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ISOMETRY GROUPS OF SIMPLY CONNECTED MANIFOLDS OF NONPOSITIVE CURVATURE

BY

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Introduction

Let H be a complete simply connected Riemannian manifold of nonpositive sectional curvature, and let D be a subgroup of $I(H)$, the group of isometries of H. Let $H(\infty)$ denote the set of points at infinity for H (Section 1). In this paper we consider subgroups D that satisfy the *duality condition* and investigate the effects on and relationships between the algebraic structure of D, the structure of the orbits of D in $H(\infty)$ and the geometry of H or H/D if the latter is a smooth manifold. The idea of a *flat point* in $H(\infty)$ (Section 3) plays an important part in this investigation.

In the context of homogeneous or symmetric spaces it is interesting to ask if the duality condition on isometry groups and the idea of flat points at infinity can be related to other properties of such spaces that have been studied. Heintze [18] has shown that if H is a symmetric space and if $D \subseteq I_0(H)$ satisfies the Selberg property (S), then D satisfies the duality condition. It is unknown under what conditions the converse is true. The description of flat points at infinity is trivial if H is symmetric (Section 3) but has not been considered if H is homogeneous but not symmetric. One may hope that the methods of Azencott-Wilson [2], [3] can provide such a description.

A subgroup $D \subseteq I(H)$ satisfies the *duality condition* if for every geodesic γ of H there exists a sequence $\{\phi_n\} \subseteq D$ such that for any point p of H, $\phi_n(p)$ converges to $\gamma(\infty)$ and $\phi_n^{-1}(p)$ converges to $\gamma(-\infty)$ (see Section 1 for definitions). If $M = H/D$ is a smooth manifold, then D satisfies the duality condition if and only if every vector in SM, the unit tangent bundle of M, is nonwandering relative to the geodesic flow. In particular D satisfies the duality condition if H/D is a smooth manifold that is either compact or has finite volume.

The duality condition may appear at first glance to be a fairly mild restriction, but actually it is quite a strong one. For example, if H is a homogeneous space, then the full isometry group $I(H)$ satisfies the duality condition if and only if H is the Riemannian product of a Euclidean space or line H_1 and a

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symmetric space H_2, where either factor may be trivial (Theorem 5.4). In particular if H is a homogeneous space but is not a Riemannian product of this type, then H admits no smooth quotient manifolds M such that every vector in SM is nonwandering relative to the geodesic flow.

A subgroup D that satisfies the duality condition must satisfy certain algebraic properties. One of the most useful (Theorem 2.4) is that any normal abelian subgroup $A \neq 1$ of D must consist of Clifford translations of H. An isometry $\phi \neq 1$ of H is a Clifford translation if the displacement function $p \to d(p, \phi p)$ is constant in H. By a theorem of J. Wolf the Clifford translations of H act as translations on the Euclidean de Rham factor of H. As a corollary to Theorem 2.4 one can show that if D admits a solvable subgroup of finite index and satisfies the duality condition, then H is isometric to a Euclidean space. This result has been proved by S-T. Yau and D. Gromoll-J. Wolf in the case that H/D is a compact smooth manifold.

When considering subgroups $D \subseteq I(H)$ that satisfy the duality condition we can for most practical purposes reduce to the case that H has no Euclidean (flat) factor in its de Rham decomposition. Express H as a Riemannian product $H_1 \times H_2$, where H_1 is the Euclidean factor and H_2 is the product of all non Euclidean factors in the de Rham decomposition of H. Each isometry ϕ of H can be written $\phi = \phi_1 \times \phi_2$, where $\phi_i \in I(H_i)$, $i = 1, 2$. For any group $D \subseteq I(H)$ we define $D_i = \{\phi_i : \phi \in D\}$, $i = 1, 2$. If D satisfies the duality condition in H, then each group D_i satisfies the duality condition in H_i, and we can usually restrict our attention to the group D_2 acting on H_2. If H has no flat de Rham factor and if $I(H)$ satisfies the duality condition, then either $I(H)$ is discrete or $I_0(H)$ is a noncompact semisimple Lie group with trivial center (Proposition 2.5).

For an arbitrary manifold H one can show that if $D \subseteq I(H)$ is any noncompact closed connected semisimple group with finite center, then $D(x) = K(x)$ for every x in the limit set $L(D) \subseteq H(\infty)$, where K is a maximal compact subgroup of D (Theorem 4.5). It follows that $I_0(H)$ always has a compact orbit in $H(\infty)$ if $I(H)$ satisfies the duality condition. Clearly this is true if $I_0(H)$ is trivial or if H has a flat de Rham factor. If H has no flat de Rham factor, then one applies the preceding two results.

If $I(H)$ satisfies the duality condition and acts minimally on $H(\infty)$, then all orbits of $I_0(H)$ in $H(\infty)$ are compact. This follows in the manner of the previous paragraph from Proposition 4.10, which states that if $I(H)$ satisfies the duality condition and acts minimally on $H(\infty)$, then either $I(H)$ is discrete or H is isometric to a Euclidean space or $I_0(H)$ is a noncompact semisimple Lie group with trivial center whose limit set is $H(\infty)$. In the last case we conjecture that H is a rank one symmetric space.

For arbitrary subgroups $D \subseteq I(H)$ the duality condition also restricts the possible D-orbits in $H(\infty)$. For example (Theorem 4.3), if D satisfies the duality condition and has a finite orbit in $H(\infty)$, then H is a Riemannian product $H_1 \times H_2$, where H_1 is either $\mathbf{R}$ or a flat Euclidean space of dimension $k \geq 2$, and each element ϕ of D can be written $\phi = \phi_1 \times \phi_2$, where ϕ_i is an isometry of H_i, $i = 1, 2$. Moreover let $D_i = \{\phi_i : \phi \in D\}$, $i = 1, 2$. Then (a) D_i satisfies the duality condition in H_i, (b) D_1 admits a subgroup D_1^* of finite index consisting of translations of H_1 and H_1/D_1^* is compact. If D has a fixed point in $H(\infty)$, then $D_1 = D_1^*$. (c) D_2 has no finite orbits in $H_2(\infty)$.

Flat points at infinity are important in describing the orbit structure in $H(\infty)$ of a group D that satisfies the duality condition. A geodesic γ of H is said to bound an imbedded *flat half plane* if there exists a totally geodesic isometric imbedding $F : [0, \infty) \times \mathbf{R} \to H$ with $F(0, t) = \gamma(t)$ for all t. An asymptote class $x \in H(\infty)$ is a *flat point* (*at infinity*) if every geodesic γ belonging to x bounds an imbedded flat half plane. Spaces for which all points at infinity are flat include nontrivial Riemannian product manifolds, and noncompact symmetric spaces of rank at least two. If all sectional curvatures at a single point p of H are negative, then no point in $H(\infty)$ is a flat point.

Our main result regarding flat points (Theorem 3.2) says that if $D \subseteq I(H)$ satisfies the duality condition and if $A \subseteq H(\infty)$ is a closed set invariant under D, then any boundary point of A must be a flat point. In particular if $H(\infty)$ has no flat points, then D and hence $I(H)$ acts minimally on $H(\infty)$. Theorem 3.2 and its corollary have various consequences. For example, if $H(\infty)$ has no flat points and if every vector in the unit tangent bundle SM of $M = H/D$ is nonwandering relative to the geodesic flow, then the geodesic flow has a dense orbit in SM (Theorem 5.15). Another consequence is that if $I_0(H)$ satisfies the duality condition and if some point of $H(\infty)$ is not flat, then H is isometric to a rank one symmetric space (Corollary 4.14).

The paper is in six sections. The first contains definitions and preliminary results. The second section discusses the duality condition for subgroups $D \subseteq I(H)$ and relates it to the existence of Euclidean factors in the de Rham decomposition of H. In the third section we consider flat points at infinity and in the fourth the properties of orbits in $H(\infty)$ of isometry groups D satisfying the duality condition. The fifth section contains results on the structure of $I(H)$ under various conditions and on the fundamental groups, geodesic flows and isometry groups of quotient manifolds $M = H/D$ whose deckgroup satisfies the duality condition. The sixth section concludes with some open questions.

