ such that $\bar{J}_\alpha \le C(\bar{J}_r) \cap C(\bar{J}_s) \cap C(J_{\alpha_i}^w)$. If W is not of type F_4, then α can be chosen conjugate to r.*

Proof. This is proved by direct check. The following table gives the relevant information. The first column gives the type of W, the second gives the element w. The third column lists the roots, α_i, with $J_{\alpha_i} \le C(\bar{J}_r)$, and the last column gives the corresponding roots α.

E_6	$(s_3s_5)^{s_4s_2}$	$\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$	$\alpha_3, \alpha_3, \alpha_3+\alpha_4+\alpha_5, \alpha_5, \alpha_5$
E_7	$(s_2s_5)^{s_4s_3s_1}$	$\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$	$\alpha_2, \alpha_2, \alpha_2+\alpha_3+\alpha_4, \alpha_3, \alpha_5, \alpha_2$
E_8	$(s_3s_2)^{s_4s_5s_6s_7s_8}$	$\alpha_1, \ldots, \alpha_7$	$\alpha_3, \alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_3, \alpha_3+\alpha_4+\alpha_5,$ $\alpha_3, \alpha_3, \alpha_3$
F_4	$s_3^{s_2s_1}$	$\alpha_2, \alpha_3, \alpha_4$	$\alpha_2+2\alpha_3, \alpha_2+\alpha_3, \alpha_2$
D_n	$(s_3s_1)^{s_2}$	$\alpha_3, \ldots, \alpha_n$	$\alpha_n, \alpha_n, \ldots, \alpha_n, \alpha_{n-2}+\alpha_{n-1}+\alpha_n, \alpha_n, \alpha_{n-1}$
C_n	s_1	$\alpha_2, \ldots, \alpha_n$	$\alpha_n, \ldots, \alpha_n, \alpha_n+2\alpha_{n-1}+2\alpha_{n-2},$ $\alpha_n+2\alpha_{n-1}$
B_n	$(s_3s_1)^{s_2}$	$\alpha_3, \ldots, \alpha_n$	$\alpha_{n-1}, \ldots, \alpha_{n-1}, \alpha_{n-1}+2\alpha_n, \alpha_{n-1}+\alpha_n$

We will also consider roots not conjugate to r. If Σ has roots of different lengths, let γ be the short root in Σ^+ of highest height. Let δ be the short root of highest height in the root system of D. So $J_\delta \le D$ and $J_\delta \sim J_\gamma$ in A.

(6.4) *Suppose $\tilde{A} \cong F_4(q)$. Let $P = E(C_A(J_\gamma))$.* Then

$$P = \langle J_{\alpha_1}, J_{\alpha_2}, J_{\alpha_3} \rangle \cong Sp(6, q), \quad P = E(C_A(Y))$$

for Y a $(q+1)$-Hall subgroup of $J_\gamma \cong SL(2, q)$. Also $Z=\langle J_{\alpha_1+\alpha_2+\alpha_3}, J_s\rangle \cong Sp(4, q)$.

Proof. This follows from the fact that a graph automorphism of $F_4(q)$ interchanges J_r and J_γ.

(6.5) *Suppose $\tilde{A} \cong PSp(n, q)$ with $n \geq 6$. Let*

$$P=O^{2'}(C_A(J_\gamma \times J_{\alpha_1})) \quad and \quad Z=O^{2'}(C_A(P)).$$

Then $P=\langle J_{\alpha_n}, \ldots, J_{\alpha_3}\rangle \leq D$, $Z=\langle J_{\alpha_1}, J_s\rangle \cong Sp(4, q)$, and $P=E(C_A(Y))$, where Y is a $(q+1)$-Hall subgroup of

$$J_{\alpha_1} \times J_\gamma \cong SL(2, q) \times SL(2, q).$$

Proof. This can be checked using the natural module V for the group $Sp(n, q)$. The involutions in J_γ and J_{α_1} are of type a_2 in the notation of §7 of [1]. One shows that $J_\gamma \times J_{\alpha_1}$ induces the identity on a non-degenerate $(n-4)$-subspace of V. The result follows.

(6.6) *Let $\tilde{A} \cong PSU(n, q)$ with $n \geq 6$. Let*

$$P=O^{2'}(C_A(J_\gamma)) \quad and \quad Z=O^{2'}(C_A(P)).$$

Then

$$P=\langle \hat{J}_{\alpha_n}, J_{\alpha_{n-1}}, \ldots, J_{\alpha_3}\rangle, \quad Z=\langle J_{\alpha_1}, J_s\rangle \cong SU(4, q) \quad and \quad P=O^2(C_A(Y)),$$

where Y is a (q^2+1)-Hall subgroup of $J_\gamma \cong SL(2, q^2)$.

Proof. As in (6.5) this is checked using the natural module V for $SU(n, q)$. We may regard the group J_γ as acting on V. Then J_γ is trivial on a non-degenerate $(n-4)$-space of V and acts faithfully on a non-degenerate 4-space, V_0, of V stabilizing complementary isotropic 2-spaces. The group Y is fixed-point-free on V_0. From the structure of $SU(4, q)$ we see that no involution in $SU(4, q)$ centralizes an element of order q^2+1. It follows that

$$O^{2'}(C_A(J_\gamma))=O^{2'}(C_A(Y)) \cong SU(n-4, q).$$

Since the commutator relations imply that $\langle \hat{J}_{\alpha_n}, \ldots, J_{\alpha_3}\rangle \cong SU(n-4, q)$ is contained in $C_A(Y)$ we have $P=\langle \hat{J}_{\alpha_n}, \ldots, J_{\alpha_3}\rangle$. Similarly $\langle J_{\alpha_1}, J_s\rangle \leq O^{2'}(C_A(P))$ and $C_A(P)$ must stabilize V_0. The result follows.

(6.7) *Let $\tilde{A} \cong {}^2E_6(q)$. Let*

$$P=O^{2'}(C_A(J_\gamma)) \quad and \quad Z=O^{2'}(C_A(P)).$$

Then $P=\langle J_{\alpha_2}, J_{\alpha_1}, J_{\alpha_2}^{s_3}\rangle \cong O^+(6, q)' \cong PSL(4, q)$, $Z=J_\gamma$, and $P=O^2(C(Y))$, where Y is a (q^2+1)-Hall subgroup of $J_\gamma \cong SL(2, q^2)$.

Proof. $J_\gamma=\langle U_\gamma, U_{-\gamma}\rangle$, so we first look at $C_A(U_\gamma)$. Using (4.6) of [6] we consider the structure of the parabolic subgroup $\langle B, s_1, s_2, s_3\rangle = I$. This group satisfies $O^{2'}(I)=QD$, where $Q=O_2(I)$ and $D=\langle J_{\alpha_1}, J_{\alpha_2}, J_{\alpha_3}\rangle \cong O^-(8, q)'$.

Moreover, Q contains a subgroup $Q_1 < I$ such that Q_1 is elementary of order q^8 and D preserves a non-degenerate quadratic form on Q_1. Then Q_1 becomes an orthogonal space and in this space U_γ is an anisotropic 2-space. Since $Q_1 \leq Z(Q)$, $C(U_\gamma) \cap QD = QD_1$ where $D_1 \cong O^+(6, q)'$. But $\langle J_{\alpha_2}, J_{\alpha_1}, J_{\alpha_2}^{s_3}\rangle$ centralizes U_γ, so $D_1 = \langle J_{\alpha_2}, J_{\alpha_1}, J_{\alpha_2}^{s_3}\rangle$. Therefore

$$P = O^{2'}(C(J_\gamma)) = O^{2'}(C(U_\gamma)) \cap O^{2'}(C(U_{-\gamma})) = \langle J_{\alpha_2}, J_{\alpha_1}, J_{\alpha_2}^{s_3}\rangle.$$

Next we check that $O^{2'}(C_A(P)) = J_\gamma$, as follows. We know that

$$O^{2'}(C_A(J_r)) = \langle J_{\alpha_2}, J_{\alpha_3}, J_{\alpha_4}\rangle.$$

Also, $\alpha_2 \sim \alpha_1 \sim \alpha_2^{s_3} \sim r$ in W. We can then check

$$C_A(J_{\alpha_2}) \cap C_A(J_{\alpha_1}) \cap C_A(J_2^{s_3})$$

to get the result.

Finally consider $Y \leq J_\gamma$ and $C_A(Y)$. Clearly $P \leq C_A(Y)$. Also the 2-central involutions in P are root involutions in A and so also in $C_A(Y)$. If u is a root involution in $C_A(Y)$, then we can use the information in (4.6) of [6] to see that $C_A(Y) \cap C_A(u) = C_P(u)$. Now $C_P(u)$ is the centralizer of a transvection, when P is regarded as $SL(4, q)$. It follows that u is a 2-central involution in $C_A(Y)$ and that the Sylow 2-subgroups of $C_A(Y)$ are isomorphic to those of $SL(4, q) \cong P$. Setting $Z = \langle P^{C_A(Y)}\rangle$, we use Theorem 1 of [17] to conclude $P = Z = O^{2'}(C_A(Y))$.

7. Generating subgroups

In this section we will construct certain subgroups of G. In later sections these subgroups will be shown to generate a subgroup $G_0 \leq G$ such that $\tilde{G}_0$ is isomorphic to one of the groups in the main theorem. To this end we will establish some commutator relations among the constructed subgroups.

Let X, D be as in §6.

(7.1) *Let bars denote images in $C_G(X)/XO(C_G(X))$. Then $\bar{D}$ is a standard subgroup of $\overline{C_G(X)}$ and $\bar{D} \ntrianglelefteq \overline{C_G(X)}$.*

Proof. This is (4.9)–(4.12) of [13].

(7.2) (i) $D \leq E(C_G(X))$.

(ii) $R = \langle t\rangle \ntrianglelefteq E(C_G(X))$.

(iii) $|Z(E(C_G(X)))|$ *is odd.*

(iv) *The pair* $(D, E(C_G(X))^\sim)$ *is one of the pairs listed in the main theorem.*

Proof. Look at the group $C_G(X)/X$ and apply (5.1) and (6.2). This gives the structure of $E(C_G(X)/X)$. Now apply (3.1).

Let $E = E(C_G(X))$. The action of t on E shows that $t^G \cap tD \neq \{t\}$. Consequently we may assume that we are not in the situation of (3.5)(ii) of [13]. In particular, we may now assume X to be of odd order.

(7.3) *Notation.* Recall, that if A is an orthogonal group, then $\bar{J}_r = J_r \times J_{\alpha_1}$. Otherwise $\bar{J}_r = J_r$. Except for the case $\tilde{A} \cong O^+(8, q)'$, X is a $(q+1)$-Hall subgroup of $\bar{J}_r$. For each $\alpha \in \Sigma^+$ with $a \sim r$ in W, choose $w \in W$ with $\alpha = r^w$, and regarding $w \in G$ set $\bar{J}_\alpha = \bar{J}_r^w$, $X_\alpha = X^w$ and $E_\alpha = E^w$. Fix notation so that $w = 1$ if $\alpha = r$ and w is as in (6.3) if $\alpha = s$.

For each of the possible pairs $(\tilde{D}, \tilde{E})$ there is a subgroup K_s of E, such that $J_s \leq K_s$, K_s is t-invariant, and

$$K_s \cong SL(2, q^2),\ SL(2, q) \text{ or } SL(2, q) \times SL(2, q).$$

Indeed, if $\tilde{E} \cong \tilde{A} \times \tilde{A}$, set K_s to be the group generated by the root involution in the projections of J_s to the components of E. Otherwise, one checks that the involutions in J_s are root involutions in E and we set K_s to be the group generated by the involutions of the root subgroups of E containing V_s and V_{-s}.

Finally, we note that $K_s = J_s \cong SL(2, q)$ only if $\tilde{D} \cong Sp(n, q)$ for n even and $\tilde{E}$ is one of $L_n(q)$, $L_{n+1}(q)$, $PSU(n, q)$, $PSU(n+1, q)$, or $PSO^{\pm}(n+2, q)'$.

(7.4) *Suppose $\tilde{A} \not\cong O^{\pm}(8, q)'$ or $O^{\pm}(10, q)'$, and also suppose that $(\tilde{D}, \tilde{E})$ is not $(PSp(n, q), O^{\pm}(n+2, q)')$, with $n \geq 4$. Let $\alpha \in \Sigma$ be conjugate to r. Then $\bar{J}_\alpha \leq C_G(E_\alpha)$, so $E_\alpha = E(C_G(\bar{J}_\alpha))$.*

Proof. It will suffice to prove this for $\alpha = r$. Here $X = X_r$ and $E = E_r = E_\alpha$. The structure of $\tilde{E}$ is known by (6.2) and Table 2. Let s be as in the remark following (6.2) and $J_s = \langle V_s, V_{-s} \rangle \leq E$. By (4.3), $D \leq C(\bar{J}_r)$.

Suppose $(\tilde{D}, \tilde{E}) \neq (PSp(4, q), PSU(4, q))$, $(PSp(4, q), PSL(4, q))$. We claim that $t \notin Z^*(C(\bar{J}_r))$. Suppose otherwise. Since $\bar{J}_r$ and $\bar{J}_s$ are conjugate by an element of A, we have $t \in Z^*(C_G(\bar{J}_s))$. Hence, $t \in Z^*(Y\langle t \rangle)$, where $Y = C_E(\bar{J}_s)$. But a direct check shows this to be false. Thus the claim holds, and, consequently, $DO(C(\bar{J}_r)) \ntrianglelefteq C(\bar{J}_r)$. Now argue as in the proof of (6.2) and then use (5.1) to obtain the structure of $E(C(\bar{J}_r))$.

Now $C(\bar{J}_r) \leq C(X)$ and D is standard in each of $E(C(\bar{J}_r))$ and $E(C(X)) = E$. By (5.2), either (7.4) holds or $(\tilde{D}, E(C(\bar{J}_r)), \tilde{E})$ is one of

$$(PSp(n, q), PSL(n, q), PSL(n+1, q)) \quad \text{or} \quad (PSp(n, q), PSU(n, q), PSU(n+1, q)).$$

Suppose one of the latter holds and let w be as in (6.3). Then w interchanges $X \times \bar{J}_s$ and $X^w \times \bar{J}_r$. So $O^{2'}(C(X\bar{J}_s)) \sim O^{2'}(C(\bar{J}_r X^w)) = O^{2'}(C(\bar{J}_r \bar{J}_s))$. Comparing centralizers of $\bar{J}_s$ in $C(X)$ and in $C(\bar{J}_r)$ we obtain a contradiction. Suppose, now, that

$$(\tilde{D}, \tilde{E}) = (PSp(4, q), PSU(4, q)) \text{ or } (PSp(4, q), PSL(4, q)).$$

Then $Y=J_{\alpha_3}\times I$, where $I/Z(E)\cong Z_{q+1}$ or Z_{q-1}, respectively. Let X_0 be a $(q+1)$-Hall subgroup of J_{α_3}. Then $X_0 \sim_A X$ and $J_r \leq C(X_0)$. In fact, $J_r = E(E(C(X_0))\cap C(X))$ (recall that $q>2$ here). Consequently, $N_G(J_r)\geq \langle D, I\rangle = E$, and the result follows.

Hypothesis (7.5). (i) $s\sim r$ in W.
(ii) $\tilde{A}\not\cong O^{\pm}(n, q)'$, with $n=8$, 10, or 12.
(iii) $(\tilde{D}, \tilde{E})\neq (PSp(n, q), O^{\pm}(n+2, q)')$, with $n\geq 4$.

Remark. As stated in §6 we distinguish the pairs

$$(PSp(4, q), PSU(4, q)), (PSp(4, q), O^{-}(6, q)')$$

and also the pairs

$$(PSp(4, q), PSL(4, q)), (PSp(4, q), O^{+}(6, q)').$$

So in each case the first pair is not ruled out in Hypothesis (7.5).

(7.6) *Assume Hypothesis* (7.5). *Then* $K_s \leq C_G(E_s)$.

Proof. This is clear from (7.4) if $K_s = J_s \cong SL(2, q)$. So suppose $J_s < K_s$. Assume first that $q\geq 4$. Then there is an easy argument as follows. Since $K_s\cong SL(2, q^2)$ or $SL(2, q)\times SL(2, q)$, there is a subgroup $\hat{X}_s \leq K_s$ such that $\hat{X}_s$ is an abelian Hall subgroup of K_s and $\hat{X}_s\cap J_s$ is an A-conjugate of the subgroup $X\leq J_r$. Moreover $\hat{X}_s$ centralizes a $(q+1)$-Hall subgroup of $\bar{J}_s$ if $\bar{J}_s > J_s$. So $\hat{X}_s\leq N_G(E_s)$ (recall the definition of E_s). But $K_s=\langle J_s, \hat{X}_s\rangle$, so $K_s\leq N_G(E_s)$. As $J_s\not\leq C_G(E_s)\cap K_s \ntrianglelefteq K_s$ we must have $K_s\leq C_G(E_s)$ as described.