1. Preliminaries

We establish some notation and basic facts. For details see [14]. A *Hadamard manifold* is a complete simply connected Riemannian manifold of nonpositive sectional curvature and will be denoted by H. There is a unique geodesic joining any two distinct points of a Hadamard manifold. M will always denote a complete nonsimply connected Riemannian manifold of nonpositive sectional curvature. Both H and M will be connected and C^∞, and all geodesics will have unit speed. SH and SM will denote the unit tangent bundles of H and M, and Ω will denote the subset of SM consisting of those vectors that are nonwandering relative to the geodesic flow in SM.

Geodesics γ and σ of H are *asymptotes* if $d(\gamma t, \sigma t) \leq c$ for some $c > 0$ and all $t \geq 0$. An equivalence class of asymptotes is a *point at infinity* for H, and $H(\infty)$ denotes the set of all points at infinity. The space $\bar{H} = H \cup H(\infty)$ together with the *cone topology* is a compactification of H that is homeomorphic to the closed ball of dimension $n =$ dimension H. For any geodesic γ of H the points $\gamma(\infty)$ and $\gamma(-\infty)$ are the asymptote classes of the geodesics γ and $\gamma^{-1}: t \to \gamma(-t)$ respectively. For any points $p \in H$ and $x \in H(\infty)$ we define γ_{px} to be the unique geodesic with $\gamma(0) = p$ that belongs to x. We define $V(p, x)$ to be the unit vector $\gamma_{px}'(0)$. For any point $p \in H$ and any points q, r in $\bar{H}$ distinct from p we define $\measuredangle_p(q, r)$, the angle subtended at p by q and r, to be the angle subtended at p by the unique geodesics joining p to q and p to r.

Two points $x \neq y$ of $H(\infty)$ can be *joined* (by a geodesic of H) if there exists a geodesic γ of H such that $\gamma(\infty) = x$ and $\gamma(-\infty) = y$. H satisfies the *Visibility axiom* if any two distinct points of $H(\infty)$ can be joined, and H or any of its quotient manifolds is called a *Visibility manifold.* For an equivalent definition of this axiom see Definition 4.2 of [14]. If the sectional curvature is uniformly bounded above by a negative constant for all 2-planes, then H satisfies the Visibility axiom.

$I(H)$ and $I_0(H)$ will denote the isometry group of H and the connected component containing the identity respectively. Both are Lie groups with the compact open topology. $I(M)$ and $I_0(M)$ have analogous meanings. For each element $\phi \in I(H)$ there is an associated displacement function $g_\phi: p \to d(p, \phi p)$. An isometry ϕ is called *elliptic*, *axial* or *parabolic* if g_ϕ has zero minimum, positive minimum or no minimum respectively. Isometries of H extend to homeomorphisms of $\bar{H}$ by defining $\phi(\gamma(\infty)) = (\phi \circ \gamma)(\infty)$ for any isometry ϕ and any point $\gamma(\infty)$ in $H(\infty)$. A subgroup $D \subseteq I(H)$ determines a *limit set* $L(D) \subseteq H(\infty)$ that is closed in $H(\infty)$ and invariant under D. By definition $L(D)$ is the set of points in $H(\infty)$ that are cluster points of an orbit $D(p)$. The definition of $L(D)$ does not depend on the choice of the point $p \in H$. It is straightforward to show that $L(D)$ is nonempty if and only if $\bar{D}$ is noncompact in $I(H)$. If D and D^* are subgroups of $I(H)$ such that D^* is a normal subgroup of D, then $L(D^*)$ is invariant under D.

Every point $x \in H(\infty)$ determines a family $\{f_{px}: H \to \mathbf{R}\}$ of *Busemann functions* at x, one for each point p of H. The difference of any two Busemann functions at the same point $x \in H(\infty)$ is constant in H. Any Busemann function f at a point x is convex [12], C^2 [19], and grad $f(p) = -V(p, x)$ for any point p in H [14, Section 3]. The *horosphere* determined by p and x is the set

$$L(p, x) = \{q \in H : f(q) = f(p)\},$$

and the *interior* $N(p, x)$ is the set $\{q \in H : f(q) < f(p)\}$. The closed convex set $B(p, x)$ is defined to be

$$L(p, x) \cup N(p, x) = \{q \in H : f(q) \leq f(p)\}.$$

If γ is any geodesic belonging to x, then $f(\gamma s) - f(\gamma t) = t - s$ for all numbers s, t.

2. The duality condition and flat de Rham factors

In this section we describe some situations under which a Hadamard manifold H admits a flat factor in its de Rham decomposition. Our starting point is a theorem of J. Wolf [29] which we state as follows:

THEOREM 2.1. *Let H be a Hadamard manifold and let $H = H_1 \times H_2$ be the de Rham decomposition of H into the Euclidean factor H_1 and the product H_2 of irreducible non-Euclidean factors. Let $\phi \neq 1$ be an isometry of H. Then the following are equivalent:*

(1) *ϕ is a Clifford translation.*
(2) *ϕ is a bounded isometry.*
(3) *The action of ϕ is an ordinary translation of the Euclidean factor.*
(4) *The associated vector field X in H is parallel.*

We recall from [29] that ϕ is a *Clifford translation* if $d(p, \phi p)$ is constant for all $p \in H$ and is a *bounded isometry* if $d(p, \phi p) \leq c$ for all $p \in H$ and some $c > 0$. The associated vector field X is determined by the condition $\exp_p(X(p)) = \phi(p)$ for every p. Note that X has constant length if and only if ϕ is a Clifford translation.

If Y is a parallel vector field in H, then Y is a Killing vector field of constant norm, and by Corollary 5.4 of [4] each integral curve of Y is a constant speed geodesic of H. By Proposition 4.2 of [4] each flow transformation ϕ_t is a Clifford translation of H. By Proposition 6.7 of [14] all geodesic integral curves belong to a single asymptote class $x \in H(\infty)$, assuming Y to be normalized, and therefore $Y(p) = V(p, x)$ for all points p in H. Conversely if Y is a vector field in H whose flow transformations $\{\phi_t\}$ are Clifford translations of H, then all integral curves of Y must be constant speed geodesics in H, and by Corollary 5.4 of [4] Y must have constant norm. It follows that Y must be the vector field X associated to the Clifford translation ϕ_1, and by Theorem 2.1, Y must be parallel.

The previous paragraph shows that a vector field Y in H is parallel if and only if the flow transformations of Y are Clifford translations of H. Moreover, if Y is a parallel vector field of norm one, then $Y(p) = V(p, x)$ for some $x \in H(\infty)$ and all points p in H. We extend this observation slightly. Let us say that a point $x \in H(\infty)$ has an *antipodal point* $y \in H(\infty)$ if $\gamma(-\infty) = y$ whenever $\gamma(\infty) = x$ for any geodesic γ of H. Clearly x is an antipodal point of y if y is an antipodal point of x. In Euclidean space every point $x \in H(\infty)$ has an antipodal point. Using the discussion above together with Propositions 5.1 and 6.7 of [14] we obtain the next result.

PROPOSITION 2.2. *Let H be a Hadamard manifold, and let x be any point of $H(\infty)$. Then the following are equivalent:*

(1) *The point x has an antipodal point y.*

(2) *The vector field $p \to V(p, x)$ is a parallel vector field in H.*
(3) *The flow transformations of $p \to V(p, x)$ are Clifford translations of H.*

The following result will be useful.