For the remainder of the proof we assume $q=2$. Recall that $\tilde{A}\not\cong O^{\pm}(n, q)'$ for $n=8, 10$, or 12. Let $r^w = s$, where w is as in (6.3). Choose α_i with $J_{\alpha_i}\leq C_A(\bar{J}_r)$. Then $J_{\alpha_i}^w \leq C_A(\bar{J}_s)$. By (6.3) there exists a root $\alpha\in\Sigma$ such that $\bar{J}_\alpha \leq C(\bar{J}_r)\cap C(\bar{J}_s)\cap C(J_{\alpha_i}^w)$. Suppose, for the moment, that W is not of type F_4. Then, by (6.3), we may take $\alpha\sim r$. From the definition of K_s one checks that $\bar{J}_\alpha\leq C(K_s)$. We claim that $J_{\alpha_i}^w\leq C(K_s)$. Clearly $K_s, J_{\alpha_i}^w \leq C(\bar{J}_\alpha)$. Also, $J_s, J_{\alpha_i}^w \leq E_\alpha = E(C(\bar{J}_\alpha))$. This is because E_α and E_r are conjugate by an element of W (considered as an element of A). If $K_s\not\cong S_3\times S_3$, then $K_s\cong L_2(4)$ and we must have $K_s\leq E_\alpha$ (since $K_s\leq N(K_s\cap E_\alpha)$ and $K_s\cap E_\alpha\geq J_s$). Suppose $K_s\nleq E_\alpha$. Then $K_s\cong S_3\times S_3$ and $\tilde{E}\cong \tilde{D}\times\tilde{D}$. Because of our standing assumptions on $\tilde{A}$ we see, from the structure of $\tilde{E}$, that either $\tilde{D}\cong Sp(6, 2)$ or $K_s\leq C_E(\bar{J}_\alpha)^{(\infty)}$. As we are assuming $K_s\nleq E_\alpha = C(\bar{J}_\alpha)^{(\infty)}$, we must have $\tilde{D}\cong Sp(6, 2)$. Since $K_s\leq N(K_s\cap E_\alpha)$ and $J_s\leq K_s\cap E_\alpha$, we must have $K_s=(K_s\cap E_\alpha)\langle u\rangle$, where u is an involution satisfying $[u, t]=v$ and $\langle v\rangle = V_s$. Since $\operatorname{Aut}(Sp(6, 2)) = Sp(6, 2)$, v interchanges the components of E_α. So tu stabilizes each component of E_α. In particular, tu stabilizes the intersection of $O_3(K_s)$ with each component of E_α. But then $v=(tu)^2$ centralizes $O_3(K_s)$, a contradiction. So we necessarily have $K_s\leq E_\alpha$.

Let $L=O^{2'}(C_A(\bar{J}_\alpha\bar{J}_s\bar{J}_r))$. Considering $T=C(\bar{J}_\alpha\bar{J}_r L)$ as a subgroup of $C(\bar{J}_r)$

we have $O^{2'}(T)=\bar{K}_s$, where $\bar{K}_s=K_s$ or $K_s\times K_s^x$, according to whether or not $\bar{J}_s=J_s$ or $\bar{J}_s>J_s$. Let $Y=E(C_{E_\alpha}(\bar{J}_s))$. Then from the structure of $E_\alpha\sim E$ we check that

$$O^{2'}(C_{E_\alpha}(Y))=O^{2'}(C_{E_\alpha}(\bar{J}_rL))\cong\bar{K}_s.$$

As $K_s\leq O^{2'}(C_{E_\alpha}(\bar{J}_rL))$ and as $J_{\alpha_i}^w\leq Y$, we conclude that $J_{\alpha_i}^w\leq C(K_s)$. Thus, the claim holds.

We show that this also holds if W is of type F_4. Consider the possible values of s_i^w, using the table in (6.3). If $i=2$ or 3, then $J_{\alpha_i}^w=J_{\alpha_i}$ and $J_{\alpha_i}\leq C(K_s)$ (view this in E). Suppose $i=4$. The corresponding value of α is $\alpha=\alpha_2\sim r$, and the above arguments apply here. So in all cases we have $J_{\alpha_i}^w\leq C(K_s)$.

At this stage we have

$$C_G(K_s)\geq\langle C_E(K_s), J_{\alpha_i}^w\colon J_{\alpha_i}\leq E\rangle=\langle C_E(K_s), D^w\rangle=Y_1.$$

Since we know the structure of $N(K_s)\cap C(\bar{J}_r)$ we can apply induction and (5.2) to see that $Y_1=E_s$. It follows that $K_s\leq C_G(E_s)$, as desired.

(7.7) *Assume Hypothesis* (7.5).

(i) *If $\tilde{A}$ is not an orthogonal group, then for $a_1, a_2\in A$, $[J_s^{a_1}, J_s^{a_2}]=1$ if and only if $[K_s^{a_1}, K_s^{a_2}]=1$.*

(ii) *If $\tilde{A}$ is an orthogonal group, then for a_1, a_2 in A $[K_s^{a_1}, K_s^{a_2}]=1$, provided $[\bar{J}_s^{a_1}, \bar{J}_s^{a_2}]=1$.*

Proof. This is clear if $J_s=K_s$, so suppose $J_s<K_s$. Also, since $J_s\leq K_s$ it will be sufficient to assume $[\bar{J}_s^{a_1}, \bar{J}_s^{a_2}]=1$ and to prove $[K_s^{a_1}, K_s^{a_2}]=1$. So set $a=a_2a_1^{-1}\in A$ and assume $[\bar{J}_s, \bar{J}_s^a]=1$. Then $\bar{J}_s^a\leq C(\bar{J}_s)$, so $\bar{J}_s^a\leq E_s\leq C_G(K_s)$ by (7.6). So $K_s\leq C_G(\bar{J}_s^a)$. Also, $J_s\leq E(C_G(\bar{J}_s^a))$ so as in (7.6) either $K_s\leq E(C_G(\bar{J}_s^a))\leq C(K_s^a)$ (by (7.6)), or $E(C_G(\bar{J}_s^a))\cong\tilde{D}\times\tilde{D}$ and $K_s=(K_s\cap E(C(\bar{J}_s^a)))\langle u\rangle$, where $[u, t]=v\in V_s^\#$. In the latter case argue as follows. By (7.6), $C(K_s)\cap C(\bar{J}_s^a)\geq E_s\cap C(\bar{J}_s^a)$. But this does not coincide with the structure of $C(\bar{J}_s^a)\cap C(K_s)$ obtained from the embedding of K_s in $C(\bar{J}_s^a)$. Therefore, we must have $[K_s, K_s^a]=1$, as required.

(7.8) *Assume Hypothesis* (7.5).

(i) $K_s\leq C_G(E_s)$.

(ii) *If $K_s>J_s$, $K_s\not\cong S_3\times S_3$, and if $\tilde{A}$ is not an orthogonal group, then* $K_s=E(C_G(E_s))$.

(iii) *If $w\in N$ (regarded as an element of W) and $J_s^w=J_s$, then $K_s^w=K_s$.*

Proof. Consider $O^{2'}(C_G(E_s))\geq J_s$. We may assume that $K_s>J_s$. (i) follows from (7.6). Assume $\tilde{A}$ is not an orthogonal group. We have $K_s\leq O^{2'}(C_G(E_s))$. If $J_s\not\cong S_3$, then J_s is a standard subgroup of $C_G(E_s)$. Using the main theorem of [10] and (2.1), we obtain (ii). Suppose $J_s\cong S_3$ and let $V_s<I\in Syl_2(K_s)$. We are assuming that $K_s\not\cong S_3\times S_3$, so $K_s\cong L_2(4)$. We claim that $I\in Syl_2(E(C_G(E_s)))$. Otherwise, there is an element $x\in E(C_G(E_s))$ with

$x \notin I$, $x^2 \in I$, and x normalizing $I\langle t\rangle$. Since $t \notin C(E_s)$, $t^x \notin C(E_s)$ and hence $t^x \in tI$. But then $t^x \in t^I$ and $x \in I(C(t) \cap C(E_s)) = IJ_s\langle t\rangle$, a contradiction. From here we obtain $K_sO(C_G(E_s)) = L(C_G(E_s))$, and arguing as in the proof of (5.1) we have the result.

Suppose $w \in N$ and $J_s^w = J_s$. Assume $\tilde{A}$ is not an orthogonal group. We have $w \in J_s \times C_A(J_s)$. So we may assume $w \in C_A(J_s)$, for, otherwise, replace w by $w_1 = gw$ with $g \in W \cap J_s$. Then $C_A(J_s) = E(C_A(J_s)) \leq E_s \leq C(K_s)$ (by (i)). So $K_s^w = K_s$ and (iii) holds. Suppose that $\tilde{A}$ is an orthogonal group. Write $s = r^{w_1}$ where $w_1 = s_2s_3s_1s_2$. Then

$$w \in (\bar{J}_r \times D)^{w_1} = J_s \times J_{\alpha_3} \times D^{w_1}.$$

Now $J_{\alpha_3} \leq C(K_s)$, so we may assume $w \in D^{w_1} \leq E_r^{w_1} = E_s$ and again the result follows from (i).

At this point we know that, given Hypothesis (7.5), we can define a subgroup K_α for each $\alpha \in \Sigma$ with $\alpha \sim r$. Namely for such a root α choose $w \in W$ with $s^w = \alpha$. Then regard w as an element of A and set $K_\alpha = K_s^w$. By (7.8)(iii) this is well defined. Also, $K_\alpha^t = K_\alpha$. Moreover, (7.7) gives certain commutator relations among the K_α. For example, we have:

(7.9) *Assume Hypothesis* (7.5) *and that* $\tilde{A}$ *is not an orthogonal group. Let* $\alpha, \beta \in \Sigma$ *and* $\alpha \sim \beta \sim r \sim s$. *Then* $[K_\alpha, K_\beta] = 1$ *if and only if* $[J_\alpha, J_\beta] = 1$.

(7.10) *Assume that Hypothesis* (7.5) *holds. Let* $\tilde{A} \cong PSp(n, q)$ *with* $n \geq 8$, $PSU(n, q)$ *with* $n \geq 6$, *or* $PSp(6, q)$ *with* $\tilde{E} \cong PSp(4, q^2)$, $PSU(5, q)$, *or* $PSp(4, q) \times PSp(4, q)$. *Then the following hold:*

(i) *There exists* $g \in E$ *with* $t \neq t^g \in C(Z)$ (*notation as in* (6.5) *and* (6.6)).

(ii) $C_G(Z)$ *contains* $P = \langle \hat{J}_{\alpha_l}, J_{\alpha_{l-1}}, \ldots, J_{\alpha_3}\rangle$ *as a standard subgroup,*

$$PO(C_G(Z)) \ntrianglelefteq C_G(Z),$$

and $\langle t\rangle \in Syl_2(C_G(Z) \cap C_G(P))$.

(iii) $\langle J_{\alpha_1}^{C(Z)}\rangle \leq E$, *and* $\langle J_{\alpha_1}^{C(Z)}\rangle = E(C_G(Z))$ *unless* $\tilde{A} \cong PSp(8, 2)$.

Proof. To get (i) we consider the action of t on E and use the results of §19 of [1]. In most cases it follows that if $v \in D$ is a transvection, then $t \sim tv$ by an element of E. Otherwise $t \sim tv$ for v a product of two commuting transvections. Since

$$C_A(Z) \geq \langle \hat{J}_{\alpha_l}, \ldots, J_{\alpha_3}\rangle,$$

we may choose v so that $t^g = tv$ satisfies (i). Also, it is easy to check that $\langle t\rangle \in Syl_2(C_G(Z) \cap C_G(P))$.

Suppose that $\tilde{A} \cong PSp(n, q)$ or $PSU(n, q)$, with $n \geq 8$. Notice that if $\tilde{A} \cong PSp(8, q)$, then (7.5)(iii) shows that $\tilde{E} \ncong L_6(q)$ or $U_6(q)$. Let $r \sim \eta \in \Sigma$ and choose η such that $[J_\eta, Z] = 1$. Let $L = O^{2'}(C_A(J_\eta Z))$. Then $\tilde{L} \cong PSp(n-6, q)$ or $PSU(n-6, q)$. Then $L \times Z \leq E_\eta$ and we check that $t \notin Z^*(C_{E_\eta}(Z)\langle t\rangle)$. Consequently, $t \notin Z^*(C_G(Z))$. This proves (ii). As $J_{\alpha_l} \leq E$

and $C(Z) \leq C(J_r) \leq N(E)$, certainly $\langle J_{\alpha_l}^{C(Z)} \rangle \leq E$. If $\tilde{A} \not\cong PSp(8, 2)$, then $J_{\alpha_l} \leq C_A(Z)^{(\infty)}$ and an easy argument gives the rest of (iii).

In the remaining cases let V be the usual module for $Sp(6, q)$, $SU(6, q)$, or $SU(7, q)$ and consider A^g acting, projectively, on V as $(A^g)^{\sim}$. Since $g \in C(J_r)$, $J_r \leq A^g$. As $Z < N(A^g)$ and $Z = \langle J_r^Z \rangle$, we must have $Z \leq A^g$. Also, $g \in C(J_r)$ implies that V_r is a root subgroup of A^g for a long root. So the elements of $V_r^{\#}$ are transvections in their action on V. $C_Z(V_r) = Q(J_s \times H_0)$, where $Q = O_2(C_Z(V_r))$, $C_Z(V_r)$ acts irreducibly on the elementary group Q/V_r, and $H_0 \cong 1$ or Z_{q+1}, depending on whether $Z \cong Sp(4, q)$ or $SU(4, q)$. Consider $C_{A^g}(V_r)$. This group has as normal subgroup $O_2(C_X(V_r))I$, where $I \cong Sp(4, q)$, $SU(4, q)$, or $SU(5, q)$. Moreover, we may assume $J_s \leq I$. From the structure of the parabolic subgroups of X (see §3 of [5]) we conclude that $Q \leq O_2(C_X(V_r))$.

Now we claim that Z stabilizes a non-degenerate 4-space of V_1. From the embedding of $J_r \leq A^g$ we see that $J_r \times J_s$ must stabilize a non-degenerate 4-space, V_2, of V. Moreover $V_2 = V_3 \perp V_4$ where V_3 and V_4 are non-degenerate 2-spaces, J_r trivial on V_4, and J_s trivial on V_3. Let $\{v_{31}, v_{32}\}$ be a hyperbolic pair for V_3 chosen so that $[V_r, V_3] = \langle v_{31} \rangle$. Then $O_2(C_X(V_r))$ is trivial on $\langle v_{31} \rangle^{\perp}/\langle v_{31} \rangle$. Apply the 3-subgroup theorem to J_s, Q, and $\langle v_{32} \rangle$. We have

$$[J_s, \langle v_{32} \rangle, Q] = 1 \quad \text{and} \quad [J_s, Q, \langle v_{32} \rangle] = [Q, \langle v_{32} \rangle].$$

Since QJ_s normalizes $[Q, \langle v_{32} \rangle, J_s]\langle v_{31} \rangle$, we conclude that

$$[Q, \langle v_{32} \rangle] \leq [Q, \langle v_{32} \rangle, J_s]\langle v_{31} \rangle \leq V_2.$$

So Q stabilizes V_2 and hence $Z = \langle J_r, J_s, Q \rangle$ stabilizes V_2, proving the claim. From here we see that $C_{A^g}(Z)$ contains $D \cong Sp(2, q)$, $SU(2, q)$, or $SU(3, q)$ as a normal subgroup. In the first two cases $q > 2$, and so $[D, t] = D$. As $D \leq C(Z)$, we see that $t \notin Z^*(C_G(Z)\langle t \rangle)$. This also holds for $\tilde{A} \cong U_7(q)$, if $q > 2$. If $\tilde{A} \cong U_7(2)$ and $t \in Z^*(C_G(Z)\langle t \rangle)$, then

$$D \cong SU(3, 2) \quad \text{and} \quad [D, t] = O_3(D) \leq O(C_G(Z)).$$

Viewing $C_G(Z) \leq C_G(J_r) \cap C_G(J_s)$, we see that this is impossible. This proves (ii), and (iii) follows.

(7.11) *Assume that the hypothesis of* (7.10) *hold and choose notation as in* (6.5) *and* (6.6). *Then*

$$O^2(E_r \cap E_s) = C_G(Y)_A = C_G(J_r \times J_s)_A = C_G(Z)_A.$$

Proof. We have $Y \leq Z$ and $J_r \times J_s \leq Z$. So

$$C_G(Z)_A \leq C_G(J_r \times J_s)_A \quad \text{and} \quad C_G(Z)_A \leq C_G(Y)_A.$$

By (7.10)(ii), P is a standard subgroup of $C_G(Z)$ and $PO(C_G(Z)) \ntrianglelefteq C_G(Z)$. From (6.5) and (6.6), P is standard in $C_G(Y)$, and by direct check we have P

standard in $C_G(J_r \times J_s)$. By (5.2) we conclude that

$$C_G(Z)_A = C_G(J_r \times J_s)_A = C_G(Y)_A$$

unless, possibly, $E(C_A(Z))^{\sim} \cong PSp(n, q)$, $E(C_G(Z))^{\sim} \cong PSU(n, q)$ (respectively $PSL(n, q)$), and one of $E(C_G(Y))$ or $E(C_G(J_r \times J_s))^{\sim}$ is isomorphic to $PSU(n+1, q)$ (respectively $PSL(n+1, q)$). Suppose that this exceptional case occurs. Let $I = Y$ or $J_r \times J_s$, so that $E(C_G(I))^{\sim} \cong PSU(n+1, q)$ (or $PSL(n+1, q)$).

Let $\delta_1 = r^{s_1 s_2}$. Then considering $C(J_{\delta_1}) \geq Z$ we see that

$$(C(J_{\delta_1}) \cap C(Z))_A = (C(J_{\delta_1}) \cap C(Y))_A = (C(J_{\delta_1}) \cap C(J_r J_s))_A.$$

Reading this in the groups $C(Z)_A$, $C(Y)_A$, and $C(J_r J_s)_A$ we see that $n = 2$. But then $PO(C_G(Z)) = J_{\alpha_3} O(C_G(Z)) \trianglelefteq C_G(Z)$, a contradiction.

Finally, $E_r = C_G(J_r)_A$ and $E_s = C_G(J_s)_A$, so $E_r \cap E_s \geq C_G(J_r \times J_s)_A$. Checking the embedding of J_s in E_r we get the equality, completing the proof of (7.11).

(7.12) *Assume that* $\tilde{A} \cong F_4(q)$. *Let* Y, Z *be as in* (6.4). *Choose* X_1 *a* $(q+1)$-*Hall subgroup of* J_s *and* Y_1 *a* $(q+1)$-*Hall subgroup of* J_η, *where* $\eta = \alpha_1 + \alpha_2 + \alpha_3$. *Then:*

(i) $X \times X_1$ *and* $Y \times Y_1$ *are* $(q+1)$-*Hall subgroups of* Z.

(ii) $Q = \langle J_{\alpha_2}, J_{\alpha_3} \rangle$ *is a standard subgroup of* $C_G(Z)$ *with*

$$\langle t \rangle \in Syl_2(C_G(Z) \cap C_G(Q)).$$

(iii) $(C_G(X \times X_1))_A$ *is* Z-*conjugate to* $(C_G(Y \times Y_1))_A$.

(iv)

$$(C_G(Z))_A = (C_G(X \times X_1))_A = (C_G(Y \times Y_1))_A = (C_G(J_r \times J_s))_A = (C_G(J_\gamma \times J_\eta))_A,$$

provided $t \notin Z^*(C_E(Q))$ *or* $t \notin Z^*(C_{E^0}(Q))$, *where* $E^0 = E(C_G(Y))$.

Proof. By order considerations (i) holds. So by Wielandt [18], $X \times X_1$ and $Y \times Y_1$ are conjugate. This proves (i) and (iii). We have (ii) by inspection. We have Z containing each of the groups $X \times X_1$, $Y \times Y_1$, $J_r \times J_s$ and $J_\gamma \times J_\eta$. Therefore (iv) will follow as in the proof of (7.11), once we show that $t \notin Z^*(C_G(Z))$.

Now $Q^g = Z$ for $g = s_1 s_4 s_2 s_3 s_2 s_1 s_3 s_4 \in A$. So it suffices to show that $t \notin Z^*(C_G(Q))$, and each of the conditions in (iv) immediately implies that this is the case. This completes the proof of (7.12).

8. $\tilde{A} \cong E_n(q)$, $D_n(q)$, and ${}^2D_n(q)$

We are now in a position to construct the subgroup G_0. The method for all the groups is essentially the same, although there are certain differences. The hardest cases are when the Dynkin diagram of A has a double bond.

(8.1) *Suppose that* $\tilde{A} \cong E_n(q)$, $n = 6, 7$, *or* 8. *Let* $w_1 \in W$ *be the element* $s_2s_4s_3$, $s_1s_3s_4$, $s_8s_7s_6$, *respectively. Let* $G_0 = \langle E, E^{w_1}\rangle$. *Then* G_0 *is semi-simple,* $|Z(G_0)|$ *is odd, and* $\tilde{G}_0 \cong E_n(q^2)$ *or* $E_n(q) \times E_n(q)$.

Proof. We give the proof for $n = 8$, the other cases being similar. $\tilde{E} \cong E_7(q^2)$ or $E_7(q) \times E_7(q)$ and $E = \langle K_{\alpha_1}, \ldots, K_{\alpha_7}\rangle$ (see Table 2). Then

$$E^{w_1} = \langle K_{\alpha_1}^{w_1}, \ldots, K_{\alpha_7}^{w_1}\rangle = \langle K_{\alpha_1}, \ldots, K_{\alpha_4}, K_{\alpha_5}^{s_6}, K_7, K_8\rangle,$$

by (7.8). So $G_0 = \langle K_{\alpha_1}, \ldots, K_{\alpha_8}\rangle$.

First assume that $\tilde{E} \cong E_7(q^2)$. Here we claim that $\tilde{G}_0 \cong E_8(q^2)$. To do this we must first know the commutator relations existing between K_{α_8} and the groups $K_{\alpha_1}, \ldots, K_{\alpha_7}$. By (7.9), $[K_{\alpha_8}, K\alpha_i] = 1$ for $i = 1, \ldots, 6$. Also

$$\langle K_{\alpha_6}, K_{\alpha_7}\rangle^{w_1} = \langle K_{\alpha_7}, K_{\alpha_8}\rangle \cong SL(3, q^2).$$

So we can label the elements of $\langle K_{\alpha_7}, K_{\alpha_8}\rangle$ by elements of $\mathbf{F}_{q^2}$. However this must be done in such a way that the elements of K_{α_7} have the same labeling in E as in $\langle K_{\alpha_7}, K_{\alpha_8}\rangle$. This can be done by relabeling $\langle K_{\alpha_7}, K_{\alpha_8}\rangle$ using a field automorphism (see §11 of [7]). Once this has been done Theorem 1.4 of Curtis [4] shows that G_0 is a homomorphic image of a certain group G^*, where $\tilde{G}^* \cong E_8(q^2)$ and G^* is generated by groups isomorphic to $K_{\alpha_1}, \ldots, K_{\alpha_8}$, subject to certain relations determined by the groups $\langle K_{\alpha_i}, K_{\alpha_j}\rangle$, $1 \le i, j \le 8$. This proves the claim. Also, note that $|Z(G_0)|$ is odd, because otherwise $C(A)$ would contain a klein subgroup.

Next, suppose $\tilde{E} \cong E_7(q) \times E_7(q)$ and write $E = E_1E_2$ with $E_2 = E_1^t$, E_1 a perfect central extension of $E_7(q)$, and $[E_1, E_2] = 1$. For $i = 1, \ldots, 7$, write $K_{\alpha_i}^1 = K_{\alpha_i} \cap E_1$ and $K_{\alpha_i}^2 = K_{\alpha_i} \cap E_2$. Then $K_{\alpha_i} = K_{\alpha_i}^1 \times K_{\alpha_i}^2$ and $K_{\alpha_i}^2 = (K_{\alpha_i}^1)^t$ for $i = 1, \ldots, 7$. Also for $i = 1, 2$ we have $E_i = \langle K_{\alpha_1}^i, \ldots, K_{\alpha_7}^i\rangle$.

Now $\langle K_{\alpha_6}, K_{\alpha_7}\rangle = \langle K_{\alpha_6}^1, K_{\alpha_7}^1\rangle \times \langle K_{\alpha_6}^2, K_{\alpha_7}^2\rangle \cong SL(3, q) \times SL(3, q)$. Conjugating this by w_1 we get a similar decomposition for $\langle K_{\alpha_7}, K_{\alpha_8}\rangle = \langle K_{\alpha_6}, K_{\alpha_7}\rangle^{w_1} = Y$. Write $Y = Y_1 \times Y_2$ where $K_{\alpha_7}^1 \le Y_1$ and $K_{\alpha_7}^2 \le Y_2$. Then set $K_{\alpha_8}^i = K_{\alpha_8} \cap Y_i$ for $i = 1, 2$. Finally for $i = 1, 2$ write $G_i = \langle K_{\alpha_1}^i, \ldots, K_{\alpha_8}^i\rangle$. We have $G_1^t = G_2$ and arguing as before we have $[G_1, G_2] = 1$, $G_0 = G_1G_2$, $\tilde{G}_1 \cong \tilde{G}_2 \cong E_8(q)$, and $|Z(G_0)|$ odd. This completes the proof of (8.1).

(8.2) *Let* $\tilde{A} \cong O^{\pm}(n, q)'$ *with* $n \ge 14$ *and* n *even. Let*

$$w_1 = s_2s_3s_4s_1s_2s_3.$$

Then $G_0 = \langle E, E^{w_1}\rangle$ *is semi-simple,* $|Z(G_0)|$ *is odd, and* $\tilde{G}_0 \cong O^+(n, q^2)'$ *or* $\tilde{G}_0 \cong \tilde{A} \times \tilde{A}$.

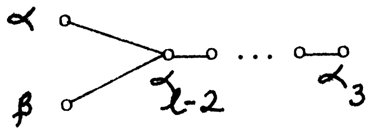
Proof. The argument is similar to that of (8.1). Write

$$A = \langle J_{\alpha_l}, \ldots, J_{\alpha_1}\rangle,$$

so $A \cap E = \langle J_{\alpha_l}, \ldots, J_{\alpha_3}\rangle$. Now

$$\tilde{E} \cong O^+(n-4, q^2)' \quad \text{or} \quad \tilde{E} \cong \tilde{D} \times \tilde{D}.$$

In the wreathed case we write $E=\langle K_{\alpha_l},\ldots,K_{\alpha_3}\rangle$, where $J_{\alpha_i}=C_{K_{\alpha_i}}(t)$ and $K_{\alpha_i}\cong J_{\alpha_i}\times J_{\alpha_i}$. For $\tilde{A}\cong O^+(n,q)'$ or $O^-(n,q)'$, label the Dynkin diagram of E



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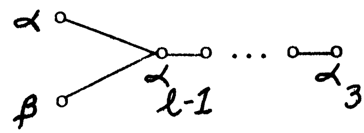


respectively. Then write

$$E=\langle K_\alpha, K_\beta, K_{\alpha_{l-2}},\ldots,K_{\alpha_3}\rangle \quad\text{or}\quad \langle K_\alpha, K_\beta, K_{\alpha_{l-1}},\ldots,K_{\alpha_3}\rangle,$$

respectively. Here,

$$K_\alpha = K_{\alpha_{l-1}} \quad\text{and}\quad K_\beta = K_{\alpha_l} \quad\text{if}\quad \hat{A}\cong O^+(n,q)'$$

and

$$J_{\alpha_l}=C(t)\cap K_\alpha K_\beta \quad\text{if}\quad \tilde{A}\cong O^-(n,q)'.$$

We then have $E^{w_1}=\langle\ldots,K_{\alpha_2},K_{\alpha_1}\rangle$ and

$$G_0=\langle K_\alpha,K_\beta,\ldots,K_{\alpha_3},K_{\alpha_2},K_{\alpha_1}\rangle \quad\text{or}\quad \langle K_{\alpha_l},\ldots,K_{\alpha_2},K_{\alpha_1}\rangle,$$

depending on whether $\tilde{E}\cong O^+(n-4,q^2)'$ or $\tilde{D}\times\tilde{D}$.

From (7.8)(ii) we have

$$K_{\alpha_4}^{s_2s_3s_4}=K_{\alpha_3},\quad K_{\alpha_3}^{s_2s_3s_4}=K_{\alpha_2},\quad K_{\alpha_3}^{s_1s_2s_3}=K_{\alpha_2},\quad\text{and}\quad K_{\alpha_2}^{s_1s_2s_3}=K_{\alpha_1}.$$

Therefore, $\langle K_{\alpha_4},K_{\alpha_3}\rangle^{s_2s_3s_4}=\langle K_{\alpha_3},K_{\alpha_2}\rangle$ and $\langle K_{\alpha_3},K_{\alpha_2}\rangle^{s_1s_2s_3}=\langle K_{\alpha_2},K_{\alpha_1}\rangle$. First, relabel elements in $\langle K_{\alpha_3},K_{\alpha_2}\rangle$ so that elements of K_{α_3} are labeled the same in E and in $\langle K_{\alpha_3},K_{\alpha_2}\rangle$. Once this has been done relabel the elements of $\langle K_{\alpha_2},K_{\alpha_1}\rangle$ so that the labeling of K_{α_2} agrees with that in $\langle K_{\alpha_3},K_{\alpha_2}\rangle$.

We can complete the proof as in (8.1) once we check that certain commutator relations hold. Suppose first that $\tilde{A}\cong O^+(n,q)'$. Then the necessary relations follow from (7.7)(ii) (such as $[K_{\alpha_l},K_{\alpha_1}]=1$). Suppose that $\tilde{A}\cong O^-(n,q)'$.

First assume that $\tilde{E}\cong O^+(n-4,q^2)'$. Then the relations not obtainable from (7.7)(ii) directly are

$$[K_\alpha,K_{\alpha_1}]=[K_\alpha,K_{\alpha_2}]=[K_\beta,K_{\alpha_1}]=[K_\beta,K_{\alpha_2}]=1.$$

Consider the group $Y=\langle K_\alpha,K_\beta,K_{\alpha_{l-1}}\rangle$. Then $\tilde{Y}\cong L_4(q^2)$ and t induces a graph-field automorphism on Y, with $C_Y(t)=\langle J_{\alpha_l},J_{\alpha_{l-1}}\rangle$. It follows that $\langle J_{\alpha_l},K_{\alpha_{l-1}}\rangle=Y$. So we need only show that

$$\langle J_{\alpha_l},K_{\alpha_{l-1}}\rangle\le C(K_{\alpha_1})\cap C(K_{\alpha_2}).$$

However,

$$J_{\alpha_l}\le C(K_{\alpha_1})\cap C(K_{\alpha_2})$$

as $J_{\alpha_l} \leq E_{\alpha_1} \cap E_{\alpha_2}$, and

$$K_{\alpha_{l-1}} \leq C(K_{\alpha_1}) \cap C(K_{\alpha_2})$$

by (7.7)(ii).

If $\tilde{E} \cong \tilde{D} \times \tilde{D}$ the same arguments apply. Here use the facts that $\langle K_{\alpha_l}, K_{\alpha_{l-1}} \rangle = \langle J_{\alpha_l}, K_{\alpha_{l-1}} \rangle \leq C(K_{\alpha_1}) \cap C(K_{\alpha_2})$. This shows that $[K_{\alpha_l}, K_{\alpha_1}] = [K_{\alpha_l}, K_{\alpha_2}] = 1$, the desired relations. The proof of (8.2) is then complete.

To handle the orthogonal groups of lower dimensions we must work a bit harder.

(8.3) *Let* $\tilde{A} \cong O^{\pm}(10, q)'$ *or* $O^{\pm}(12, q)'$ *and set*

$$w_1 = s_2 s_3 s_4 s_1 s_2 s_3.$$

Then $G_0 = \langle E, E^{w_1} \rangle$ *is semi-simple,* $|Z(G_0)|$ *is odd, and* $\tilde{G}_0 \cong \tilde{A} \times \tilde{A}$, $O^+(10, q^2)'$, *or* $O^+(12, q^2)'$.

Proof. Choose notation for E as in (8.2). The difficulty here is that (ii) of Hypothesis (7.5) does not hold. Consequently, we cannot apply (7.7). Let $K_{\alpha_2} = K_{\alpha_3}^{s_2 s_3}$ and $K_{\alpha_1} = K_{\alpha_2}^{s_1 s_2}$.

Let $I \leq \bar{J}_r = J_{\alpha_1} \times J_r$ be cyclic of order $q+1$ and such that I corresponds to the centralizer of a non-degenerate $(n-2)$-subspace of the usual module for $O^{\pm}(n, q)$ ($n = 10$ *or* 12). We may choose $I \leq X$. Then $E(C_A(I)) \cong O^{\mp}(n-2, q)'$. Let $P = E(C_A(I))$. It is easy to check that P is a standard subgroup of $C_G(I)$ and

$$\langle t \rangle \in Syl_2(C_G(I) \cap C_G(P)).$$

Also $E \leq C_G(I)$, so $t \notin Z^*(C_G(I))$. As $I \leq X$, $(C_G(I) \cap C_G(X))_A = E$ so by induction and (5.5), $E(C_G(I)) \cong O^+(n-2, q^2)'$ or $\tilde{P} \times \tilde{P}$. Except for the case $E(C_G(I)) \cong \tilde{P} \times \tilde{P} \cong O^-(n-2, q)' \times O^-(n-2, q)'$ the Dynkin diagram of $E(C_G(I))$ is of type D_k for $k = \frac{1}{2}(n-2)$ (or the union of two such diagrams).

Let $\delta_1 = r^{s_2 s_1 s_3 s_2}$ and note that $\bar{J}_r \sim_A \bar{J}_{\delta_1} = J_{\alpha_3} \times J_{\delta_1}$. Also

$$t \notin Z^*(C(\bar{J}_{\delta_1}) \cap E(C_G(I))\langle t \rangle).$$

Consequently $t \notin Z^*(C_G(\bar{J}_r))$. It follows from (5.2) that $E = E(C_G(X)) = E(C_G(\bar{J}_r))$, so $\bar{J}_r \leq C_G(E)$. Define a subgroup, $L \leq E$, as follows. If $\tilde{A} \cong O^+(n, q)'$, set $L = K_{\alpha_4} \times K_{\alpha_5}$ or $K_{\alpha_5} \times K_{\alpha_6}$, depending on whether $n = 10$ or 12. If $\tilde{A} \cong O^-(n, q)'$, set $L = K_{\alpha_3} \times K_{\alpha_3}^{s_4}$ or $K_{\alpha_4} \times K_{\alpha_4}^{s_5}$, depending on whether $n = 10$ or 12.

From the embedding of $L \leq E \leq E(C_G(I))$ we have the structure of

$$Z = (E(C_G(I)) \cap C_G(L))_A.$$

If $E(C_G(I)) \cong O^+(n-2, q^2)'$, then $\tilde{Z} \cong O^+(4, q^2)'$ or $O^+(6, q^2)'$, depending on whether $n = 10$ or 12. Then $C_Z(t) \cong O^{\mp}(4, q)'$ or $O^{\mp}(6, q)'$, according to $\tilde{A} \cong O^{\pm}(n, q)'$, and depending on whether $n = 10$ or 12. Similarly, we have the structure of Z and $C_Z(t)$ if $E(C_G(I))$ is wreathed. Now, $C_G(L) \geq \langle \bar{J}_r, C_Z(t)\rangle$. Also, $\langle \bar{J}_r, C_Z(t)\rangle = C_A(L \cap A)$ (use the Lie structure or argue as in the proof of (2A) in Wong [19]). There exists $a \in A$ such that $L^a \cap A = \bar{J}_s$ and $L^a \geq K_s$. Then

$$C_G(K_s) \geq C_G(L^a) \geq \langle C_A(L^a \cap A), Z^a\rangle = \langle C_A(\bar{J}_s), Z^a\rangle.$$

So $E(C_A(\bar{J}_s))$ is standard in $C_G(L^a)$ and $t \notin Z^*(C_G(L^a))$. From (5.2) and the fact that $C_G(L^a) \leq C_G(\bar{J}_s)$ we conclude that $K_s \leq L^a \leq C(E_s)$. Once we have this, we can prove (7.7)(ii) and complete the proof as in (8.2).

In dealing with the orthogonal groups $O^{\pm}(8, q)'$, $q \geq 4$, we must introduce a certain subgroup as follows. Let I be a $(q-1)$-Hall subgroup of $\bar{J}_r = J_{\alpha_1} \times J_r$, normalized by s_1, and $I \leq H$. Let $I_1 < I$ be such that

$$|I: I_1| = q-1 \quad \text{and} \quad C_A(I_1) \geq \langle J_{\alpha_l}, \ldots, J_{\alpha_2}\rangle.$$

If $\tilde{A} \cong O^+(8, q)'$ we may take $I_1 = X$, where X is as in (4.1) of [13].