PROPOSITION 2.3. *Let $D \subseteq I(H)$ be a subgroup with $\bar{D}$ noncompact, and let Z be the centralizer of D in $I(H)$. Then:*

(1) *If $\phi \neq 1$ lies in Z then ϕ fixes every point of $L(D)$. Every point of $L(Z)$ is a fixed point of D.*
(2) *If $L(D) = H(\infty)$ then every element of Z is a Clifford translation of H.*

Proof. (1) Let $1 \neq \phi \in Z$ and $x \in L(D)$ be given; we observed earlier that $L(D)$ is nonempty since $\bar{D}$ is noncompact. If $\{\phi_n\} \subseteq D$ is a sequence so that $\phi_n p \to x$ for any point $p \in H$, then

$$\phi x = \lim_{n \to \infty} \phi\phi_n p = \lim_{n \to \infty} \phi_n(\phi p) = x.$$

It follows similarly that $\psi x = x$ for any $\psi \in D$ and $x \in L(Z)$.

(2) Fix $p \in H$. Let $1 \neq \phi \in Z$ and $q \in H$, $q \neq p$, be given. It suffices to show that $d(p, \phi p) \geq d(q, \phi q)$. Let γ be the geodesic ray starting at p and passing through q. Let $x = \gamma(\infty)$. Since $1 \neq \phi \in Z$ we see by (1) that ϕ fixes x, and therefore the geodesics γ and $\phi \circ \gamma$ are asymptotic. The function $f: t \to d^2(\gamma t, \phi\gamma t)$ is bounded for $t \geq 0$ and convex by Proposition 4.2 of [4]. Hence f is nonincreasing, $d^2(q, \phi q) = f(c)$ where $c = d(p, q)$, and $f(c) \leq f(0) = d^2(p, \phi p)$.

Remark. For any Hadamard manifold H the subgroup C of Clifford translations is normal in $I(H)$, and as a consequence $L(C)$ is invariant under $I(H)$ by the discussion of Section 1.

DEFINITION OF THE DUALITY CONDITION. Points x, y in $H(\infty)$ are *dual* relative to a subgroup D of $I(H)$ if there exists a sequence $\{\phi_n\} \subseteq D$ such that $\phi_n p \to x$ and $\phi_n^{-1} p \to y$ for every $p \in H$. A subgroup D of $I(H)$ will be said to satisfy the *duality condition* if $\gamma(\infty)$ and $\gamma(-\infty)$ are dual relative to D for any geodesic γ of H.

We remark that distinct points x, y that are dual relative to a group D cannot always be joined by a geodesic of H. Note also that $L(D) = H(\infty)$ if D satisfies the duality condition, but the converse is not true as we shall see. The duality condition arises from the study of geodesic flows and acts as an extension of and substitute for compactness. If D acts freely and properly discontinuously on H, then D satisfies the duality condition if and only if every vector in the unit tangent bundle of $M = H/D$ is nonwandering relative to the geodesic flow. See Proposition 4.9 of [12]. In particular, D satisfies the duality condition if H/D is compact or has finite volume. Because of these special cases we regard the duality condition as a natural hypothesis and shall require it in most of our results.

If H is a symmetric space and if $D \subseteq I_0(H)$ satisfies the Selberg condition (S) [5], then D satisfies the duality condition as Heintze proves in [18]. It is not clear under what conditions the converse is true.

The main result of this section is the following:

THEOREM 2.4. *Let $G \subseteq I(H)$ be a subgroup with nontrivial center A such that the normalizer of G in $I(H)$ satisfies the duality condition. Then A consists of Clifford translations.*

Let D be the normalizer of A in $I(H)$. Then D contains the normalizer of G and hence also satisfies the duality condition. It suffices therefore to consider the case that $G = A$, an abelian subgroup.

We shall need several lemmas.

LEMMA 2.4a. *Let $x \in H(\infty)$ be arbitrary and let $y \in H(\infty)$ be a point that can be joined to x. If $z \in H(\infty)$ is any point that can be joined to x, then $z \in \overline{D(y)}$.*

Proof. Let γ and σ be geodesics of H with $\gamma(\infty) = y$, $\sigma(\infty) = z$ and $\gamma(-\infty) = \sigma(-\infty) = x$. There exists a sequence $\{\phi_n\} \subseteq D$ such that $\phi_n p \to z$ and $\phi_n^{-1} p \to x$ for any point p in H since D satisfies the duality condition. If we choose p to lie on γ, then

$$\measuredangle_p(\phi_n p, \phi_n y) = \measuredangle_{\phi_n^{-1}p}(p, y) \leq \measuredangle_p(\phi_n^{-1}p, x) \to 0$$

since

$$\measuredangle_p(\phi_n^{-1}p, y) + \measuredangle_{\phi_n^{-1}p}(p, y) \leq \pi.$$

It follows that $\phi_n y \to z$.

LEMMA 2.4b. *The subgroup A contains no elliptic elements except the identity.*

Proof. Let E be the set of elliptic elements in A. We show that E has a fixed point $q \in H$. Assuming this has been established, it follows that E fixes every point of $D(q)$ and hence every point of the segments $\gamma_{q,gq}$, $g \in D$. It follows that E fixes every point of H since $L(D) = H(\infty)$, and therefore E contains only the identity element of D.

To show that E has a fixed point $q \in H$, let $\phi \in E$ be arbitrary and let C_ϕ be the set of fixed points of ϕ in H. Now C_ϕ is a closed totally geodesic submanifold of H [22], and C_ϕ is invariant under the commuting set E. Let H^* be a closed totally geodesic submanifold of H of smallest dimension that is invariant under E. If $\psi \in E$ is arbitrary, then $C_\psi \cap H^*$ is nonempty by Lemma 1 of [15] and is clearly a closed, totally geodesic submanifold of H that is invariant under E. Hence H^* is a point since it is contained in each C_ψ.

LEMMA 2.4c. *For each point $x \in L(A)$ there exists a unique point $y \in L(A)$ such that x can be joined to y.*

Proof. We prove uniqueness first. Suppose that $x \in L(A)$ can be joined to $y \in L(A)$, and let $\{\phi_n\} \subseteq A$ be any sequence such that $\phi_n p \to x$ for all $p \in H$. Let γ be a geodesic of H joining x to y, and let p be a point of γ. Recalling that A fixes every point of $L(A)$ by Proposition 2.3 we observe that

$$\measuredangle_p(\phi_n^{-1}p, y) = \measuredangle_p(\phi_n^{-1}p, \phi_n^{-1}y) = \measuredangle_{\phi_n p}(p, y) \leq \measuredangle_p(\phi_n p, x) \to 0$$

by the argument of Lemma 2.4a. Therefore y is unique.

To show the existence of y let x and $\{\phi_n\}$ be as above and let $\phi_n^{-1}p \to y \in L(A)$ by passing to a subsequence if necessary. We show that x can be joined to y. Let z be any point of $H(\infty)$ that can be joined to y. Then as above,

$$\measuredangle_p(\phi_n p, \phi_n z) = \measuredangle_{\phi_n^{-1}p}(p, z) \to 0,$$

which implies that $\phi_n z \to x$. Now $L(A)$ is left invariant by D, the normalizer of A in $I(H)$, and it follows from Lemma 2.4a that z can be joined only to points in $\overline{D(y)} \subseteq L(A)$. Therefore $\phi_n z$ can be joined only to points in $L(A)$ for each n. Fix a point $p \in H$ and let $y_n^* = \gamma_{p,\phi_n z}(-\infty)$. Then $y_n^* \in L(A)$ and $y_n^* \to y^* = \gamma_{px}(-\infty)$. Therefore x can be joined to $y^* \in L(A)$ and by the uniqueness assertion $y^* = y$.

LEMMA 2.4d. *For every point $x \in L(A)$ the C^1 vector field $p \to V(p, x)$ is parallel in H and the flow transformations $\phi_t: p \to \gamma_{px}(t)$ are Clifford translations.*

Proof. The vector field $p \to V(p, x)$ is C^1 since $V(p, x) = -\operatorname{grad} f(p)$ for any Busemann function f at x [14]. Busemann functions are C^2 by [19]. Now let $x \in L(A)$ be given and let y be the unique point of $L(A)$ that can be joined to x. By Proposition 2.2 it suffices to show that y is an antipodal point of x. If $z \in H(\infty)$ is any point that can be joined to y, then $z \in \overline{D(x)} \subseteq L(A)$ by Lemma 2.4a. Therefore $z = x$ by Lemma 2.4c and x, y are antipodal points.

Proof of Theorem 2.4. By Lemma 2.4d, $I(H)$ contains Clifford translations, and therefore we may write H as a product $H_1 \times H_2$, where H_1 is a flat Euclidean space and H_2 has no flat factor. The factor H_2 may be trivial and the proof then continues as in the next paragraph. Assume now that H_2 is not trivial. It is not difficult to show that the flat subspaces $H_1 \times \{q\}$, $q \in H_2$, are permuted by the elements of $I(H)$. Therefore each element ϕ of D can be written $\phi = \phi_1 \times \phi_2$, where $\phi_i = \pi_i(\phi)$ is an isometry of H_i. Now $A_2 = \pi_2(A)$ is normal in $D_2 = \pi_2(D)$, and it is easy to show that D_2 satisfies the duality condition in H_2 since D satisfies it in H. If A_2 is nontrivial then the argument above will show that H_2 admits a flat de Rham factor. Hence $A_2 = \{1\}$ and $A \subseteq I(H_1) \times \{1\} \subseteq I(H)$ operates only on the flat factor H_1.

No isometry of a Euclidean space is parabolic. The fact that D satisfies the duality condition in H also implies that $D_1 = \pi_1(D)$ satisfies the duality condition in H_1. It follows by Lemma 2.4b that every isometry of A is axial. Let $\phi \neq 1 \in A$ be given, and let γ be a geodesic of H such that $(\phi \circ \gamma)(t) = \gamma(t + c)$

for some $c > 0$ and all t. Let $x = \gamma(\infty) \in L(A)$. Then $\phi(p) = \phi_{x,c}(p)$ for all points $p \in \gamma$, where $\phi_{x,c}$ is the Clifford translation $q \to \gamma_{qx}(c)$. The theorem will be proved when we show that $\phi = \phi_{x,c}$.

Let $A^* \subseteq I(H)$ be the group generated by A and the Clifford translations $\{\phi_{y,t}: y \in L(A), t \in \mathbf{R}\}$. If $\psi \in A$ then

$$(\psi \circ \phi_{y,t})(p) = (\psi \circ \gamma_{py})(t) = \gamma_{\psi p,y}(t) = (\phi_{y,t} \circ \psi)(p)$$

since A fixes each point of $L(A)$. Therefore each $\phi_{y,t}$ centralizes A. Clifford translations commute with each other since they act as translations of the flat de Rham factor (Theorem 2.1), and therefore A^* is an abelian group.

It suffices to show that D^*, the normalizer of A^* in $I(H)$, contains D, the normalizer of A in $I(H)$. The group D^* will then satisfy the duality condition and since $\psi = \phi_{x,c} \circ \phi^{-1}$ is an elliptic element of A^* it will follow by Lemma 2.4b that ψ is the identity. Now let $\zeta \neq 1$ be an arbitrary element of D. If $\psi \in A$ then $\zeta\psi\zeta^{-1} \in A \subseteq A^*$. If ψ is a Clifford translation that translates a geodesic γ with endpoints in $L(A)$, then $\zeta\psi\zeta^{-1}$ is a Clifford translation that translates $\zeta \circ \gamma$, whose endpoints lie in $L(A)$. Conjugation by ζ carries the generators of A^* into A^* and therefore ζ normalizes A^*. This completes the proof of Theorem 2.4.

Let H be an arbitrary Hadamard manifold and write H as a Riemannian product $H_1 \times H_2$, where H_1 is a flat Euclidean space and H_2 has no flat de Rham factor. The group $I(H)$ is the product of the subgroups $I(H_1) \times \{1\}$ and $\{1\} \times I(H_2)$, and if $I(H)$ satisfies the duality condition in H, then $I(H_2)$ satisfies the duality condition in H_2. Therefore if $I(H)$ satisfies the duality condition, then the structure of the group $I(H)$ reduces to the case where H has no flat factor.

PROPOSITION 2.5. *Let H be a Hadamard manifold such that $I(H)$ satisfies the duality condition. The following are equivalent:*

(1) *H admits no flat de Rham factor.*
(2) *Either $I(H)$ is discrete or $I_0(H)$ is a noncompact semisimple Lie group with trivial center.*

Proof. Clearly (2) implies (1) by the discussion above. Suppose now that H has no flat de Rham factor and that $I(H)$ is not discrete. By the first part of the proof of Lemma 2.4b, $I_0(H)$ has no fixed point in H and is therefore noncompact. We assert that $I_0(H)$ is semisimple. If $I_0(H)$ were not semisimple, then there would exist an abelian subgroup A invariant under all continuous automorphisms of $I_0(H)$ [6]. In particular, A would be normal in $I(H)$ and by Theorem 2.4 would consist of Clifford translations, contradicting the assumption that H has no flat de Rham factor. For the same reason the center of $I_0(H)$ is trivial. Therefore (1) implies (2).

3. Flat points at infinity

Let M be a complete manifold of nonpositive sectional curvature, and let $\gamma: \mathbf{R} \to M$ be a unit speed geodesic. We say that γ *bounds* an *imbedded* (*immersed*) *flat half plane* if there exists an isometric totally geodesic imbedding (immersion) $F: \mathbf{R} \times [0, \infty) \to M$ with $F(t, 0) = \gamma(t)$ for all $t \in \mathbf{R}$. If the domain of F is $\mathbf{R} \times [0, c]$ for some $c > 0$, then we say that γ bounds an imbedded (immersed) *flat strip*.

If H is an arbitrary Hadamard manifold we define a point $x \in H(\infty)$ to be a *flat point* (*at infinity*) if every geodesic γ of H that belongs to x bounds an imbedded flat half plane.

Examples. We present some examples and nonexamples of flat points at infinity.

(1) Euclidean space $\mathbf{R}^n$. Clearly every point at infinity is a flat point.

(2) Product manifolds. Let $H = H_1 \times H_2$ be a nontrivial Riemannian product of two Hadamard manifolds. Then every $x \in H(\infty)$ is a flat point.

(3) Let H be a symmetric space of noncompact type and rank $k \geq 2$. Then every point $x \in H(\infty)$ is a flat point.

(4) Let H be a symmetric space of noncompact type and rank 1, or more generally let H be a Visibility manifold. Then no point in $H(\infty)$ is a flat point.

To show that every point $x \in H(\infty)$ is a flat point in Example 2, it suffices to show that every geodesic γ of H bounds a flat strip of arbitrarily large width. If $\gamma(t) = (\gamma_1(t), q)$, where γ_1 is a unit speed geodesic of H_1, then let $\gamma_n(t) = (\gamma_1(t), q_n)$, where q_n is a divergent sequence of points in H_2. Then γ_n and γ bound a flat strip of width $d(q, q_n)$. Similarly if $\gamma(t) = (p, \gamma_2(t))$. If $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, where both γ_1 and γ_2 are geodesics with positive speed, then let $\gamma_n(t) = (\gamma_1(t + n), \gamma_2 t)$. If n is fixed then $t \to d(\gamma_n t, \gamma)$ is a bounded convex function on $\mathbf{R}$ and hence is constant. Therefore γ_n and γ bound a flat strip and the width as a function of n is unbounded.

In Example 3, write H as a coset space G/K and let γ be an arbitrary unit speed geodesic of H. Choose $g \in G$ so that $g(\gamma(0)) = p$, the base point of G/K, and let $v = (g \circ \gamma)'(0)$. Then both v and the geodesic $g \circ \gamma$ are tangent to a flat totally geodesic submanifold H_0 of dimension $k \geq 2$ by Theorem 6.2 (ii) of [20, p. 210]. Therefore γ is tangent to $g^{-1}(H_0)$ and it follows that every point of $H(\infty)$ is flat.

In Example 4, the Euclidean geometry that holds in any flat half plane is incompatible with the Visibility axiom as defined in [14, p. 61].

We continue with some elementary observations.