(8.4) *Let* $\tilde{A} \cong O^+(8, q)'$ *with* $q \geq 4$; *set* $F = E(C_G(I_1))$ *and* $F^s = F^{s_1 s_2}$. *Then* $G_0 = \langle F, F^s\rangle$ *is semi-simple,* $|Z(G_0)|$ *is odd, and*

$$\tilde{G}_0 \cong O^+(8, q^2)' \quad \text{or} \quad O^+(8, q)' \times O^+(8, q)'.$$

Proof. We have $O^2(C_A(I_1)) = \langle J_{\alpha_4}, J_{\alpha_3}, J_{\alpha_2}\rangle$ and $t \notin Z^*(C_G(I))$ by (4.7) of [13]. So

$$\tilde{F} \cong O^+(6, q^2)' \quad \text{or} \quad O^+(6, q)' \times O^+(6, q)'.$$

We label $F = \langle K_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4}\rangle$, as usual. So, $J_{\alpha_i} \leq K_{\alpha_i}$ for $i = 2, 3, 4$.

Now $C(I) \cap \langle J_{\alpha_2}, J_{\alpha_3}, J_{\alpha_4}\rangle = J_{\alpha_3} \times J_{\alpha_4}$. It follows that

$$O^{2'}(C_F(I)) = K_{\alpha_3} \times K_{\alpha_4}.$$

As $C_G(I) \leq C_G(I_1)$ we have $K_{\alpha_3} \times K_{\alpha_4} = E(C_G(I))$. In particular, s_1 normalizes $K_{\alpha_3} \times K_{\alpha_4}$, and since s_1 centralizes J_{α_3} and J_{α_4} we have $K_{\alpha_3}^{s_1} = K_{\alpha_3}$ and $K_{\alpha_4}^{s_1} = K_{\alpha_4}$. Let $K_{\alpha_1} = K_{\alpha_2}^{s_1 s_2}$.

Next, we note that there is a subgroup $Z \leq A$ such that $Z(A)Z/Z(A)$ is cyclic of order $q-1$, $E(C_A(Z)) = \langle J_{\alpha_1}, J_{\alpha_2}, J_{\alpha_3}\rangle$, and Z centralizes I_1. To see this, just choose $Z = C_H(\langle J_{\alpha_1}, J_{\alpha_2}, J_{\alpha_3}\rangle)$. Then $C_F(Z) \geq \langle K_{\alpha_2}, K_{\alpha_3}\rangle$, so $t \notin Z^*(C_G(Z))$ and

$$E(C_G(Z))^{\sim} \cong L_4(q^2) \quad \text{or} \quad L_4(q) \times L_4(q)$$

depending on whether $\langle K_{\alpha_2}, K_{\alpha_3}\rangle^{\sim} \cong L_3(q^2)$ or $L_3(q) \times L_3(q)$. In any case we write

$$E(C_G(Z)) = \langle \hat{K}_{\alpha_1}, K_{\alpha_2}, K_{\alpha_3}\rangle$$

where $\hat{K}_{\alpha_1} \geq J_{\alpha_1}$, $[\hat{K}_{\alpha_1}, K_{\alpha_3}] = 1$, and $\langle \hat{K}_{\alpha_1}, K_{\alpha_2}\rangle \cong \langle K_{\alpha_2}, K_{\alpha_3}\rangle$. But then $\hat{K}_{\alpha_1} = K_{\alpha_2}^{s_1 s_2} = K_{\alpha_1}$ and $[K_{\alpha_1}, K_{\alpha_3}] = 1$. Similarly, $[K_{\alpha_1}, K_{\alpha_4}] = 1$. We now have all necessary commutator relations to determine the structure of

$\langle K_{\alpha_1}, K_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4}\rangle = G_{00}$. We conclude that $G_{00} = G_0$ and (8.4) follows.

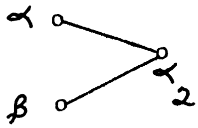
(8.5) *Let $\tilde{A} \cong O^-(8, q)'$, with $q \geq 4$. Choose I and I_1 as in the remarks preceding* (8.4), *and set $F = E(C_G(I_1))$. Then $G_0 = \langle F, F^{s_1 s_2}\rangle$ is semi-simple, $|Z(G_0)|$ is odd, and*

$$\tilde{G}_0 \cong O^+(8, q^2)' \quad or \quad O^-(8, q)' \times O^-(8, q)'.$$

Proof. $F = \langle J_{\alpha_3}, J_{\alpha_2}\rangle$. As in (4.5) of [13] we argue that $t \notin Z^*(C_G(I))$ (use the fact that $t^g \in tV^{\#}_{\alpha_3}$ for some $g \in E$). So

$$\tilde{F} \cong O^+(6, q^2)' \quad \text{or} \quad O^-(6, q)' \times O^-(6, q)'.$$

We write $F = \langle K_\alpha, K_\beta, K_{\alpha_2}\rangle$ or $\langle K_{\alpha_3}, K_{\alpha_2}\rangle$, respectively. Here, labeling corresponds to the Dynkin diagram



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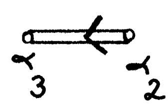


,

respectively, and in the wreathed case we really mean a union of two diagrams.

It follows from the above that $E(C_G(I)) = K_\alpha \times K_\beta$ or K_{α_3}, respectively. Let $g \in N(V_{\alpha_3}) \cap K_\alpha K_\beta$ or $g \in N(V_3) \cap K_{\alpha_3}$, with $t^g = tv$ and $v \in V^{\#}_{\alpha_3}$. Consider $C_G(t^g) \leq N(A^g)$. We have $\bar{J}_r \leq C(t^g)$, so $\bar{J}_r = \bar{J}_r^{(\infty)} \leq N(A^g)^{(\infty)} = A^g$. Also, g centralizes I, so the embedding of I in A^g is the same as that of I in A. Consider A^g acting on the subspaces of the usual module, M, for $O^-(8, q)$. Writing

$$I = (I \cap J_{\alpha_1}) \times (I \cap J_r),$$

we see that M contains 4-spaces, M_1 and M_2, such that $M = M_1 \perp M_2$, $I \cap J_{\alpha_1}$ and $I \cap J_r$ fix all the 1-spaces of M_1 and the preimage of $I \cap J_{\alpha_1}$ and $I \cap J_r$ in $O^-(8, q)$ acts fixed-point-freely on M_2. Now $J_{\alpha_1} \leq C(I \cap J_r)$ and $J_r \leq C(I \cap J_{\alpha_1})$, and these facts imply that J_{α_1} and J_r stabilize M_1 and M_2. So $\bar{J}_r$ stabilizes M_2. Hence $E(C_{A^g}(\bar{J}_r)) = E(C_{A^g}(I)) \cong L_2(q^2)$. As in the proof of (4.5) of [13] this implies that $t \notin Z^*(C_G(\bar{J}_r))$. As $C_G(\bar{J}_r) \leq C_G(I)$, we have $E(C_G(\bar{J}_r)) = K_\alpha \times K_\beta$ or K_{α_3}. Now set $K_{\alpha_1} = K_{\alpha_2}^{s_1 s_2}$ and $K_r = K_{\alpha_2}^{s_3 s_1 s_2}$. Then

$$\langle K_{\alpha_1}, K_r\rangle = K_{\alpha_1} \times K_r \geq J_{\alpha_1} \times J_r.$$

Also, there is an abelian subgroup $\hat{I} > I$ with

$$\hat{I}/I \cong Z_{q+1} \times Z_{q+1} \quad \text{or} \quad Z_{q-1} \times Z_{q-1},$$

depending on whether $F = \langle K_\alpha, K_\beta, K_{\alpha_2}\rangle$ or $\langle K_{\alpha_3}, K_{\alpha_2}\rangle$. Then $K_{\alpha_1} \times K_r = \langle \hat{I}, \bar{J}_r\rangle$. Now $\hat{I}$ normalizes $C_G(I)$, so $K_{\alpha_1} \cap K_r \cap C(E(C_G(I)))$ is a normal subgroup of $K_{\alpha_1} \times K_r$ containing $\bar{J}_r$. We must have

$$K_{\alpha_1} \times K_r \leq C(E(C_G(I))).$$

This says that $[K_{\alpha_1}, K_\alpha]=[K_{\alpha_1}, K_\beta]=1$ or $[K_{\alpha_1}, K_{\alpha_3}]=1$, depending on whether $F=\langle K_\alpha, K_\beta, K_{\alpha_2}\rangle$ or $\langle K_{\alpha_3}, K_{\alpha_2}\rangle$.

Suppose $F=\langle K_\alpha, K_\beta, K_{\alpha_2}\rangle$ and write $s_3=s_\alpha s_\beta$ for $s_\alpha \in K_\alpha$ and $s_\beta \in K_\beta$. Then $K_\alpha = K_{\alpha_2}^{s_\alpha s_2}$, so by the above,

$$\langle K_{\alpha_2}, K_{\alpha_1}\rangle \sim \langle K_{\alpha_2}^{s_\alpha}, K_{\alpha_1}\rangle \sim \langle K_{\alpha_2}^{s_\alpha s_2 s_1}, K_{\alpha_1}^{s_2 s_1}\rangle = \langle K_\alpha^{s_1}, K_{\alpha_2}\rangle = \langle K_\alpha, K_{\alpha_2}\rangle.$$

So in this case we have all necessary commutator relations to conclude that $G_{00}=\langle F, K_{\alpha_1}\rangle$ satisfies $\tilde{G}_{00}\cong O^+(8, q^2)'$. As $A \le G_{00}$ we have $G_{00}=G_0$.

Now suppose that $F=\langle K_{\alpha_3}, K_{\alpha_2}\rangle$. All that is needed here is to show that

$$\langle K_{\alpha_2}, K_{\alpha_1}\rangle^\sim \cong L_3(q)\times L_3(q).$$

Let $L=\langle J_{\alpha_3}, J_{\alpha_2}\rangle \cap C(J_{\alpha_2}\times J_{\alpha_2}^{s_3})$. Then $L/L\cap Z(\langle J_{\alpha_3}, J_{\alpha_2}\rangle)$ is cyclic of order $q+1$. Regarding $\langle J_{\alpha_3}, J_{\alpha_2}\rangle$ as $O^-(6, q)'$ acting on its usual module, L acts trivially on a non-degenerate 4-space of index 2. Since the $(q+1)$-Hall subgroup of $J_{\alpha_3}\cap H$ centralizes $J_{\alpha_2}\times J_{\alpha_2}^{s_3}$, we conclude that $L \le J_{\alpha_3}Z(A)$, so $[L, J_{\alpha_1}]=1$. Now from above we have $E(C_A(L))^\sim \cong O^+(6, q)'$ and so $E(C_A(L))=\langle J_{\alpha_2}, J_{\alpha_1}, J_{\alpha_2}^{s_3}\rangle$.

The group L is conjugate to a subgroup of X, so $t\notin Z^*(C_G(L))$ and, necessarily, $E(C_G(L))\cong L_4(q)\times L_4(q)$. Consequently,

$$E(C_G(L))=\langle \hat{K}_{\alpha_2}, \hat{K}_{\alpha_1}, \hat{K}_{\alpha_2}^{s_3}\rangle$$

where $\hat{K}_{\alpha_2}\ge J_{\alpha_2}$, $\hat{K}_{\alpha_1}\ge J_{\alpha_1}$, each $\hat{K}_{\alpha_i}$ is t-invariant and

$$\hat{K}_{\alpha_1}\cong \hat{K}_{\alpha_2}\cong L_2(q)\times L_2(q).$$

We also have $L\le J_{\alpha_3}\le K_{\alpha_3}\le C(K_{\alpha_1})$, so $K_{\alpha_1}\le E(C_G(L))$ and we must have $K_{\alpha_1}=\hat{K}_{\alpha_1}$. But then,

$$\hat{K}_{\alpha_2}=\hat{K}_{\alpha_1}^{s_2 s_1}=K_{\alpha_2} \quad \text{and} \quad \langle K_{\alpha_1}, K_{\alpha_2}\rangle = \langle \hat{K}_{\alpha_1}, \hat{K}_{\alpha_2}\rangle,$$

showing that $\langle K_{\alpha_2}, K_{\alpha_1}\rangle^\sim \cong L_3(q)\times L_3(q)$. This completes the proof of (8.5).

9. $\tilde{A}\cong PSp(n, q)$ or $PSU(n, q)$

In this section and the next we assume that $\tilde{A}\cong PSp(n, q)$ or $PSU(n, q)$. In the present section we also assume that either $\tilde{E}\cong \tilde{D}\times\tilde{D}$ or that the pair $(\tilde{D}, \tilde{E})$ is of type (7), (12), or (13) in Table 2. This implies that the Dykin diagram for E is the same as that of D (or the union of two such, in the wreathed case). Let $\tilde{A}$ have Lie rank l.

For any root $\alpha\in\Sigma$ with $U_\alpha\le E$ we have associated a root subgroup $\hat{U}_\alpha\le E$ such that $U_\alpha\le \hat{U}_\alpha$ ($\hat{U}_\alpha$ is a direct product in the wreathed case). Moreover $J_\alpha\le \hat{J}_\alpha\le\langle \hat{U}_\alpha, \hat{U}_{-\alpha}\rangle=\hat{K}_\alpha$. If the components of E are not odd-dimensional unitary groups, then $\hat{K}_\alpha=K_\alpha$. In the exceptional cases, $\alpha\sim s$ and $\hat{K}_\alpha\cong SU(3, q)$ or $SU(3, q)\times SU(3, q)$. With this notation, we have $E=\langle \hat{K}_{\alpha_l}, K_{\alpha_{l-1}}, \dots, K_{\alpha_2}\rangle$.

Set $K_{\alpha_1}=K_{\alpha_2}^{s_1s_2}$, $E^0=E^{s_1s_2}$, and $G_0=\langle E, E^0\rangle$. We will show that

$$G_0=\langle \hat{K}_{\alpha_l}, K_{\alpha_{l-1}}, \ldots, K_{\alpha_1}\rangle$$

and that G_0 satisfies the necessary commutator relations.

(9.1) *Suppose that $n\geq 8$. Then G_0 is semi-simple, $|Z(G_0)|$ is odd, and either*

$$\tilde{G}_0\cong \tilde{A}\times\tilde{A}$$

or

$$\tilde{A}\cong PSp(n, q) \quad \text{and} \quad \tilde{G}_0\cong PSp(n, q^2), PSU(n, q), \text{ or } PSU(n+1, q).$$

Proof. By (7.11),

$$C_G(Z)_A=C_G(J_r\times J_s)_A=\langle \hat{K}_{\alpha_l}, K_{\alpha_{l-1}}, \ldots, K_{\alpha_3}\rangle=P.$$

In particular, $s_1\in J_{\alpha_1}\leq C_G(P)$ and it follows that

$$E^0=\langle \hat{K}_{\alpha_k}, \ldots, K_{\alpha_4}, K_{\alpha_3}^{s_2}, K_{\alpha_1}\rangle.$$

Also we have

$$\begin{aligned}[J_{\alpha_3}, K_{\alpha_1}]&=[J_{\alpha_3}, K_{\alpha_2}^{s_1s_2}]\sim[J_{\alpha_3}^{s_2s_1}, K_{\alpha_2}]\sim[J_{\alpha_3}^{s_2s_1s_3s_2}, K_{\alpha_3}]\\&=[J_{\alpha_1}, K_{\alpha_3}]=1.\end{aligned}$$

In particular, $s_3\in C(K_{\alpha_1})$. This implies that

$$\langle K_{\alpha_2}, K_{\alpha_1}\rangle\sim\langle K_{\alpha_2}^{s_3}, K_{\alpha_1}\rangle=\langle K_{\alpha_2}^{s_3}, K_{\alpha_2}^{s_1s_2}\rangle\sim\langle K_{\alpha_2}^{s_3s_2s_1}, K_{\alpha_2}\rangle=\langle K_{\alpha_3}, K_{\alpha_2}\rangle.$$

Finally, we have the relation

$$[K_{\alpha_3}, K_{\alpha_1}]=[K_{\alpha_4}^{s_3s_4}, K_{\alpha_1}]\sim[K_{\alpha_4}, K_{\alpha_1}^{s_4s_3}]=[K_{\alpha_4}, K_{\alpha_1}]=1.$$

With the above relations we argue as in §8 that

$$G_{00}=\langle \hat{K}_{\alpha_l}, K_{\alpha_{l-1}}, \ldots, K_{\alpha_1}\rangle$$

is semi-simple $|Z(G_{00})|$ is odd, and

$$\tilde{G}_{00}\cong\tilde{A}\times\tilde{A},\ PSp(n, q^2),\ PSU(n, q), \text{ or } PSU(n+1, q).$$

Since $A\leq G_{00}$, we have $G_{00}=G_0$, and the proof of (9.1) is complete.

(9.2) *Suppose that $\tilde{A}\cong PSp(6, q)$ or $PSU(6, q)$, with $q\geq 4$, or that $\tilde{A}\cong PSU(7, q)$. Then G_0 is semi-simple, $|Z(G_0)|$ is odd, and either*

$$\tilde{G}_0\cong\tilde{A}\times\tilde{A}$$

or

$$\tilde{A}\cong PSp(6, q) \quad \text{and} \quad \tilde{G}_0\cong PSp(6, q^2), PSU(6, q), \text{ or } PSU(7, q).$$

Proof. The argument is similar to that of (9.1) although we must work more to get some of the commutator relations. As in (9.1) we need only show that $G_{00}=\langle \hat{K}_{\alpha_3}, K_{\alpha_2}, K_{\alpha_1}\rangle$ satisfies the necessary commutator relations.

First we claim that $[\hat{K}_{\alpha_3}, K_{\alpha_1}]=1$. Note that

$$[J_{\alpha_3}, K_{\alpha_1}]=[J_{\alpha_3}, K_{\alpha_2}^{s_1s_2}]\sim[J_{\alpha_3}^{s_2s_1}, K_{\alpha_2}]=[J_r, K_{\alpha_2}]=1.$$

If $\tilde{D}\cong PSp(4, q)$ and $\tilde{E}\cong PSU(4, q)$, then $\hat{K}_{\alpha_3}=J_{\alpha_3}$ so the claim holds. Consider the other cases. Using (7.8)(i) and the above we have $[K_{\alpha_3}, K_{\alpha_1}]=1$. So we may assume $\hat{K}_{\alpha_3}>K_{\alpha_3}$; that is

$$(\hat{K}_{\alpha_3})^{\sim}\cong PSU(3, q) \quad\text{or}\quad PSU(3, q)\times PSU(3, q).$$

By (7.11) $\hat{K}_{\alpha_3}=C_G(Z)_A=E(C_G(Z))$. Let $Y=C_G(\hat{K}_{\alpha_3})$. Then Z is a standard subgroup of Y.