PROPOSITION 3.1. *The set of flat points in $H(\infty)$ is closed in $H(\infty)$ and invariant under $I(H)$. If $I(H)$ satisfies the duality condition and if γ is any geodesic of H, then either both endpoints $\{\gamma(\infty), \gamma(-\infty)\}$ are flat points at infinity or neither endpoint is a flat point at infinity.*

Proof. The first assertion is clear. Suppose now that $I(H)$ satisfies the duality condition, and let γ be an arbitrary geodesic of H. Assume further that one endpoint, say $\gamma(\infty) = x$, is a flat point. Let y denote $\gamma(-\infty)$ and let $p \in H$ be arbitrary. If $z = \gamma_{py}(-\infty)$, then by Lemma 2.4a the point z lies in $\overline{I(H)(x)}$ and hence is a flat point. Therefore the maximal geodesic $\gamma_{pz} = \gamma_{py}$ bounds an imbedded flat half plane, which shows that y is a flat point at infinity.

The main result of this section is:

THEOREM 3.2. *Let $D \subseteq I(H)$ be a subgroup that satisfies the duality condition, and let A be a closed subset of $H(\infty)$ that is invariant under D. If A has nonempty boundary, ∂A, then every point of ∂A is a flat point at infinity.*

We need some preliminary results.

LEMMA 3.2a. *Let x, y be distinct points of $H(\infty)$ and p any point of H. Let f be a Busemann function at x, and let $C = B(p, x) \cap B(p, y)$. If f has a local minimum in C at a point q, then q lies on a geodesic joining x to y.*

Remark. f always has a maximum on C at p so the lemma does not hold for local maxima.

Proof. Suppose that $\measuredangle_q(x, y) < \pi$ and choose a geodesic γ starting at q so that $\gamma'(0)$ makes an angle less than $\pi/2$ with both $V(q, x)$ and $V(q, y)$. Let g be any Busemann function at y. It follows that $(f \circ \gamma)(t)$ and $(g \circ \gamma)(t)$ are strictly decreasing on $[0, \varepsilon]$ for some $\varepsilon > 0$ since grad $f(q) = -V(q, x)$ and (grad g) $(q) = -V(q, y)$. For any $t \in (0, \varepsilon]$ we have $f(\gamma t) < f(q) \leq f(p)$ and $g(\gamma t) < g(q) \leq g(p)$. Hence $\gamma(t) \in C$ and $f(\gamma t) < f(q)$, a contradiction. Therefore $\measuredangle_q(x, y) = \pi$ and q lies on a geodesic joining x to y.

LEMMA 3.2b. *Let γ be a maximal geodesic of H that does not bound an imbedded flat half plane, and let p be a point of γ. If $x = \gamma(\infty)$ and $y = \gamma(-\infty)$, then we can find neighborhoods $U \subseteq H(\infty)$ of x and $V \subseteq H(\infty)$ of y and a number $R > 0$ such that for any points $x^* \in U$ and $y^* \in V$ there exists a geodesic γ^* joining x^* to y^* with $d(p, \gamma^*) \leq R$.*

Remark. By modifying the proof of this lemma we can also prove that (a) the lemma remains true if U, V are neighborhoods in the larger space $\bar{H} = H \cup H(\infty)$, (b) $x \in H(\infty)$ is a flat point if and only if $B(p, x) \cap B(p, y)$ is noncompact for all $p \in H$ and all $y \neq x \in H(\infty)$.

Proof. Suppose that the assertion is false. Then we can find sequences $\{x_n\} \subseteq H(\infty)$ and $\{y_n\} \subseteq H(\infty)$ such that $x_n \to x$, $y_n \to y$ and no geodesic γ_n with $d(p, \gamma_n) \leq n$ joins x_n to y_n. By the preceding lemma the set $B(p, x_n) \cap B(p, y_n)$ is not contained in the closed (compact) ball of radius n and center p. Hence we may choose $q_n \in B(p, x_n) \cap B(p, y_n)$ with $d(p, q_n) > n$. The geodesic segment

γ_{pq_n} is contained in the convex set $B(p, x_n) \cap B(p, y_n)$ since the endpoints are in this set. If $q_n \to z \in H(\infty)$, passing to a subsequence, then $\gamma_{pz}[0, \infty) \subseteq B(p, x) \cap B(p, y)$ by continuity. It follows from the fact that γ does not bound an imbedded flat half plane and from Proposition 5.1 of [14] that for some $t_0 > 0$ there exists no geodesic through $\gamma_{pz}(t_0)$ that joins x to y. Let f be any Busemann function at x. We will obtain a contradiction from Lemma 3.2a when we show that $f \equiv f(p)$ in $B(p, x) \cap B(p, y)$. If $q \neq p$ is any point of $B(p, x) \cap B(p, y)$, then the convex function f is nonincreasing on the geodesic segment $\sigma = \gamma_{pq}$ and hence

$$0 \geq (f \circ \sigma)'(0) = \langle \sigma'(0), \operatorname{grad} f(p) \rangle = -\langle \sigma'(0), V(p, x) \rangle.$$

It follows that $\measuredangle_p(q, x) \leq \pi/2$. Similarly $\measuredangle_p(q, y) \leq \pi/2$ since any Busemann function g at y is also nonincreasing on σ. Therefore $\measuredangle_p(q, x) = \measuredangle_p(q, y) = \pi/2$ since these angles sum to π. Consequently we have $0 = (f \circ \sigma)'(0)$, which implies that f is constant on σ since $(f \circ \sigma)''(t) \geq 0$ and $f \circ \sigma$ is nonincreasing.

Proof of Theorem 3.2. We show that an arbitrary point $x \in \partial A$ is a flat point. Let γ be an arbitrary geodesic that belongs to x. If γ does not bound an imbedded flat half plane, then by Lemma 3.2b there exists a neighborhood $U \subseteq H(\infty)$ of x such that y can be joined to every point $x^* \in U$. By Lemma 2.4a, $U \subseteq \overline{D(x)} \subseteq A$. However, U must contain points of $H(\infty) - A$ since $x \in \partial A$, a contradiction. Therefore γ bounds an imbedded flat half plane, and x is a flat point at infinity.

4. Orbit structure

In this section we discuss the orbits in $H(\infty)$ of certain subgroups D of $I(H)$, in particular subgroups that are semisimple or satisfy the duality condition. We obtain characterizations of flat Euclidean spaces and rank one symmetric spaces in terms of the action of $I(H)$ on $H(\infty)$.

Our first result is a restatement of Theorem 3.2.

PROPOSITION 4.1. *Let $D \subseteq I(H)$ be a subgroup satisfying the duality condition, and let $A \subseteq H(\infty)$ be a closed subset invariant under D. If $x \in A$ is not a flat point at infinity, then x is an interior point of A.*

The next two results describe the simplest case where an orbit is a point or a finite set of points. We remark that if $H = H_1 \times H_2$ is a Riemannian product manifold, then each geodesic γ of H_1 beginning at a point p_1 determines a geodesic of H: $t \to (\gamma t, p_2)$ beginning at (p_1, p_2) for any point p_2 of H_2. A similar remark applies to the geodesics of H_2. We then obtain natural inclusions of $H_1(\infty)$ and $H_2(\infty)$ into $H(\infty)$.

THEOREM 4.2. *Let $D \subseteq I(H)$ satisfy the duality condition. Then the following are equivalent:*

(1) *D has a centralizer $Z \neq 1$ in $I(H)$.*
(2) *D has a fixed point in $H(\infty)$.*
(3) *H is a Riemannian product $H_1 \times H_2$ such that H_1 is either $\mathbf{R}$ or flat Euclidean space of dimension $k \geq 2$, and each element ϕ of D can be written $\phi = \phi_1 \times \phi_2$, where ϕ_1 is a translation of H_1 and ϕ_2 is an isometry of H_2. The quotient space H_1/D_1 is compact, where $D_1 = \{\phi_1 : \phi \in D\}$.*

Under any of the conditions above the set of points fixed in $H(\infty)$ by D is $H_1(\infty)$, and the centralizer Z of D consists of the translations of H_1. Equivalently $D_2 = \{\phi_2 : \phi \in D\}$ has trivial centralizer in the isometry group of H_2 and has no fixed points in $H_2(\infty)$.