We first show that $t\notin Z^*(Y)$. Suppose otherwise. If $\tilde{E}\cong\tilde{D}\times\tilde{D}$ then $K_r\leq Y$ and $t\notin Z^*(K_r\langle t\rangle)$. So suppose that $\tilde{E}\cong PSU(5, q)$. If $q>4$, let $I=C_E(\hat{K}_{\alpha_3}\circ J_s)$. Then $I/Z(E)$ is cyclic of order $(q+1)/d$ for $d=(5, q+1)$. If $q=4$ and

$$O^2(C(J_r)/C(J_rE))\cong PSU(5, q),$$

set $I=1$. Finally, if $q=4$ and

$$O^2(C(J_r)/C(J_rE))\cong PGU(5, q),$$

then we may choose $I=\langle x\rangle$ where $I\leq C(J_r)$ and I induces an outer diagonal automorphism of E of order 5 and centralizing $\hat{K}_{\alpha_3}\circ J_s$. Since I centralizes $J_r\times J_s$, and since we are assuming that $ZO(Y)\trianglelefteq Y$, we have $[Z, I]\leq O(Y)$.

Also, $\hat{K}_{\alpha_3}$ contains a subgroup I_1, with $[J_{\alpha_3}, I_1]=1$, $I_1\geq Z(\hat{K}_{\alpha_3})$, and $I_1/Z(\hat{K}_{\alpha_3})$ is cyclic of order $(q+1)/e$, where $e=(3, q+1)$. Note that for this case $\tilde{A}\cong PSp(6, q)$, so $q\geq 4$, $q+1>3$, and $I_1\neq Z(\hat{K}_{\alpha_3})$. Now $[II_1, Z]\leq O(Y)$ and II_1 acts on $E(C_G(J_{\alpha_3}))=E^{s_1s_2}$, centralizing $J_r\times J_s$. It follows that II_1 induces a group of inner automorphisms of $E^{s_1s_2}$ of order dividing $q+1$. Consequently, there is a subgroup $I_0\leq II_1$ with $I_0\leq C(E^{s_1s_2})$ and $I_0\not\leq Z(E)$. So $I_0^{s_2s_1}$ centralizes $J_r\times E$.

In particular $I_0^{s_2s_1}\leq C(\hat{K}_{\alpha_3})=Y$. Since $I_0^{s_2s_1}$ also centralizes $J_r\times J_s$ we have $I_0^{s_2s_1}\leq O(Y)$. We want to have $I_0^{s_2s_1}\leq C(Z)$, and to get this it will certainly suffice to show that $[Z, O(Y)]=1$. Let $O=O(Y)$ and let $v\in V_r$ be an involution. Then

$$O=C_0(t)C_0(tv)C_0(v).$$

Now $C_0(t)\leq N(A)\cap C(J_{\alpha_3})\leq N(Z)$, so $[C_0(t), Z]\leq Z\cap O(Y)\leq Z(Z)$ and $C_0(t)\leq C_0(v)$. Also there is an element $g\in\hat{K}_{\alpha_3}^{s_2s_1}$ with $t^g=tv$. $C_0(tv)$ normalizes A^g and, as $q\geq 4$, $C_Z(tv)\times J_{\alpha_3}\leq A^g$. Since $C_0(tv)\leq O(Y)$ we conclude that $C_0(tv)\leq C_0(v)$. We then have $v\in C_G(O(Y))$, so $Z\leq\langle v^Y\rangle\leq C_G(O(Y))$, as needed. In particular, $I_0^{s_2s_1}\leq C_G(Z)$, which implies $C_G(I_0^{s_2s_1})\geq\langle Z, J_{\alpha_2}, J_{\alpha_3}\rangle=A$. So $I_0^{s_2s_1}=I_0$, whereas $I_0^{s_2s_1}\leq C(E)$ and $I_0\not\leq C(E)$. This contradiction shows that $t\notin Z^*(Y)$.

Let $Q=E(Y)$. As $Y\leq C(J_{\alpha_3})\sim C(J_r)$ and since $(C(J_r)\cap Y)_A=K_s$ we apply the theorem of [9] and obtain

$$\tilde{Q} \cong PSU(4, q) \quad \text{or} \quad PSU(4, q) \times PSU(4, q),$$

depending on whether $\tilde{E} \cong PSU(5, q)$ or $PSU(5, q) \times PSU(5, q)$. So we may write $Q = \langle K_\alpha, K_s \rangle$, where

$$K_\alpha \cong SL(2, q^2) \quad \text{or} \quad SL(2, q^2) \times SL(2, q^2)$$

and $K_\alpha \geq J_{\alpha_1}$. Now $K_\alpha \leq C(\hat{K}_{\alpha_3}) \leq C(J_{\alpha_3})$, so $K_\alpha \leq E^{s_1 s_2}$. Since K_α also centralizes $C(J_{\alpha_3}) \cap \hat{K}_{\alpha_3}$ (which is just I_1 if $\tilde{E} \cong PSU(5, q)$) we conclude from the action of $PSU(5, q)$ on its usual module, that $K_\alpha = K_{\alpha_1}$. In particular, we have now proved that $[K_{\alpha_1}, \hat{K}_{\alpha_3}] = 1$.

What remains is the structure of $\langle K_{\alpha_2}, K_{\alpha_1} \rangle$. For this start with $\langle J_{\alpha_2}, J_{\alpha_1} \rangle$ and notice that since $q \geq 4$, $C = C_A(\langle J_{\alpha_2}, J_{\alpha_1} \rangle) \not\leq Z(A)$. So we consider $C_G(C)$. Then $\langle J_{\alpha_2}, J_{\alpha_1} \rangle$ is standard in $C_G(C)$ and

$$\langle t \rangle \in Syl_2(C_G(C) \cap C(\langle J_{\alpha_2}, J_{\alpha_1} \rangle)).$$

Choose $v \in V_{\alpha_1}^{\#}$. Then there is an element $g \in K_{\alpha_1}$ with $t^g = tv$. Then C normalizes A^g and it is not difficult to see that $C_{A^g}(C)$ is not 2-constrained. From here the argument in (4.5) of [13] shows that $t \notin Z^*(C_G(C))$.

Apply the main theorem of [12] and conclude that

$$E(C_G(C)) \cong L_3(q^2) \quad \text{or} \quad L_3(q) \times L_3(q) \quad \text{if} \quad \tilde{A} \cong PSp(6, q)$$

and

$$E(C_G(C)) \cong L_3(q^4) \quad \text{or} \quad L_3(q^2) \times L_3(q^2) \quad \text{if}$$
$$\tilde{A} \cong PSU(6, q) \quad \text{or} \quad PSU(7, q).$$

Now $C \leq H$ and so $C \leq N(J_r) \cap C(J_{\alpha_2})$. Viewing this in $N_G(J_r)$ we conclude that $C \leq C(K_{\alpha_2})$. It follows that

$$E(C_G(C))^\sim \cong \langle J_{\alpha_2}, J_{\alpha_1} \rangle^\sim \times \langle J_{\alpha_2}, J_{\alpha_1} \rangle^\sim \quad \text{if} \quad \tilde{E} \cong \tilde{D} \times \tilde{D}$$

and otherwise $E(C_G(C))^\sim \cong L_3(q^2)$. We know that $K_{\alpha_2} \leq E(C_G(C))$, so we must have $E(C_G(C)) = \langle K_{\alpha_2}, K_{\alpha_1} \rangle$. From here we easily derive the necessary commutator relations. This completes the proof of (9.2).

10. $\tilde{A} \cong PSp(n, q)$ or $PSU(n, q)$ (continued)

We continue the assumption that $\tilde{A} \cong PSp(n, q)$ or $PSU(n, q)$. Here we also assume that the pair $(\tilde{D}, \tilde{E})$ is of type (5), (6), (8), (9), (10), or (11) in Table 2. Set $E^0 = E^{s_1 s_2}$ and $G_0 = \langle E, E^0 \rangle$.

(10.1) *Assume that $\tilde{A} \cong PSp(n, q)$ with $n \geq 8$ and that $\tilde{E} \cong O^-(n, q)'$. Then G_0 is quasisimple, $|Z(G_0)|$ is odd, and $\tilde{G}_0 \cong O^+(n+2, q)'$.*

Proof. Write $E = \langle K_{\alpha_l}, \ldots, K_{\alpha_2} \rangle$, where $l = n/2$ and $J_{\alpha_l} \leq K_{\alpha_l} \cong SL(2, q^2)$

and $J_{\alpha_i}=K_{\alpha_i}$ for $i=2,\ldots,l-1$. We choose the K_{α_i} satisfying the usual commutator relations for $PSO^-(n,q)$. In particular, $\langle K_{\alpha_l}, K_{\alpha_{l-1}}\rangle \cong PSU(4,q)$ and $[K_{\alpha_l}, K_{\alpha_i}]=1$ for $i=2,\ldots,l-2$. We point out that (7.4) fails to hold in this case.

Let $\varepsilon=\alpha_l+2\alpha_{l-1}+\cdots+2\alpha_3+\alpha_2$ and $\gamma=\varepsilon+\alpha_2+\alpha_1$. Then

$$C_E(J_{\alpha_2}\times J_\varepsilon)=\langle K_{\alpha_l},\ldots,K_{\alpha_4}\rangle.$$

So $t\notin Z^*(C_G(J_{\alpha_2}\times J_\varepsilon))$ and hence $t\notin Z^*(C_G(J_{\alpha_1}\times J_\gamma))$ (because $\alpha_2^{s_1s_2}=\alpha_1$ and $\varepsilon^{s_1s_2}=\gamma$). It follows from (5.2) that

$$C_G(J_{\alpha_1}J_\gamma)_A=C_G(Y)_A$$

(Y as in (6.5)). On the other hand, $C_G(Y)_A\sim C_G(XX_1)_A=C_E(X_1)_A$, and from the embedding of D in E we have $C_E(X_1)_A\cong O^+(n-2,q)'$. Consequently, we write

$$C_G(J_{\alpha_1}J_\gamma)_A=L=\langle J_\alpha, J_\beta, J_{l-1},\ldots,J_{\alpha_3}\rangle$$

where $J_\beta=J_\alpha^t\cong SL(2,q)$, $[J_\alpha,J_\beta]=1$, $\langle J_\alpha, J_{l-1}\rangle^{\sim}\cong L_3(q)$, and $[J_\alpha, J_{\alpha_i}]=1$ for $i=3,\ldots,l-2$. Finally $C(t)\cap J_\alpha J_\beta=J_{\alpha_l}$.

It will suffice to show that $[J_\alpha, J_{\alpha_2}]=[J_\beta, J_{\alpha_2}]=1$, for once these relations are checked we have $\langle J_\alpha, J_\beta, J_{\alpha_{l-1}},\ldots,J_{\alpha_1}\rangle=G_{00}$ satisfying the defining relations for $O^+(n+2,q)'$. Since $G_{00}\geq A$ we have $G_{00}=G_0$, completing the proof. There is a subgroup $P\leq J_\alpha\times J_\beta$ such that P is a t-invariant $(q+1)$-Hall subgroup of $J_\alpha\times J_\beta$ and $P_0=C_P(t)=X^{s_1\cdots s_{l-1}}$. Notice that $J_\alpha J_\beta=\langle P, J_{\alpha_l}\rangle$, so it will suffice to show that $P\leq C(J_{\alpha_2})$.

We have $P\leq C_G(P_0)=C_G(X)^w$, where $w=s_1\cdots s_{l-1}$. Also

$$E^w=\langle K_{\alpha_l}^w, J_{\alpha_{l-2}},\ldots,J_{\alpha_1}\rangle$$

and P centralizes $J_{\alpha_1}\times J_\gamma\times\langle J_{\alpha_{l-2}},\ldots,J_{\alpha_3}\rangle=I$. Consider the group $O^-(n,q)'$ acting on its usual module M. There is a homomorphism φ from E^w onto $O^-(n,q)'$. Then $(I)\varphi$ has as its fixed space an anisotropic 2-space of M. From there we can determine $C_{E^w}(I)$. If $l\neq 5$ (that is, $n\neq 10$) then $C_{E^w}(I)$ is cyclic of order $q+1$. If $l=5$, then

$$C_{E^w}(I)\cong Z_{q+1}\times L_2(q) \quad\text{and}\quad C_{E^w}(I)\geq J_{\alpha_3}^{s_4s_5s_4}.$$

For $l\neq 5$ set $I_1=I$ and for $l=5$ set $I_1=I\times J_{\alpha_3}^{s_4s_5s_4}$. Since P centralizes I we must have $P\leq E^wC(E^w)$, and, the projection of P to E^w must centralize I_1. Now $(I_1)\varphi$ defines a unique non-degenerate $(n-2)$-subspace, M_0, of M, on which the stabilizer in $O^-(n,q)'$ induces $O^+(n-2,q)'$. We already know that

$$C_G(P)_A\cong O^+(n-2,q)'$$

and the commutator relations imply that $\langle J_{\alpha_{l-2}}^{s_{l-1}s_ls_{l-1}}, J_{\alpha_{l-2}},\ldots,J_{\alpha_1}\rangle=Q$ satisfies $\tilde{Q}\cong O^+(n-2,q)'$ and $(Q)\varphi$ acts on M_0. It follows that $P\leq C(Q)$. In particular, $P\leq C(J_{\alpha_2})$, as required.

(10.2) *Assume that $\tilde{A} \cong PSp(6, q)$ with $q \geq 4$ and $\tilde{E} \cong O^-(6, q)'$. Then G_0 is quasisimple, $|Z(G_0)|$ is odd, and $\tilde{G}_0 \cong O^+(8, q)'$.*

Proof. Let C be a $(q-1)$-Hall subgroup of J_r. Then

$$O^{2'}(C_A(C)) = \langle J_{\alpha_3}, J_{\alpha_2}\rangle.$$

Also, $C^{s_1s_2} \leq J_{\alpha_3} \leq K_{\alpha_3}$, where $E = \langle K_{\alpha_3}, J_{\alpha_2}\rangle$. So $C_E(C^{s_1s_2})$ involves $O^-(4, q)' \cong L_2(q^2)$ and so $t \notin Z^*(C_G(C))$. Since

$$F = C_G(C) \cap C(X^{s_2s_1})$$

satisfies $\tilde{F} \cong L_2(q^2)$ we must have $C_G(C)\tilde{_A} = O^+(6, q)'$. Write

$$I = C_G(C)_A = \langle J_\alpha, J_\beta, J_{\alpha_2}\rangle$$

where $[J_\alpha, J_\beta] = 1$, $J_\beta = J_\alpha^t$, $\langle J_\alpha, J_{\alpha_2}\rangle^\sim \cong L_3(q)$, and $J_{\alpha_3} = C(t) \cap J_\alpha J_\beta$.

One checks that $C_I(J_\alpha J_\beta)/Z(I)$ is cyclic of order $q-1$ and contained in $J_{\alpha_3}^{s_2} Z(I)$. So, let $C_1 = C(J_\alpha J_\beta) \cap J_{\alpha_3}^{s_2}$, and let P be the t-invariant $(q-1)$-Hall subgroup of $J_\alpha J_\beta$ with $C_P(t) = C^{s_1s_2}$. Then

$$Q = C_G(C^{s_1s_2})_A = \langle J_\alpha^{s_1s_2}, J_\beta^{s_1s_2}, J_{\alpha_1}\rangle \quad \text{and} \quad A \cap Q = \langle J_{\alpha_3}^{s_2}, J_{\alpha_1}\rangle.$$

Now, P normalizes Q, and since P centralizes $C \times C_1$ we conclude that $P \leq QC_G(Q)$ and P projects into a Cartan subgroup of Q normalizing J_{α_1}. It follows that $J_\alpha J_\beta = \langle J_{\alpha_3}, P\rangle \leq N(J_{\alpha_1})$ and hence $J_\alpha J_\beta \leq C(J_{\alpha_1})$.

We now conclude that if $G_{00} = \langle J_\alpha, J_\beta, J_{\alpha_2} J_{\alpha_1}\rangle$, then $A \leq G_{00}$ and $\tilde{G}_{00} \cong O^+(8, q)'$. Then $C_{G_{00}}(X)\tilde{_A} \cong C_G(X)^\sim$, so $E \leq G_{00}$ and we have $G_{00} = G_0$. This completes the proof of (10.2).

Similar methods will be used to handle the case $(\tilde{D}, \tilde{E})$ of type 10).

(10.3) *Assume that $\tilde{A} \cong PSp(n, q)$, $n \geq 8$, and $\tilde{E} \cong O^+(n, q)'$. Then G_0 is quasisimple, $|Z(G_0)|$ is odd, and $\tilde{G}_0 \cong O^-(n+2, q)'$.*

Proof. Write $E = \langle J_\alpha, J_\beta, J_{\alpha_{l-1}}, \ldots, J_{\alpha_2}\rangle$, where $l = n/2$, $J_\beta = J_\alpha^t$, $[J_\alpha, J_\beta] = 1$, $J_{\alpha_l} = C(t) \cap (J_\alpha \times J_\beta)$, $\langle J_\alpha, J_{\alpha_{l-1}}\rangle^\sim \cong L_3(q)$, and $[J_\alpha, J_{\alpha_i}] = 1$ for $i = 2, \ldots, l-2$. Let

$$\varepsilon = \alpha_l + 2\alpha_{l-1} + \cdots + 2\alpha_3 + \alpha_2$$

as in the proof of (10.1). Then

$$C_E(J_{\alpha_2} \times J_\varepsilon) = \langle J_\alpha, J_\beta, J_{\alpha_{l-1}}, \ldots, J_{\alpha_4}\rangle.$$

Consequently, $t \notin Z^*(C_G(J_{\alpha_2} \times J_\varepsilon))$ and so $t \notin Z^*(C_G(J_{\alpha_1} \times J_\gamma))$.

Now $C_G(J_{\alpha_1} J_\gamma) \leq C_G(Y)$, where Y is as in (6.5). As $Y \sim XX_1$, in A, we have

$$C_G(Y)\tilde{_A} \sim C_G(XX_1)\tilde{_A} = C_E(X_1)\tilde{_A} \cong O^-(n-2, q)'.$$

By the above and (5.2), $E(C_G(J_{\alpha_1} J_\gamma)) = E(C_G(Y))$, Set $P = E(C_G(J_{\alpha_1} J_\gamma))$. Then $\tilde{P} \cong O^-(n-2, q)'$ and we write

$$P = \langle \hat{K}_{\alpha_l}, J_{\alpha_{l-1}}, \ldots, J_{\alpha_3}\rangle,$$

where $J_{\alpha_l} \le \hat{K}_{\alpha_l} \cong L_2(q^2)$, $[\hat{K}_{\alpha_l}, J_{\alpha_i}] = 1$ for $i = 3, \ldots, l-2$, and

$$\langle \hat{K}_{\alpha_l}, J_{\alpha_{l-1}} \rangle^{\sim} \cong PSU(4, q) \cong O^-(6, q)'.$$

If we can show that $[\hat{K}_{\alpha_l}, J_{\alpha_2}] = 1$, then $G_{00} = \langle P, J_{\alpha_2}, J_{\alpha_1} \rangle$ will satisfy the defining relations of $O^-(n+2, q)'$. It will then follow that $G_{00} = G_0$, and the proof will be complete. So it suffices to show $[\hat{K}_{\alpha_l}, J_{\alpha_2}] = 1$. Let $I = I^t$ be cyclic of order $q+1$, with

$$I \le N(C(V_{\alpha_l}) \cap \hat{K}_{\alpha_l}) \cap N(C(V_{-\alpha_l}) \cap \hat{K}_{\alpha_l}).$$

Then I normalizes each of the root subgroups of P in the natural root system for P and it follows that I must centralize

$$\langle J^{s_l}_{\alpha_{l-1}}, J_{\alpha_{l-1}}, J_{\alpha_{l-2}}, \ldots, J_{\alpha_3} \rangle = F.$$

So $C_G(I) \ge J_{\alpha_1} \times J_\gamma \times F$.

On the other hand, I is conjugate in $\hat{K}_{\alpha_l}$ to a cyclic subgroup of J_{α_l} of order $q+1$, which in turn, is conjugate to X. So $E(C_G(I))^{\sim} \cong O^+(n, q)'$. As $E(C_G(I)) \cap C(t) \ge J_{\alpha_1} \times J_\gamma \times F$, we have $E(C_G(I)) \le A$. Regard $\tilde{A}$ as $O(n+1, q)'$. Then $\tilde{A}$ acts on a module M of dimension $n+1$ over $\mathbf{F}_q$ and $\tilde{A}$ preserves a quadratic form. Also there is a unique 1-space, M_0, of M with $(M_0, M) = 0$. It is easily checked that $\langle F, J_{\alpha_2} \rangle \cong O^+(n, q)'$ and that $\langle F, J_{\alpha_2} \rangle$ stabilizes a unique complement, M_1, to M_0. Moreover, M_1 is the unique complement to M_0 stabilized by $J_{\alpha_1} \times J_\eta \times F$. It is also easy to see that $E(C_G(I))$ must stabilize a complement to M_0. Consequently $E(C_G(I)) = \langle F, J_{\alpha_2} \rangle$. In particular, $J_{\alpha_2} \le C_G(I)$. So $C(J_{\alpha_2}) \ge \langle J_{\alpha_l}, I \rangle = \hat{K}_{\alpha_l}$ as needed.

(10.4) *Assume that $\tilde{A} \cong PSp(6, q)$ with $q \ge 4$ and $\tilde{E} \cong O^+(6, q)'$. Then G_0 is quasisimple, $|Z(G_0)|$ is odd, and $\tilde{G}_0 \cong O^-(8, q)'$.*

Proof. As in the proof of (10.2), let C be a $(q-1)$-Hall subgroup of J_r. Then $O^{2'}(C_A(C)) = \langle J_{\alpha_3}, J_{\alpha_2} \rangle$. We claim that

$$E(C_G(C))^{\sim} \cong O^-(6, q)' \cong U_4(q) \quad \text{or} \quad U_5(q).$$

(For consider $C^{s_1 s_2} \le J_{\alpha_3}$. From the known structure of $E(C_G(X))$, we have

$$E(C_G(XC^{s_1 s_2}))^{\sim} \cong L_2(q) \times L_2(q) \quad \text{and} \quad t \notin Z^*(C_G(XC^{s_1 s_2}) \langle t \rangle).$$

So $t \notin Z^*(C_G(C))$. Also, since $\langle J_{\alpha_3}, J_{\alpha_2} \rangle$ is standard in $C_G(C)$ and $X^{s_2 s_1} \le J_r^{s_2 s_1} \le C_G(C)$, we use the above and induction to get the claim.) Write $E(C_G(C)) \ge \langle \hat{K}_{\alpha_3}, J_{\alpha_2} \rangle$, where $J_{\alpha_3} \le \hat{K}_{\alpha_3} \cong L_2(q^2)$ and $\langle \hat{K}_{\alpha_3}, J_{\alpha_2} \rangle \cong U_4(q)$.

There is a subgroup $I \le \hat{K}_{\alpha_3}$ such that I is cyclic of order $q+1$, and I is in a Cartan subgroup of $\langle \hat{K}_{\alpha_3}, J_{\alpha_2} \rangle$ normalizing each of the root subgroups in the root system spanned by $\pm\alpha_2$ and $\pm\alpha_3$. Then $C_G(I) \ge J_{\alpha_2} \times J^{s_3}_{\alpha_2} \times C$. Now I is conjugate in K_{α_3} to $X^{s_1 s_2}$, so $E(C_G(I))^{\sim} \cong O^+(6, q)'$. As t centralizes $J_{\alpha_2} \times J^{s_3}_{\alpha_2} \times C$ we must have $t \in C(E(C_G(I)))$. For otherwise, t induces a graph automorphism on $E(C_G(I))$ and $[C, E(C_G(I))] = 1$. But then

$$Sp(4, q) = O^{2'}(C_A(I)) \le O^{2'}(C_A(C)) = \langle J_{\alpha_3}, J_{\alpha_2} \rangle \cong Sp(4, q),$$

whereas $[I, J_{\alpha_3}] \neq 1$. Now argue as in the proof of (10.3) to obtain

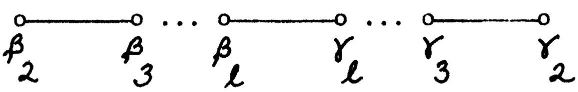
$$E(C_G(I)) = \langle J_{\alpha_2}, J_{\alpha_1}, J_{\alpha_2}^{s_3} \rangle.$$

Therefore $C(J_{\alpha_1}) \geq \langle J_{\alpha_3}, I \rangle \geq \hat{K}_{\alpha_3}$, and so $\langle \hat{K}_{\alpha_3}, J_{\alpha_2}, J_{\alpha_1} \rangle^{\sim} \cong O^{-}(8, q)'$. It follows that $G_0 = \langle \hat{K}_{\alpha_3}, J_{\alpha_2}, J_{\alpha_1} \rangle$, and the proof of (10.4) is complete.

(10.5) *Assume that* $\tilde{A} \cong PSp(n, q)$ *or* $PSU(n+1, q)$ *with* $n \geq 8$ *and that* $\tilde{E} \cong PSL(n-1, q)$ *or* $PSL(n-1, q^2)$, *respectively. Then* G_0 *is quasisimple,* $|Z(G_0)|$ *is odd, and*

$$\tilde{G}_0 \cong PSL(n+1, q) \quad or \quad PSL(n+1, q^2).$$

Proof. Write $E = \langle K_{\beta_2}, \ldots, K_{\beta_l}, K_{\gamma_l}, \ldots, K_{\gamma_2} \rangle$, where each of the generating subgroups is isomorphic to $SL(2, q)$ or $SL(2, q^2)$, depending on whether $\tilde{A} \cong PSp(n, q)$ or $PSU(n+1, q)$. Notation is chosen to correspond with the following labeling of the Dynkin diagram:



Also, for $i = 2, \ldots, l-1$, $J_{\alpha_i} = C(t) \cap K_{\beta_i} K_{\gamma_i}$ and $\hat{J}_{\alpha_l} = C(t) \cap \langle K_{\beta_l}, K_{\gamma_l} \rangle$. Finally, $K_{\gamma_i} = K_{\beta_i}^t$ for $i = 2, \ldots, l$.

Set $K_{\beta_1} = K_{\beta_2}^{s_1 s_2}$, $K_{\gamma_1} = K_{\gamma_2}^{s_1 s_2}$, and $G_{00} = \langle E, K_{\beta_1}, K_{\gamma_1} \rangle$. Then $A \leq G_{00}$, so $G_{00} = G_0$. We will show that $\tilde{G}_{00}$ satisfies the necessary commutator relations. Apply the results of §7. Set $s = r^{s_1}$ and K_s the corresponding subgroup of E (so $K_s \sim K_{\beta_2}$). Then by (7.8), $K_s \leq C_G(E_s)$. Setting $K_r = K_s^{s_1}$ we have $K_r \geq J_r$ and $K_r \leq C_G(E)$. Next, we apply (7.11) to get

$$C_G(Z)_A = \langle K_{\beta_3}, \ldots, K_{\beta_l}, K_{\gamma_l}, \ldots, K_{\gamma_3} \rangle.$$

In particular, $s_1 \in Z$, so s_1 centralizes $C_G(Z)_A$ and

$$E^0 = \langle K_{\beta_1}, K_{\beta_3}^{s_2}, K_{\beta_4}, \ldots, K_{\beta_l}, K_{\gamma_l}, \ldots, K_{\gamma_4}, K_{\gamma_3}^{s_2}, K_{\gamma_1} \rangle.$$

Set $P = \langle K_{\beta_4}, \ldots, K_{\gamma_4} \rangle$.

Then $C_G(P) \geq \langle Z, K_{\beta_2}, K_{\gamma_2} \rangle \geq \langle Z, J_{\alpha_2} \rangle = \langle J_s, J_{\alpha_2}, J_{\alpha_1} \rangle$ and

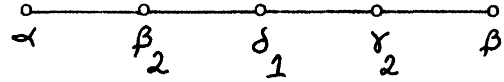
$$\langle Z, J_{\alpha_2} \rangle^{\sim} \cong PSp(6, q) \quad \text{or} \quad PSU(6, q),$$

depending on whether $\tilde{A} \cong PSp(n, q)$ or $PSU(n, q)$. We also know that

$$C_E(P) \geq \langle K_{\beta_2}, J_s, K_{\gamma_2} \rangle = \langle K_{\beta_2}, J_{\delta_1}, K_{\gamma_2} \rangle \quad \text{where} \quad \delta_1 = s^{s_2} = r^{s_1 s_2}.$$

In particular, $t \notin Z^*(C_G(P))$. Since $C_G(P) \cap C(J_r) \geq C_E(P)$ we conclude that $E(C_G(P))^{\sim} \cong PSL(6, q)$ or $PSL(6, q^2)$, depending on whether $\tilde{A} \cong PSp(n, q)$ or $PSU(n+1, q)$.

Choose notation so that $E(C_G(P)) = \langle K_\alpha, K_{\beta_2}, J_{\delta_1}, K_{\gamma_2}, K_\beta \rangle$, corresponding to the labeling



of the Dynkin diagram of $E(C_G(P))$. Here $K_\beta = K_\alpha^t$ and $J_{\alpha_1} = C(t) \cap K_\alpha K_\beta$. Also, notice that $K_{\beta_1} \times K_{\gamma_1} \leq C_G(P)$. As $K_{\beta_2} \leq E \leq C_G(K_r)$, we have $[K_{\beta_2}, K_r] = 1$, and hence $1 = [K_{\beta_2}^{s_1 s_2}, K_r^{s_1 s_2}] = [K_{\beta_1}, K_{\delta_1}]$. Similarly, $[K_{\gamma_1}, K_{\delta_1}] = 1$.

We next note that

$$\langle K_{\beta_1}, K_r \rangle \sim \langle K_{\beta_2}, K_r^{s_2 s_1} \rangle = \langle K_{\beta_2}, K_\delta \rangle,$$

so $\langle K_{\beta_1}, K_r \rangle^\sim \cong L_3(q)$ or $L_3(q^2)$. Similarly, $\langle K_{\gamma_1}, K_r \rangle^\sim \cong L_3(q)$ or $L_3(q^2)$. With these facts we conclude that $\langle K_{\beta_1}, K_r, K_{\gamma_1} \rangle \leq E(C_G(P))$ and is a covering group of $PSL(4, q)$ or $PSL(4, q^2)$. Since $\langle K_{\beta_1}, K_r, K_{\gamma_1} \rangle \leq C(K_{\delta_1})$ we have

$$\langle K_{\beta_1}, K_r, K_{\gamma_1} \rangle = E(C(K_{\delta_1}) \cap E(C_G(P))) = \langle K_\alpha, K_r, K_\beta \rangle.$$

By (5.3) we have $\{K_{\beta_1}, K_{\gamma_1}\} = \{K_\alpha, K_\beta\}$.

Suppose $K_\alpha = K_{\gamma_1}$ and $K_\beta = K_{\beta_1}$. The looking in $E(C_G(P))$ we have $K_{\beta_1}^{s_2} = K_{\gamma_2}^{s_1}$. But $K_{\beta_1}^{s_2} = K_{\beta_2}^{s_1 s_2 s_2} = K_{\beta_2}^{s_1}$. This is impossible. Therefore $K_{\beta_1} = K_\alpha$ and $K_{\gamma_1} = K_\beta$.

Therefore

$$\langle K_{\beta_1}, K_{\beta_2} \rangle^\sim \cong \langle K_{\gamma_2}, K_{\gamma_1} \rangle^\sim \cong PSL(3, q) \quad \text{or} \quad PSL(3, q^2)$$

and

$$[K_{\beta_1}, K_{\gamma_2}] = [K_{\beta_2}, K_{\gamma_1}] = 1.$$

From the structure of E^0 we have $[K_{\beta_1}, K_{\gamma_3}^{s_2}] = 1$. Write $s_3 = xy$, with $x \in K_{\beta_3}$ and $y = x^t \in K_{\gamma_3}$. Then $K_{\beta_1}^{s_2 y s_2} = K_{\beta_1}$ implies $K_{\beta_2}^{s_1 s_2 s_2 y s_2} = K_{\beta_2}^{s_1 s_2}$ and $y \in N(K_{\beta_2}^{s_1})$. Therefore,

$$\begin{aligned}[K_{\beta_1}, K_{\gamma_3}] &= [K_{\beta_2}^{s_1 s_2}, K_{\gamma_2}^{s_3 s_2}] \sim [K_{\beta_2}^{s_1}, K_{\gamma_2}^{s_3}] = [K_{\beta_2}^{s_1}, K_{\gamma_2}^{y}] \\ &\sim [K_{\beta_2}^{s_1}, K_{\gamma_2}] \sim [K_{\beta_1}, K_{\gamma_2}] = 1.\end{aligned}$$

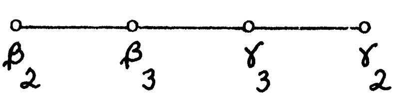
We now have

$$C(K_{\beta_1}) \geq \langle K_{\delta_1}, P, K_{\gamma_3} \rangle = \langle K_{\beta_3}, \ldots, K_{\gamma_3} \rangle.$$

So $[K_{\beta_1}, K_{\beta_3}] = 1$. Similarly, $[K_{\gamma_1}, K_{\beta_3}] = [K_{\gamma_1}, K_{\gamma_3}] = 1$. At this point we have sufficient information to determine the structure of $\tilde{G}_{00}$. This completes the proof of (10.5).

(10.6) *Let $\tilde{A} \cong PSp(6, q)$ with $q \geq 4$ or $PSU(7, q)$. Assume that $\tilde{E} \cong PSL(5, q)$ or $PSL(5, q^2)$, respectively. Then G_0 is quasisimple, $|Z(G_0)|$ is odd, and $\tilde{G}_0 \cong PSL(7, q)$ or $PSL(7, q^2)$ respectively.*

Proof. The argument is similar to that of (10.5). Write $E = \langle K_{\beta_2}, K_{\beta_3}, K_{\gamma_3}, K_{\gamma_2} \rangle$, with notation chosen to correspond to the Dynkin diagram

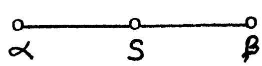


Set $D = \langle K_{\beta_3}, K_{\gamma_3} \rangle$. Then E contains a subgroup I such that $C_{\bar{E}}(\bar{D}) = \bar{K}_s \times \bar{I}$, where bars denote images in $\tilde{E}$ and $\bar{I}$ is cyclic of order $(q-1)/d$ or $(q^2-1)/d$, respectively, where $d=(5, q-1)$ or $(5, q^2-1)$. So $I \not\leq Z(E)$.

Consider $C_G(D)$. We claim that $t \notin Z^*(C_G(D))$ and that $E(C_A(D))=Z$. First note that from the structure of $E\langle t\rangle$ we have $t \sim tv$ with $v \in C_A(Z)$ and $vZ(A)$ a transvection in $\tilde{A}$ (see (19.8) of [1]). From here we see that the proofs in (7.10) and (7.11) go through, showing that $E(C_A(D)) \geq Z$. But also $E(C_A(D)) \leq E(C_A(J_{\alpha_3}))=Z$. This proves the second statement of the claim. We note that $s_1 \in J_{\alpha_1} \leq Z \leq C(\langle K_{\beta_3}, K_{\gamma_3}\rangle)$.

If $\tilde{A} \cong PSU(7, q)$, then $K_s \cong PSL(2, q^2)$ and $t \notin Z^*(C_E(D))$. Consequently, the claim holds in this case. Suppose now that $\tilde{A} \cong PSp(6, q)$ and that $t \in Z^*(C_G(D))$. Let bars denote images in $C_G(D)/O(C_G(D))$. Then $\bar{Z}=\overline{E(C_G(D))}$. Since $I \leq C_G(D)$ and I centralizes $J_r \times J_s$, it follows that $\bar{I}=1$. So $[Z, I] \leq O(C_G(D))$. Let $I_1=O(C_D(J_{\alpha_3}))$. Then $I_1 \leq C(J_{\alpha_3} \times J_s)$ and $I_1Z(D)/Z(D)$ is cyclic of order $(q-1)/e$, where $e=(3, q-1)$. Now apply the argument that occurs in the proof of (9.2) in order to get a contradiction. We use $q-1$ in place of $q+1$, but otherwise the argument is the same.