Proof. We prove the equivalence of the three statements before establishing the final assertion of the theorem.

Consider (1) ⇒ (2). Let ϕ be a nonidentity element in Z, the centralizer of D. It follows from Proposition 2.3 that ϕ is a Clifford translation since $L(D) = H(\infty)$. Now ϕ translates the geodesics γ_p joining p to ϕp for every $p \in H$, and these geodesics γ_p all belong to a single asymptote class $x \in H(\infty)$ by Proposition 6.7 of [14]. If $\psi \in D$ is arbitrary then $\psi(\gamma_p) = \gamma_{\psi_p}$ since ψ commutes with ϕ, and it follows that $\psi x = x$ since ψ permutes the oriented axes of ϕ.

The assertion (3) ⇒ (1) is obvious so it remains only to prove that (2) ⇒ (3). Let $x \in H(\infty)$ be a fixed point of D, and let $y \in H(\infty)$ be any point that can be joined to x. By Lemma 2.4a, y can be joined only to points in $\overline{D(x)} = \{x\}$, which means that x and y are antipodal points. By Proposition 2.2 the vector field $p \to V(p, x)$ is parallel in H, and its flow transformations are Clifford translations of H.

We define a distribution N in H by

$$N(p) = \text{span}\,\{V(p, x) : D \text{ fixes } x\}$$

for any point p in H. From the previous paragraph it is clear that the distribution N is spanned by the parallel vector fields $p \to V(p, x_i)$, $1 \leq i \leq k$, for suitable fixed points $x_1, \ldots, x_k$ of D, and therefore N is involutive and invariant under the holonomy group at each point. If $N^\perp$ denotes the orthogonal distribution, then $N^\perp$ is also involutive and invariant under the action of the holonomy group at each point. If H_1 and H_2 are maximal integral manifolds of N and $N^\perp$ respectively, then H is isometric to the Riemannian product $H_1 \times H_2$ by the de Rham decomposition theorem. See [23].

If V is any parallel vector field in a Riemannian manifold, then the sectional curvature of any 2-plane containing $V(p)$ is zero. It follows that either H_1 has dimension one or H_1 is isometric to $\mathbf{R}^k$, $k \geq 2$, with the usual flat metric. Because $\phi_* N(p) = N(\phi p)$ for all points p and all $\phi \in D$ it follows that each isometry $\phi \in D$ can be written as $\phi = \phi_1 \times \phi_2$, where ϕ_i is an isometry of H_i, $i = 1, 2$. If γ is a geodesic of H that starts at $p = (p_1, p_2)$ and belongs to an asymptote class $x \in H(\infty)$ fixed by D, then by the definition of N, γ is tangent to the Euclidean factor H_1 through each of its points and may be written as $\gamma(t) = (\gamma_1(t), p_2)$, where γ_1 is a geodesic of H. It follows that ϕ_1 fixes $\gamma_1(\infty) \in H_1(\infty)$. Moreover, ϕ_1 must be a translation of H_1 and H_1/D_1 must be compact since the foliation N is spanned by the vector fields $p \to V(p, x)$ where D fixes x. This completes the proof of $(2) \Rightarrow (3)$.

We establish the final assertion of the theorem. From (3) it is clear that $H_1(\infty)$ equals the fixed point set of D and that the translations of H_1 are contained in the centralizer Z of D. Conversely let ψ be a nonidentity element of Z. Now Z leaves the fixed point set of D invariant, and hence $\psi_* N(p) = N(\psi p)$ for all points p in H. Therefore ψ can be written $\psi = \psi_1 \times \psi_2$, where ψ_i is an isometry of H_i, $i = 1, 2$. If $D_i = \{\phi_i : \phi \in D\}$ and $Z_i = \{\psi_i : \psi \in Z\}$, $i = 1, 2$, then Z_i centralizes D_i. Now D_2 satisfies the duality condition in H_2 since D satisfies the duality condition in H. If Z_2 were not the identity, then by the implication $(1) \Rightarrow (2)$, D_2 and hence D would have a fixed point in $H_2(\infty)$, which is disjoint from $H_1(\infty)$. Therefore Z_2 is the identity. The group $Z_1 = Z$ must consist of translations since D_1 is a group of translations and H_1/D_1 is compact. This completes the proof of the theorem.

THEOREM 4.3. *Let $D \subseteq I(H)$ satisfy the duality condition. Then the following are equivalent:*

(1) *D has a finite orbit in $H(\infty)$.*

(2) *H is a Riemannian product $H_1 \times H_2$ such that H_1 is either $\mathbf{R}$ or a flat Euclidean space of dimension $k \geq 2$, and each element ϕ of D can be written as $\phi = \phi_1 \times \phi_2$, where ϕ_i is an isometry of H_i, $i = 1, 2$. Moreover,*

$$H_1(\infty) = \{x \in H(\infty) : D(x) \textit{ is a finite set}\},$$

and $D_2 = \{\phi_2 : \phi \in D\}$ has no finite orbits in $H_2(\infty)$. If

$$D^* = \{\phi \in D : \phi \textit{ fixes every point of } H_1(\infty)\},$$

then D^* is a normal subgroup of finite index in D, $D_1^* = \{\phi_1 : \phi \in D^*\}$ is a group of translations of H_1 and the quotient space H_1/D_1^* is compact.

Proof. Clearly $(2) \Rightarrow (1)$ so it remains to prove $(1) \Rightarrow (2)$. Let

$$A = \{x \in H(\infty) : D(x) \text{ is finite}\}.$$

We show first that $p \to V(p, x)$ is a parallel vector field in H for every $x \in A$. Let $x \in A$ be given, and let $y \in H(\infty)$ be a point that can be joined to x. By Proposition 2.2 it suffices to show that x and y are antipodal points. Any point $z \in H(\infty)$ to which y can be joined must lie in $\overline{D(x)} = D(x)$ by Lemma 2.4a, and in particular y can be joined to at most finitely many points in $H(\infty)$. If $\gamma_{py}(-\infty) = x$ and $\gamma_{qy}(-\infty) = z$ then $\gamma_{\sigma t,y}(-\infty) = x_t$ is a curve in $H(\infty)$ from x

to z if σ is any curve in H from p to q. Therefore y can be joined to each of the points x_t, which implies that $z = x$. Hence x and y are antipodal points.

Now define a distribution in H by $N(p) = \text{span } \{V(p, x) : x \in A\}$. By the argument of the previous theorem, H is a Riemanniaan product $H_1 \times H_2$, where for each point $p = (p_1, p_2)$ we have $T_{p_1}(H_1) = N(p)$ and $T_{p_2}(H_2) = N^{\perp}(p)$, the orthogonal distribution. Similarly each isometry ϕ of D can be written $\phi = \phi_1 \times \phi_2$, where ϕ_i is an isometry of H_i, $i = 1, 2$. The factor H_1 is again either $\mathbf{R}$ or a flat Euclidean space.

Now choose points $x_1, \ldots, x_k$ in A so that the foliation N is spanned by the parallel vector fields $p \to V(p, x_i)$, $1 \leq i \leq k$. If D_i^* is the stability group of x_i in D, then each D_i^* has finite index in D, and therefore $D^* = \bigcap_{i=1}^{k} D_i^*$ has finite index in D. Clearly $A \subseteq H_1(\infty)$ by the definition of N and H_1. We assert that $D^* = \{\phi \in D : \phi$ fixes every point of $H_1(\infty)\}$. From this it will follow that $A = H_1(\infty)$ since D^* has finite index in D. This implies immediately that D_2 has no finite orbits in $H_2(\infty)$.

If ϕ is an element of D that fixes every point of $H_1(\infty)$, then ϕ clearly lies in D^*. Conversely let $\phi \in D^*$ and $x \in H_1(\infty)$ be given. Write ϕ as a product $\phi_1 \times \phi_2$, where ϕ_i is an isometry of H_i. Since ϕ fixes the points $x_1, \ldots, x_k$ in $A \subseteq H_1(\infty)$ it follows that $\phi_1 \in D_1^*$ also fixes these points. Therefore ϕ_1 is a translation of H_1 since the vector fields $p \to V(p, x_i)$, $1 \leq i \leq k$, span the tangent space of H_1. It follows that ϕ_1 and hence ϕ fixes every point of $H_1(\infty)$. Moreover D_1^* consists of translations of H_1 and H_1/D_1^* is compact. It remains only to show that D^* is a normal subgroup of D. If $\psi \in D$ is arbitrary, then $\psi^{-1}D^*\psi = \{\phi \in D : \phi$ fixes every point of $\psi^{-1}H_1(\infty)\}$. However $\psi^{-1}H_1(\infty) = \psi^{-1}(A) = A = H_1(\infty)$, which shows that $\psi^{-1}D^*\psi = D^*$.

We investigate next the orbits in $H(\infty)$ of a semisimple group $G \subseteq I(H)$. We recall that if G is a connected noncompact semisimple Lie group with finite center and if K is a maximal compact subgroup of G, then the coset space $H = G/K$ is a symmetric space of noncompact type and in particular a Hadamard manifold. H is equipped with a Riemannian metric such that for each $g \in G$ we have an isometry τ_g given by $\tau_g(hK) = ghK$. The map τ is a homomorphism of G onto the semisimple group $G^* = I_0(H)$, and $K^* = \tau(K)$ is a maximal compact subgroup of G^*.

We assert that Z^*, the center of G^*, is trivial. The group G^* satisfies the duality condition as is shown in the proof of Theorem 5.4 below, and therefore if Z^* is nontrivial then it is not discrete by Theorem 4.2, contradicting the semisimplicity of G^*.

PROPOSITION 4.4. *Let H be a symmetric space of noncompact type. Let $G = I_0(H)$ and let K be a maximal compact subgroup of G. Then $G(x) = K(x)$ for every $x \in H(\infty)$. In particular every orbit of G in $H(\infty)$ is compact.*

Remark. This result is essentially contained in Lemma 5 of [25] expressed in different language and notation.

Proof. Let $p \in H$ be a fixed point of K. The maximality of K in G implies that K is the isotropy subgroup of G at p. It will suffice to show that given $g \in G$ and $x \in H(\infty)$ we can find $k \in K$ such that $gx = kx$. Let g, x be given and choose $X \in \mathfrak{G}$, the Lie algebra of G, so that the unit speed geodesic $t \to \exp(tX)(p)$ belongs to the asymptote class x. For each $t > 0$ let ϕ_t denote $\exp(tX)$ and let $x_t \in H(\infty)$ be the asymptote class of the geodesic ray joining p and $g(\phi_t p)$. From the one parameter subgroup of G that translates γ_{px_t} we may choose an element ψ_t so that $\psi_t(p) = g(\phi_t p)$ and $\psi_t(x_t) = x_t$. The element $k_t = \psi_t^{-1} \circ g \circ \phi_t$ fixes p and hence lies in K. We observe that

$$\measuredangle_p(k_t x, x_t) = \measuredangle_{\psi_t p}(\psi_t k_t x, \psi_t x_t) = \measuredangle_{g\phi_t p}(g\phi_t x, x_t)$$
$$= \measuredangle_{g\phi_t p}(gx, x_t) = \measuredangle_{g\phi_t p}(p, gp) \to 0 \quad \text{as} \quad t \to +\infty$$

by the law of cosines [14, p. 48]. Now

$$x_t \to gx \quad \text{as} \quad t \to +\infty$$

since $\measuredangle_p(gx, x_t) = \measuredangle_p(gx, g\phi_t p) \to 0$ as $t \to +\infty$. Therefore $k_t x \to gx$ as $t \to +\infty$ by the set of inequalities above, and hence there exists $k \in K$ with $kx = gx$ since K is compact.

We can extend this result as follows for an arbitrary space H.

THEOREM 4.5. *Let $G \subseteq I(H)$ be a noncompact closed connected semisimple group with finite center. Let K be a maximal compact subgroup of G. Then $G(x) = K(x)$ for every x in $L(G)$.*

Proof. The set $L(G)$ is nonempty since G is closed and noncompact. Let $p \in H$ be a fixed point of K. Let $g \in G$ and $x \in L(G)$ be given. The orbit $H^* = G(p)$ is a noncompact symmetric space G/K relative to the metric induced from H, and H^* is thus a Hadamard manifold imbedded in H. Let d^* denote the metric in H^*, and let x^* denote a typical point in $H^*(\infty)$. Now choose a sequence $\{\phi_n\} \subseteq G$ such that $\phi_n p \to x$ in the cone topology of $\bar{H} = H \cup H(\infty)$. Let γ_n^* be the unit speed H^*-geodesic such that $\gamma_n^*(0) = p$ and $\gamma_n^*(t_n) = \phi_n p$, where $t_n = d^*(p, \phi_n p) \to +\infty$. Let $x_n^* \in H^*(\infty)$ be the asymptote class of γ_n^*. By the preceding result we may choose $k_n \in K$ so that $k_n(x_n^*) = g(x_n^*)$. Now let $k_n \to k \in K$ by passing to a subsequence.

We assert that $kx = gx$. Let α_n, β_n denote the H^*-geodesics $g \circ \gamma_n^*$, $k_n \circ \gamma_n^*$ respectively. The function $t \to d^*(\alpha_n t, \beta_n)$ is a continuous convex function by Proposition 4.7 of [4] and is bounded for $t \geq 0$ hence nonincreasing in t since α_n and β_n are asymptotes in H^*. If $\{s_n\} \subseteq \mathbf{R}$ is a sequence for which $d^*(\alpha_n t_n, \beta_n) = d^*(\alpha_n t_n, \beta_n s_n)$, then it follows from the monotonicity of $t \to d^*(\alpha_n t, \beta_n)$ that

$$d^*(\alpha_n t_n, \beta_n s_n) \leq d^*(\alpha_n(0), \beta_n) \leq d^*(\alpha_n(0), \beta_n(0)) = d^*(gp, p).$$

Furthermore, $|s_n - t_n| = |d^*(\beta_n s_n, p) - d^*(\alpha_n t_n, gp)| \le 2d^*(gp; p)$ by the triangle inequality. This implies that

$$d(\alpha_n t_n, \beta_n t_n) \le d^*(\alpha_n t_n, \beta_n t_n) \le 3d^*(gp, p).$$

Finally

$$\begin{aligned} kx &= \lim_{n\to\infty} k_n(\phi_n p) = \lim_{n\to\infty} k_n(\gamma_n^* t_n) = \lim_{n\to\infty} \beta_n(t_n) \\ &= \lim_{n\to\infty} \alpha_n(t_n) = \lim_{n\to\infty} g(\gamma_n^* t_n) = gx. \end{aligned}$$

All limits in the last step take place in the cone topology of H.

COROLLARY 4.6. *Let $I(H)$ satisfy the duality condition. Then $I_0(H)$ has a compact orbit in $H(\infty)$.*

Proof. If $I_0(H)$ is the identity then $I_0(H)$ clearly has a compact orbit in $H(\infty)$. If H has a flat de Rham factor H_1, then $H_1(\infty)$ is a compact orbit of $I_0(H)$. If H has no flat de Rham factor and $I_0(H)$ is not the identity, then $I_0(H)$ has a compact orbit by Proposition 2.5 and Theorem 4.5. The set $L(I_0(H))$ is nonempty by the argument in the first paragraph of the proof of Proposition 4.10 below.

Minimal actions on $H(\infty)$. Except for a point or a finite set of points, the simplest orbit of a subgroup $D \subseteq I(H)$ is a dense subset of $H(\infty)$. If every orbit of D in $H(\infty)$ is dense, then by definition D acts *minimally* on $H(\infty)$. For applications to geodesic flows we shall frequently use the fact that if $M = H/D$ is a smooth manifold, then D acts minimally on $H(\infty)$ if and only if the geodesic flow has a dense orbit in the unit tangent bundle of M. For a proof see Theorem 4.14 of [12], which includes the next result in the case that H/D is a smooth manifold.