Continue the assumption that $\tilde{A} \cong PSp(6, q)$. The argument of (9.2) actually shows that $E(C_G(D))^{\sim}$ must contain a non-trivial cyclic subgroup of order dividing $q-1$ and centralizing $J_r \times J_s$. Checking the possibilities for $E(C_G(D))^{\sim}$ we have $E(C_G(D))^{\sim} \cong PSL(4, q)$. If $\tilde{A} \cong PSU(6, q)$, then since $[J_r, K_s]=1$ we must have $E(C_G(D))^{\sim} \cong PSL(4, q^2)$. Choose notation so that $E(C_G(D))=\langle K_\alpha, K_s, K_\beta\rangle$ corresponding to the labeling



of the Dynkin diagram of $E(C_G(D))$. Also, $K_\beta=K_\alpha^t$ and $J_{\alpha_1}=C(t) \cap K_\alpha K_\beta$.

Note that $\langle K_\alpha, K_s, K_\beta\rangle \leq E(C_G(J_{\alpha_3}))=E^{s_1s_2}=\langle K_{\beta_1}, K_{\beta_3}^{s_2}, K_{\gamma_3}^{s_2}, K_{\gamma_1}\rangle$, where $K_{\beta_1}=K_{\beta_2}^{s_1s_2}$ and $K_{\gamma_1}=K_{\gamma_2}^{s_1s_2}$. It is easy to see that in the usual action on the subspaces of a 5-dimensional $\mathbf{F}_q$-space (or $\mathbf{F}_{q^2}$-space) for $\tilde{E}^{s_1s_2}$, $K_\alpha \times K_\beta$ acts on the unique 4-space preserved by J_{α_1}. From here it follows that $\langle K_\alpha, J_s, K_\beta\rangle=\langle K_{\beta_1}, J_s, K_{\gamma_1}\rangle$, so by (5.3), $\{K_\alpha, K_\beta\}=\{K_{\beta_1}, K_{\gamma_1}\}$. We may choose notation so that $K_\alpha=K_{\beta_1}$ and $K_\beta=K_{\gamma_1}$.

In the (B, N)-decomposition for $D=\langle K_{\beta_3}, K_{\gamma_3}\rangle$ let t_3, v_3 be involutions generating the Weyl group of D and chosen so that $v_3=t_3^t$. Here $v_3 \in K_{\beta_3}$ and $t_3 \in K_{\gamma_3}$. We then have

$$\begin{aligned}\langle K_{\beta_1}, K_{\beta_2}\rangle &= \langle K_{\beta_2}^{s_1s_2}, K_{\beta_3}^{s_2v_3}\rangle \sim \langle K_{\beta_2}^{s_1s_2}, K_{\beta_3}^{s_2}\rangle \quad (\text{as } K_{\beta_1} \leq C(D))\\ &\sim \langle K_{\beta_2}^{s_1}, K_{\beta_3}\rangle \sim \langle K_{\beta_2}, K_{\beta_3}\rangle.\end{aligned}$$

Similarly

$$\langle K_{\beta_1}, K_{\gamma_2}\rangle \sim \langle K_{\beta_2}, K_{\gamma_3}\rangle, \langle K_{\gamma_1}, K_{\beta_2}\rangle \sim \langle K_{\gamma_2}, K_{\beta_3}\rangle \quad \text{and} \quad \langle K_{\gamma_1}, K_{\gamma_2}\rangle \sim \langle K_{\gamma_2}, K_{\gamma_3}\rangle.$$

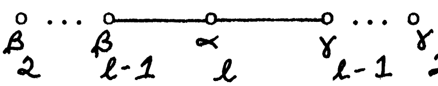
At this point we have the necessary commutator relations to conclude that $G_{00}=\langle K_{\beta_1}, K_{\beta_2}, K_{\beta_3}, K_{\gamma_3}, K_{\gamma_2}, K_{\gamma_1}\rangle$ satisfies $\tilde{G}_{00} \cong PSL(7, q)$ or $PSL(7, q^2)$ and $A \leq G_{00}$. It follows that $G_{00}=G_0$ and (10.6) holds.

(10.7) *Let $\tilde{A} \cong PSp(n, q)$ or $PSU(n, q)$ with $n \geq 8$ and assume that $\tilde{E} \cong PSL(n-2, q)$ or $PSL(n-2, q^2)$, respectively. Then G_0 is quasisimple, $|Z(G_0)|$ is odd, and $\tilde{G}_0 \cong PSL(n, q)$ or $PSL(n, q^2)$, respectively.*

Proof. The argument here is very similar to that of (10.5). The differences are only notational. Write

$$E = \langle K_{\beta_2}, \ldots, K_{\beta_{l-1}}, K_{\alpha_l}, K_{\gamma_{l-1}}, \ldots, K_{\gamma_2} \rangle,$$

where each of the generating subgroups is isomorphic to $SL(2, q)$ or $SL(2, q^2)$, depending on whether $\tilde{A} \cong PSp(n, q)$ or $PSU(n, q)$. Notation corresponds to the following labeling of the Dynkin diagram:



Also, $K_{\gamma_i} = K^t_{\beta_i}$ for $i = 2, \ldots, l-1$, $J_{\alpha_i} = C(t) \cap K_{\beta_i} K_{\gamma_i}$ for $i = 1, \ldots, l-1$, and $J_{\alpha_l} = C(t) \cap K_{\alpha_l}$. Set $P = \langle K_{\beta_4}, \ldots, K_{\alpha_l}, \ldots, K_{\gamma_4} \rangle$ and proceed as in (10.5).

Our final result of §7 is the following.

(10.8) *Let $\tilde{A} \cong PSp(6, q)$ or $PSU(6, q)$, with $q \geq 4$. Assume that $\tilde{E} \cong PSL(4, q)$ or $PSL(4, q^2)$. Then G_0 is quasisimple, $|Z(G_0)|$ is odd, and $\tilde{G}_0 \cong PSL(6, q)$ or $PSL(6, q^2)$.*

Proof. Write

$$E = \langle K_{\beta_2}, K_{\alpha_3}, K_{\gamma_2} \rangle$$

with

$$K^t_{\beta_2} = K_{\gamma_2}, \quad J_{\alpha_2} = C(t) \cap K_{\beta_2} K_{\gamma_2} \quad \text{and} \quad J_{\alpha_3} = C(t) \cap K_{\alpha_3}.$$

Now $J_r = J^{s_2 s_1}_{\alpha_3}$ and by (7.8), $K_r \leq E(C_G(E))$. So

$$[K^{s_1 s_2}_{\beta_2}, K_{\alpha_3}] \sim [K_{\beta_2}, K_r] = 1.$$

Set $K_{\beta_1} = K^{s_1 s_2}_{\beta_2}$ and $K_{\gamma_1} = K^{s_1 s_2}_{\gamma_2}$. Then $[K_{\beta_1}, K_{\alpha_3}] = 1$ and, similarly, $[K_{\gamma_1}, K_{\alpha_3}] = 1$.

The group A contains a subgroup I such that $IZ(A)/Z(A)$ is cyclic of order $q-1$ or $(q+1)/(3, q+1)$ (depending on whether $\tilde{A} \cong PSp(6, q)$ or $PSU(6, q)$) and such that

$$I \leq C(\langle J_{\alpha_1}, J_{\alpha_2} \rangle) \cap H.$$

We claim that $\langle J_{\alpha_1}, J_{\alpha_2} \rangle$ is standard in $C_G(I)$,

$$\langle t \rangle \in Syl_2(C(I) \cap C(\langle J_{\alpha_1}, J_{\alpha_2} \rangle)),$$

and $t \notin Z^*(C_G(I))$. The first two assertions are routine. For the other part first note that from the structure of $E^{s_1 s_2}\langle t \rangle$ it is clear that $t \sim tv$, where $v \in J^{\#}_{\alpha_1}$. Write $tv = t^g$. Then $I \leq C_A(J_{\alpha_1})$, so I normalizes A^g. It follows that

$C_{A^g}(I)$ is not 2-constrained. From here we argue as in (4.5) of [13] to get the conclusion. Now, we will argue as in (9.2).

Apply the main theorem of [14] and conclude that

$$E(C_G(I)) \cong L_3(q^2) \quad \text{or} \quad L_3(q) \times L_3(q) \quad \text{if} \quad \tilde{A} \cong PSp(6, q)$$

and that

$$E(C_G(I)) \cong L_3(q^4) \quad \text{or} \quad L_3(q^2) \times L_3(q^2) \quad \text{if} \quad \tilde{A} \cong PSU(6, q).$$

Now I normalizes J_r and centralizes J_{α_2}. Viewing this in $N_G(J_r) = N_G(K_r)$ we conclude that $K_{\beta_2} \times K_{\gamma_2} \leq E(C_G(I))$. Consequently

$$E(C_G(I)) \cong L_3(q) \times L_3(q) \quad \text{or} \quad L_3(q^2) \times L_3(q^2).$$

Similarly, I normalizes $J_{\alpha_3} = J_r^{s_1 s_2}$, and we look at $E^{s_1 s_2}$ to conclude $K_{\beta_1} \times K_{\gamma_1} \leq E(C_G(I))$. It follows that

$$E(C_G(I)) = \langle K_{\beta_1}, K_{\beta_2} \rangle \circ \langle K_{\gamma_1}, K_{\gamma_2} \rangle \quad \text{or} \quad \langle K_{\beta_1}, K_{\gamma_2} \rangle \circ \langle K_{\gamma_1}, K_{\beta_2} \rangle.$$

If the latter case holds, then $K_{\beta_2}^{s_1 s_2} = K_{\gamma_1}$, whereas $K_{\beta_2}^{s_1 s_2} = K_{\beta_1}$. This is impossible. So the first case must hold, and setting

$$G_{00} = \langle K_{\beta_1}, K_{\beta_2}, K_{\alpha_3}, K_{\gamma_2}, K_{\gamma_1} \rangle$$

we have, as usual, $A \leq G_{00} = G_0$, and the result holds.

11. $\tilde{A} \cong F_4(q)$

In this section we assume that $\tilde{A} \cong F_4(q)$. To get the necessary commutator relations we must consider the groups $E = E(C_G(X))$ and also $E^0 = E(C_G(Y))$ (notation as in §6). Recall, $P = E(C_A(Y))$. Once we show that E and E^0 "pair up" in an acceptable way we set $G_0 = \langle E, E^0 \rangle$ and show that G_0 has the desired properties.

(11.1) *One of the following holds.*
(i) $\tilde{E} \cong \tilde{D} \times \tilde{D} \cong \tilde{E}^0$.
(ii) $\tilde{E} \cong PSp(6, q^2) \cong \tilde{E}^0$.
(iii) $\tilde{E} \cong PSU(6, q)$ *and* $\tilde{E}^0 \cong O^+(8, q)'$.
(iv) $\tilde{E} \cong PSL(6, q)$ *and* $\tilde{E}^0 \cong O^-(8, q)' \cong \tilde{P}$.

Proof. We know the possibilities for the structure of E and E^0, and the respective embedding of D and P. Since

$$(C_G(X \times X_1))_A \quad \text{and} \quad (C_G(Y \times Y_1))_A$$

are Z-conjugate (see (7.12)), we know that the embedding of $\langle J_{\alpha_2}, J_{\alpha_3} \rangle$ is the same in each of $(C_G(X \times X_1))_A$ and $(C_G(Y \times Y_1))_A$. Checking possibilities, we have the result.

(11.2) *Assume that* (11.1)(i) *or* (11.1)(ii) *holds and set* $G_0 = \langle E, E^0 \rangle$.

Then G_0 *is semisimple,* $|Z(G_0)|$ *is odd, and* $\tilde{G}_0 \cong \tilde{A} \times \tilde{A}$ *or* $F_4(q^2)$, *respectively.*

Proof. Write

$$E = \langle K_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4} \rangle,$$

where $J_{\alpha_i} \leq K_{\alpha_i}$, $K_{\alpha_i} \cong SL(2, q) \times SL(2, q)$ if (11.1)(i) holds, and $K_{\alpha_i} \cong SL(2, q^2)$ if (11.1)(ii) holds. Moreover,

$$\langle K_{\alpha_2}, K_{\alpha_3} \rangle^{\sim} \cong PSp(4, q) \times PSp(4, q) \quad \text{or} \quad PSp(4, q^2)$$

and

$$C_E(\langle K_{\alpha_2}, K_{\alpha_3} \rangle) = K_{\alpha_2+2\alpha_3+2\alpha_4}.$$

So $t \notin Z^*(C_E \langle K_{\alpha_2}, K_{\alpha_3} \rangle)$.

By (7.12)(iv) we conclude that

$$\langle K_{\alpha_2}, K_{\alpha_3} \rangle = (C_G(X \times X_1))_A = (C_G(Y \times Y_1))_A = C_{E^0}(Y_1)_A.$$

So we write $E^0 = \langle \bar{K}_{\alpha_1}, \bar{K}_{\alpha_2}, \bar{K}_{\alpha_3} \rangle$ where $J_{\alpha_i} \leq \bar{K}_{\alpha_i}$, $\bar{K}_{\alpha_i} \cong K_{\alpha_j}$ for $i \in \{1, 2, 3\}$ and $j \in \{2, 3, 4\}$. Then

$$\langle \bar{K}_{\alpha_2}, \bar{K}_{\alpha_3} \rangle = C_G(YY_1)_A = C_G(XX_1)_A = \langle K_{\alpha_2}, K_{\alpha_3} \rangle,$$

so by (2.3) we have $\bar{K}_{\alpha_2} = K_{\alpha_2}$ and $\bar{K}_{\alpha_3} = K_{\alpha_3}$. So

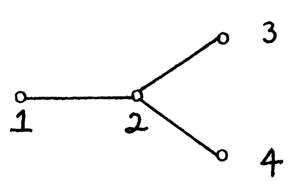
$$G_0 = \langle \bar{K}_{\alpha_1}, K_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4} \rangle.$$

At this point we need only show that $[\bar{K}_{\alpha_1}, K_{\alpha_4}] = 1$. For once we have this commutator relation, the arguments in §8 give the structure of G_0. Now $[\bar{K}_{\alpha_1}, K_{\alpha_4}] = [\bar{K}_{\alpha_1}, K_{\alpha_3}^{s_4 s_3}]$ and s_3 normalizes $\bar{K}_{\alpha_1}$ as $\bar{K}_{\alpha_1}$ and K_{α_3} commute. So it suffices to show that $[\bar{K}_{\alpha_1}, K_{\alpha_3}^{s_4}] = 1$ and for this we need only show that $s_4 \in N(\bar{K}_{\alpha_1})$. However this follows from (7.8)(iii) once we interchange the roles of X and Y. We have now completed the proof of (11.2).

(11.3) *Assume* (11.1)(iii) *holds. Then* $G_0 = \langle E, E^0 \rangle$ *is quasisimple,* $|Z(G_0)|$ *is odd, and* $\tilde{G}_0 \cong {}^2E_6(q)$.

Proof. We write $E = \langle J_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4} \rangle$ where $K_{\alpha_3} \cong K_{\alpha_4} \cong SL(2, q^2)$, $J_{\alpha_3} \leq K_{\alpha_3}$, $J_{\alpha_4} \leq K_{\alpha_4}$, $[J_{\alpha_2}, K_{\alpha_4}] = 1$, $\langle J_{\alpha_2}, K_{\alpha_3} \rangle^{\sim} \cong PSu(4, q)$, and $\langle K_{\alpha_3}, K_{\alpha_4} \rangle^{\sim} \cong PSL(3, q^2)$.

The group E^0 can be expressed $E^0 = \langle J_{\alpha_1}, J_{\alpha_2}, J_{\beta_3}, J_{\beta_4} \rangle$ where $J_{\alpha_1}, J_{\alpha_2}, J_{\beta_3}, J_{\beta_4}$ are conjugate in E^0 and the ordering corresponds to the ordering



of the Dynkin diagram of E^0. Now

$$E^0 = \langle P, (C_G(Y \times Y_1))_A \rangle$$

and $(C_G(Y \times Y_1))_A$ is Z-conjugate to $(C_G(X \times X_1))_A = \langle J_{\alpha_2}, K_{\alpha_3} \rangle$. As

$$A \leq \langle J_{\alpha_1}, J_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4} \rangle,$$

we conclude that $G_0 = \langle J_{\alpha_1}, J_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4} \rangle$.

As in (11.1) it will suffice to show that $[J_{\alpha_1}, K_{\alpha_3}] = [J_{\alpha_1}, K_{\alpha_4}] = 1$. Since $K_{\alpha_3} = K_{\alpha_4}^{s_3 s_4}$ and since s_3 and s_4 centralize J_{α_1}, we need only show that $[J_{\alpha_1}, K_{\alpha_4}] = 1$. Let I be a $(q+1)$-Hall subgroup of K_{α_4}, normalizing each of $V_{\pm\alpha_2}, \hat{V}_{\pm\alpha_3}, \hat{V}_{\pm\alpha_4}$, where $\hat{V}_{\pm\alpha_3}$ is the Sylow 2-subgroup of K_{α_3} containing $V_{\pm\alpha_3}$, and similarly for $\hat{V}_{\pm\alpha_4}$. Then I centralizes each of $J_{\alpha_2}, J_{\alpha_2}^{s_3}, J_{\alpha_2}^{s_3 s_4}$, and J_r. Also, I is inverted by t, so t normalizes $E(C_G(I)) \sim E(C_G(Y))$. Checking centralizers (see §8 and §19 of [1]), we see that t must centralize $E(C_G(I))$, so that $E(C_G(I)) \leq A$. Let $S = E(C_G(I))$. Then $\tilde{S} \cong PSO^+(8, q)'$.

We only need $[I, J_{\alpha_1}] = 1$, since $K_{\alpha_4} = \langle J_{\alpha_4}, I \rangle$. Therefore if $J_{\alpha_1} \leq S$, we are done. Suppose, then, that $J_{\alpha_1} \not\leq S$. As above we have

$$P = J_{\alpha_2} \times J_{\alpha_2}^{s_3} \times J_{\alpha_2}^{s_3 s_4} \times J_r \leq S,$$

and consequently we may write

$$S = \langle J_{\alpha_2}, J_{\alpha_2}^{s_3}, J_{\alpha_2}^{s_3 s_4}, C \rangle, \quad \text{where} \quad \langle J_{\alpha_2}, C \rangle^{\sim} \cong \langle J_{\alpha_2}^{s_3}, C \rangle^{\sim} \cong \langle J_{\alpha_2}^{s_3 s_4}, C \rangle^{\sim} \cong L_3(q).$$

We will first handle the case $q > 4$. We have $H \cap P$ isomorphic to the direct product of four copies of Z_{q-1}. Thus $H = H \cap P$. Also, $H \leq N_S(C)$. From the Theorem in [4] we conclude that C is generated by a pair of opposite root subgroups, $U_\alpha, U_{-\alpha}$, for $\alpha \in \Sigma$. As $U_\alpha \sim U_{\alpha_2}$, α is a long root and an easy check shows that $\alpha = \pm\alpha_1$. Thus $J_{\alpha_1} = C \leq S$, as needed. If $q = 4$, essentially the same argument applies. However, one must go to the proof of the theorem in [4] and check that for $F_4(4)$ all the arguments go through.

Now suppose that $q = 2$. Let $P_0 = O_3(P)$ and let $\bar{A} \cong F_4(4)$ with $A < \bar{A}$, under the natural embedding. So for each root $\alpha \in \Sigma$ there is a unique root subgroup, $\bar{U}_\alpha$, of $\bar{A}$ with $U_\alpha < \bar{U}_\alpha$. For $\alpha \in \Sigma$, let $\bar{J}_\alpha = \langle \bar{U}_\alpha, \bar{U}_{-\alpha} \rangle$. We then have the groups $\bar{P}$ and $\bar{S}$, containing P, S, respectively. With this notation, T is a Cartan subgroup of $\bar{P}$, and hence of $\bar{A}$. Also, $T \leq N(C)$ implies $T \leq N(\bar{C})$. It now follows that $\bar{P}$ is generated by all the long root subgroups in a root system of $\bar{A}$. Consequently,

$$\bar{S} \sim \langle \bar{J}_{\alpha_2}, \bar{J}_{\alpha_1}, \bar{J}_{\alpha_2}^{s_3}, \bar{J}_{\alpha_2}^{s_3 s_4} \rangle \quad \text{in} \quad \bar{A},$$

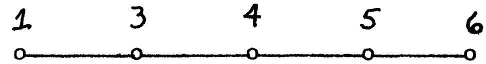
and this conjugation can be performed by an element, g, normalizing each of $\bar{J}_{\alpha_2}, \bar{J}_{\alpha_2}^{s_3}, \bar{J}_{\alpha_2}^{s_3 s_4}, \bar{J}_r$. But then $g \in \bar{P}$ (check normalizers in $F_4(4)$) and so

$$\bar{S} = \langle \bar{J}_{\alpha_2}, \bar{J}_{\alpha_1}, \bar{J}_{\alpha_2}^{s_3}, \bar{J}_{\alpha_2}^{s_3 s_4} \rangle.$$

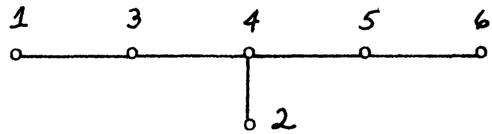
In particular, $\bar{J}_{\alpha_1} \leq \bar{S}$. So $J_{\alpha_1} = \bar{J}_{\alpha_1} \cap A \leq \bar{S} \cap A = S$, completing the proof of (11.3).

(11.4) *Assume* (11.1)(iv) *holds. Let* $G_0 = \langle E, E^0 \rangle$. *Then* G_0 *is quasisimple,* $|Z(G_0)|$ *is odd, and* $\tilde{G}_0 \cong E_6(q)$.

Proof. $\tilde{E} \cong PSL(6, q)$ and we may write $E = \langle K_{\beta_1}, K_{\beta_3}, K_{\beta_4}, K_{\beta_5}, K_{\beta_6} \rangle$ where each $K_{\beta_i} \cong SL(2, q)$ and notation is chosen to correspond to the Dynkin diagram



viewed as a subdiagram of



So $[K_{\beta_1}, K_{\beta_4}] = [K_{\beta_1}, K_{\beta_5}] = [k_{\beta_1}, K_{\beta_6}] = 1$, $\langle K_{\alpha_3}, K_{\alpha_4} \rangle^{\sim} \cong PSL(3, q)$, etc. The group $\langle t \rangle D$ is embedded in $E\langle t \rangle$ in such a way that

$$J_{\alpha_2} = K_{\beta_4}, \quad J_{\alpha_3} = C(t) \cap (K_{\beta_3} \times K_{\beta_5}), \quad J_{\alpha_4} = C(t) \cap (K_{\beta_1} \times K_{\beta_6}),$$
$$K_{\beta_1}^t = K_{\beta_6} \quad \text{and} \quad K_{\beta_3}^t = K_{\beta_5}.$$

Let I be a $(q+1)$-Hall subgroup of J_{α_4} and $\bar{I}$ a $(q+1)$-Hall subgroup of $K_{\beta_1} \times K_{\beta_6}$, containing I, with $\bar{I}$ t-invariant. Then $\bar{I}$ normalizes $C_G(I)_A = E(C_G(I))$ and centralizes $J_r \times K_{\beta_4} = J_r \times J_{\alpha_2}$. Writing $I = Y^w$, for $w = s_4 s_3 s_2 s_3 s_1 s_2 s_3$, we have

$$P = C_G(I)_A = (E^0)^w = \langle J_{\alpha_2}, J_{\alpha_1}, C \rangle,$$

where $\tilde{C} \cong L_2(q^2)$, C is t-invariant, and $C_C(t) = J_{\alpha_4}^{s_3 s_2 s_3}$. Then

$$O^{2'}(C_P(J_{\alpha_2} J_r)) = C.$$

In particular, $C \leq E$. Let I_1 be a $(q+1)$-Hall subgroup of C, chosen such that I_1 is t-invariant and I_1 normalizes each of the root subgroups, $U_{\pm\alpha_2}, U_{\pm\alpha_1}$. Then I_1 must centralize $J_{\alpha_1}, J_{\alpha_2}, J_r$. Viewing this in $C_G(J_r)$ we see that II_1 and $\bar{I}$ are each in E and project to $(q+1)$-Hall subgroups of $C_{\tilde{E}}(\tilde{J}_{\alpha_2})$. In fact, $I_1 \leq C \leq E$. Considering the group $\langle J_{\alpha_4}, I_1 \rangle$, we have $\langle J_{\alpha_4}, I_1 \rangle \leq C_E(\langle J_r, J_{\alpha_1} \rangle)$.

Using the Bruhat decomposition and the fact that $C_A(J_r) = \langle J_{\alpha_2}, J_{\alpha_3}, J_{\alpha_4} \rangle$ one checks that $E(C_A(\langle J_r, J_{\alpha_1} \rangle)) = \langle J_{\alpha_3}, J_{\alpha_4} \rangle$. So

$$C_G(\langle J_r, J_{\alpha_1} \rangle) \geq C_E(\langle J_r, J_{\alpha_1} \rangle) \geq \langle J_{\alpha_3}, J_{\alpha_4}, I_1 \rangle.$$

It follows that

$$t \notin Z^*(C_G(\langle J_r, J_{\alpha_1} \rangle))$$

so by the main theorem in [14], $L = E(C_G(\langle J_r, J_{\alpha_1} \rangle))$ satisfies $L \leq E$ and $\tilde{L} \cong L_3(q^2)$, $L_3(q) \times L_3(q)$, or $q = 2$ and $\tilde{L} \cong J_2$. However, in the last case

$C_E(t)$ contains an involution x acting on $\langle J_{\alpha_3}, J_{\alpha_4}\rangle$ as a graph automorphism. But x cannot act on A. So $\tilde{L} \cong L_3(q^2)$ or $L_3(q)\times L_3(q)$.

Suppose that $\tilde{L}\cong L_3(q^2)$. Then t induces a field automorphism on $\tilde{L}$. Let F be a cyclic subgroup of L inverted by t and such that $FZ(L)/Z(L)$ has order q^3+1. Such a subgroup exists and in E we see that $C_E(F)$ is cyclic of order dividing q^6-1 and $\mathrm{Aut}_E(F)\cong Z_6$. Let $\langle a, t\rangle$ be a klein group in $N_{E\langle t\rangle}(F)$, with $a\in E$. Then a inverts F and it follows from consideration of the usual module for $SL(6, q)$, that a is of type j_3, in the notation of §4 of [1]. Since $C_E(t)^{\sim}\cong PSp(6, q)$ we know that t centralizes a conjugate of F. Therefore, $t\sim ta$. By the results in §7 of [1] we have a being conjugate to an involution in $V^{\#}_{\alpha_2}V^{\#}_{\alpha_4}$, so $t\sim ta_1a_2$, where $a_1\in V^{\#}_{\alpha_2}$ and $a_2\in V^{\#}_{\alpha_4}$. Conjugating by an element in K_{β_1} we have $t\sim ta_1$. Finally, conjugate by an element of $C_E(t)$ to get $t\sim tv$ for $v\in V^{\#}_s$. All of the conjugation above takes place in $E\langle t\rangle$. However by (19.8) of [1] $t\not\sim tv$ in $E\langle t\rangle$. This is a contradiction. Therefore, $\tilde{L}\cong L_3(q)\times L_3(q)$. Let M be the usual module for $SL(6, q)$ and view $SL(6, q)$ as a covering group of $\tilde{E}$. Let $\overline{\langle J_{\alpha_3}, J_{\alpha_4}\rangle}$ be the preimage of $\langle J_{\alpha_3}, J_{\alpha_4}\rangle$ in $SL(6, q)$. Then $\overline{\langle J_{\alpha_3}, J_{\alpha_4}\rangle}$ stabilizes two complementary 3-spaces of M, inducing contragredient representations on the subspaces. Therefore, $\overline{\langle J_{\alpha_3}, J_{\alpha_4}\rangle}$ stabilizes precisely two proper subspaces of M. On the other hand, it is easy to see that the preimage of $\tilde{L}$ in $SL(6, q)$ must also stabilize complementary 3-spaces in M. It follows that $L=\langle K_{\beta_1}, K_{\beta_3}\rangle\langle K_{\beta_5}, K_{\beta_6}\rangle$. In particular $K_{\beta_1}, K_{\beta_3}, K_{\beta_5}, K_{\beta_6}$ all centralize J_{α_1}.

It follows that $\langle E, J_{\alpha_1}\rangle^{\sim}\cong E_6(q)$ and $A\leq\langle E, J_{\alpha_1}\rangle$. From here we get $\langle E, J_{\alpha_1}\rangle = G_0$ and (11.4) holds.

12. $\tilde{A}\cong {}^2E_6(q)$

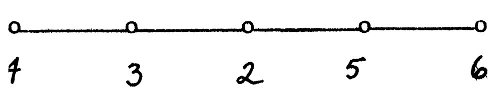
For this section assume that $\tilde{A}\cong {}^2E_6(q)$. Then

$$D=\langle J_{\alpha_2}, J_{\alpha_3}, J_{\alpha_4}\rangle \quad\text{and}\quad \tilde{D}\cong PSU(6, q).$$

Therefore, $\tilde{E}\cong PSU(6, q)\times PSU(6, q)$ or $PSL(6, q^2)$.

(12.1) *Assume $\tilde{E}\cong PSL(6, q^2)$ and let $E^0=E^{s_1s_2}$. Then $G_0=\langle E, E^0\rangle$ is quasisimple, $|Z(G_0)|$ is odd, and $\tilde{G}_0\cong E_6(q^2)$.*

Proof. Assume $\tilde{E}\cong PSL(6, q^2)$ and label the Dynkin diagram of E as follows:



Then write $E=\langle K_{\beta_4}, K_{\beta_3}, K_{\beta_2}, K_{\beta_5}, K_{\beta_6}\rangle$ with each $\tilde{K}_{\beta_i}\cong SL(2, q^2)$ and commutator relations as usual. Here

$$J_{\alpha_2}=C(t)\cap K_{\alpha_2},\quad J_{\alpha_3}=C(t)\cap(K_{\beta_3}\times K_{\beta_5}),\quad\text{and}\quad J_{\alpha_4}=C(t)\cap(K_{\beta_4}\times K_{\beta_6}).$$

Define K_{β_1} by $K_{\beta_1}=K_{\beta_2}^{s_1s_2}$. Then $K_{\beta_1}\geq J_{\alpha_1}$ and by (7.8), $K_{\beta_1}\leq C_G(E_{\alpha_1})$. We next show that $K_{\beta_3}, K_{\beta_4}, K_{\beta_5}$, and K_{β_6} are each in E_{α_1}. Consider Y_3, a (q^2+1)-Hall subgroup of J_{α_3} inverted by s_3. Then Y_3 is contained in a subgroup $\hat{Y}_3$ of $K_{\beta_3}\times K_{\beta_5}$ with $\hat{Y}_3\cong Y_3\times Y_3$ and $\hat{Y}_3$ inverted by s_3. Now $\hat{Y}_3$ normalizes $(C_G(Y_3))_A$. Also $C_E(J_{\alpha_4})\geq K_{\beta_2}$, so $t\notin Z^*(C_G(J_{\alpha_4}))$, and hence $t\notin Z^*(C_G(J_{\alpha_3}))$. By (6.7) $E(C_A(J_{\alpha_3}))=E(C_A(Y_3))$. Since $C_G(J_{\alpha_3})\leq C_G(Y_3)$, (5.2) implies that $C_G(J_{\alpha_3})_A=C_G(Y_3)_A$. Now $\langle J_{\alpha_3}, \hat{Y}_3\rangle = K_{\beta_3}\times K_{\beta_5}$, so

$$K_{\beta_3}\times K_{\beta_5}\leq N(C_G(J_{\alpha_3})_A),$$

and since $J_{\alpha_3}\leq C(C_G(J_{\alpha_3})_A)$ we must have $K_{\beta_3}\times K_{\beta_5}$ centralizing $C_G(J_3)_A$. In particular, $K_{\beta_3}\times K_{\beta_5}$ centralizes J_{α_1}. Similarly, $K_{\beta_4}\times K_{\beta_6}$ centralizes J_{α_1}. So each of $K_{\beta_3}, K_{\beta_4}, K_{\beta_5}$, and K_{β_6} are in $C(J_{\alpha_1})_A=E_{\alpha_1}\leq C(K_{\beta_1})$.

Let $t_3\in K_{\beta_3}$ be defined by $[t_3, t]=s_3$. Then $t_3\in C(K_{\beta_1})$ and so $SL(3, q^2)\cong \langle K_{\beta_3}, K_{\beta_2}\rangle^{\sim s_1s_2t_3}=\langle K_{\beta_2}, K_{\beta_1}\rangle^{\sim}$. At this point we argue as usual to conclude that $\langle E, K_{\beta_1}\rangle=\langle E, E^0\rangle=G_0$ and (12.1) holds.

(12.2) *Assume that* $\tilde{E}\cong PSU(6,q)\times PSU(6,q)$. *Set* $E^0=E^{s_1s_2}$ *and* $G_0=\langle E, E^0\rangle$. *Then* G_0 *is semisimple,* $|Z(G_0)|$ *is odd, and* $\tilde{G}_0\cong\tilde{A}\times\tilde{A}$.

Proof. Write $E=\langle K_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4}\rangle$ with $J_{\alpha_i}\leq K_{\alpha_i}$, $K_{\alpha_i}\cong J_{\alpha_i}\times J_{\alpha_i}$ for $i=1,2,3$. Set $K_{\alpha_1}=K_{\alpha_2}^{s_1s_2}$, so $J_{\alpha_1}\leq K_{\alpha_1}$. The argument in (12.1) shows that $[K_{\alpha_1}, K_{\alpha_3}]=[K_{\alpha_1}, K_{\alpha_4}]=1$. We still need the structure of $\langle K_{\alpha_1}, K_{\alpha_2}\rangle$ in order to complete the proof.

Consider J_γ as in (6.7). Then

$$P=O^2(C_A(J_\gamma))=\langle J_{\alpha_2}, J_{\alpha_1}, J_{\alpha_2}^{s_3}\rangle \quad\text{and}\quad \tilde{P}\cong L_4(q).$$

We argue as in (12.1) that for $i=1,2$ $K_{\alpha_i}\leq C(E_{\alpha_i})$, so $K_{\alpha_1}, K_{\alpha_2}$ are in $C(J_\gamma)$. Also s_3 normalizes J_γ so we have $C(J_\gamma)\geq\langle K_{\alpha_2}, K_{\alpha_1}, K_{\alpha_2}^{s_3}\rangle$. By the main theorem in [14] we conclude that $E(C(J_\gamma))^{\sim}\cong L_4(q)\times L_4(q)$. Then

$$O^{2'}(C(J_\gamma)\cap C(J_{\alpha_2}))\cong L_2(q)\times L_2(q).$$

Since $K_{\alpha_2}^{s_3}\leq C(J_{\alpha_2})$ (by 7.8), we have $K_{\alpha_2}^{s_3}=O^{2'}(C(J_\gamma)\cap C(J_{\alpha_2}))$. Let E_1 and E_2 be the components of E, D_1 and D_2 the components of $C(J_\gamma)$. We may assume that $K_{\alpha_2}^{s_3}\cap E_i=K_{\alpha_2}^{s_3}\cap D_i$, for $i=1,2$. Conjugating by s_3, we have $K_{\alpha_2}\cap E_i=K_{\alpha_2}\cap D_i$, for $i=1,2$. At this point the structure of $\langle K_{\alpha_1}, K_{\alpha_2}, K_{\alpha_3}, K_{\alpha_4}\rangle$ is determined, using the usual arguments. This completes the proof of (12.2).

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