PROPOSITION 4.7. *Let $D \subseteq I(H)$ be a subgroup such that $\bar{D}$ is noncompact. Then the following are equivalent:*

(1) *D acts minimally on $H(\infty)$.*

(2) *D satisfies the duality condition and some orbit of D in $H(\infty)$ is dense in $H(\infty)$.*

Proof. Suppose that D acts minimally on $H(\infty)$. Clearly D has a dense orbit in $H(\infty)$. Now let γ be a geodesic of H with endpoints $x = \gamma(\infty)$ and $y = \gamma(-\infty)$. The fact that $\bar{D}$ is noncompact means that $L(D)$ is nonempty in $H(\infty)$, and in fact $L(D) = H(\infty)$ since D acts minimally and leaves $L(D)$ invariant. Let $\{\varphi_n\} \subseteq D$ be a sequence such that $\varphi_n p \to x$, and let $\varphi_n^{-1} p \to z \in H(\infty)$ by passing to a subsequence. The point p in H is arbitrary. It is easy to see that the point x is dual to any point in $\overline{D(z)} = H(\infty)$ and in particular to y. Therefore D satisfies the duality condition.

Conversely let D satisfy the duality condition, and let $x \in H(\infty)$ be chosen so that $\overline{D(x)} = H(\infty)$. We shall show that $\overline{D(y)} = H(\infty)$ for an arbitrary point $y \in H(\infty)$. Let $y \in H(\infty)$ be given, and let γ be any geodesic with $\gamma(\infty) = y$. If z denotes $\gamma(-\infty)$, then by hypothesis z is dual to y.

First we show that z is dual to any point w in $H(\infty)$. Let $\{\varphi_n\} \subseteq D$ be a sequence such that $\varphi_n x \to y$. Fix a point p on γ and let $z_n = \gamma_{p,\varphi_n x}(-\infty)$. By continuity $z_n \to z$ since $\varphi_n x \to y$. Now z_n is dual to $\phi_n x$ by the duality condition and also to any point in $\overline{D(\phi_n x)} = \phi_n \overline{D(x)} = H(\infty)$. If w is an arbitrary point of $H(\infty)$, then w is dual to each z_n and hence to z since the points dual to w are a closed subset of $H(\infty)$.

Finally let w be an arbitrary point of $H(\infty)$, and let $\{\phi_n\} \subseteq D$ be a sequence such that $\phi_n p \to w$ and $\phi_n^{-1} p \to z$. Since p lies on γ it follows as in the proof of Lemma 2.4a that

$$\sphericalangle_p(\phi_n p, \phi_n y) = \sphericalangle_{\phi_n^{-1} p}(p, y) \leq \sphericalangle_p(z, \phi_n^{-1} p) \to 0.$$

Therefore $w = \lim_{n\to\infty} \phi_n p = \lim_{n\to\infty} \phi_n y$, which shows that $\overline{D(y)} = H(\infty)$.

The following observation is useful.

Proposition 4.8. *Let $D \subseteq I(H)$ be a subgroup that acts minimally on $H(\infty)$. Then any subgroup of D of finite index also acts minimally on $H(\infty)$.*

Proof. Let D^* be a subgroup of D of finite index, and let D be the union of the cosets $\{\phi_i \cdot D^*: 1 \leq i \leq n\}$. Given $x \in H(\infty)$ let $F = \overline{D^*(x)}$. It is clear that $H(\infty) = \bigcup_{i=1}^n \phi_i(F)$ since $\overline{D(x)} = H(\infty)$, and therefore F has nonempty interior. If the boundary, ∂F, is empty we are done so assume that there exists $z \in \partial F$. It follows that $G = \overline{D^*(z)} \subseteq \partial F$, and we argue as above to conclude that G contains an interior point. This contradiction shows that $F = H(\infty)$ and that D^* acts minimally.

Our final result of a general nature is:

Proposition 4.9. *If $H(\infty)$ has no flat points then the following are equivalent:*

(1) *$D \subseteq I(H)$ satisfies the duality condition.*
(2) *$D \subseteq I(H)$ has noncompact closure and D acts minimally on $H(\infty)$.*

Proof. If $\bar{D}$ is noncompact and D acts minimally on $H(\infty)$, then D satisfies the duality condition by Proposition 4.7. If $H(\infty)$ has no flat points and D satisfies the duality condition, then D has no proper closed invariant sets in $H(\infty)$ by Theorem 3.2. Clearly $\bar{D}$ is noncompact since $L(D) = H(\infty)$ when D satisfies the duality condition. Therefore (1) implies (2).

We now turn to results of a more specialized nature.

PROPOSITION 4.10. *Let $I(H)$ be noncompact and let $I(H)$ act minimally on $H(\infty)$. Then on of the following must occur:*

(1) *$I(H)$ is discrete.*
(2) *H is isometric to $\mathbf{R}^n$ with the usual metric.*
(3) *$I_0(H)$ is a noncompact semisimple Lie group with trivial center. Moreover $L(I_0(H)) = H(\infty)$ and all orbits of $I_0(H)$ in $H(\infty)$ are compact.*

Proof. By Proposition 4.7, $I(H)$ satisfies the duality condition and in particular $L(I(H)) = H(\infty)$. Suppose now that $I(H)$ is not discrete. If $I_0(H)$ were compact, then its fixed point set S in H would be a nonempty totally geodesic submanifold invariant under $I(H)$ [22]. It would follow that $S = H$ since $L(I(H)) = H(\infty)$. This contradiction implies that $I_0(H)$ is noncompact, and it follows that $L(I_0(H))$ is nonempty and in fact equals $H(\infty)$ since it is invariant under $I(H)$.

Either $I_0(H)$ is semisimple or it is not. Suppose first that $I_0(H)$ is semisimple. The center of $I_0(H)$ must consist of Clifford translations by Proposition 2.3. We assert that $I_0(H)$ has in fact a trivial center. Suppose that ϕ is a nonidentity element in the center of $I_0(H)$. The geodesics γ_p joining p to ϕp are translated by ϕ for every point p in H. These geodesics all belong to a single asymptote class $x \in H(\infty)$ and x has an antipodal point y in $H(\infty)$, by Proposition 6.7 of [14]. Therefore by Proposition 2.2, $\phi_t\colon p \to \gamma_{px}(t)$ is a Clifford translation for every t and $\phi = \phi_a$ for some number $a \neq 0$. If $\psi \in I_0(H)$ is arbitrary, then define $\psi_t = \psi \circ \phi_t \circ \psi^{-1}$ for every number t. For each point p the geodesics $t \to \phi_t p$ and $t \to \psi_t p$ must be equal since they both contain the points p and $\phi_a p = \psi_a p \neq p$. Therefore $\phi_t = \psi_t$ and $\{\phi_t\}$ lies in the center of $I_0(H)$, contradicting the fact that a semisimple group must have discrete center. It follows from Theorem 4.5 that the orbits of $I_0(H)$ in $H(\infty)$ are compact.

Suppose now that $I_0(H)$ is not semisimple, and let $A \neq 1$ be an abelian subgroup invariant under all continuous automorphisms of $I_0(H)$ [6]. In particular A is normal in $I(H)$. The argument above shows that A has no fixed points in H and that $L(A) = H(\infty)$. By Lemma 2.4d the vector field $p \to V(p, x)$ is parallel in H for every $x \in H(\infty)$. (One could also derive this from the fact that A consists of Clifford translations by Proposition 2.3.) For any point p and any 2-plane π in $T_p(H)$ choose $x \in H(\infty)$ so that $V(p, x)$ lies in π. Then $K(\pi) = 0$ since $p \to V(p, x)$ is parallel, and hence H is isometric to $\mathbf{R}^n$ with the standard flat metric.

COROLLARY 4.11. *Let $I(H)$ act minimally on $H(\infty)$. Then all orbits of $I_0(H)$ in $H(\infty)$ are compact.*

Proof. The result is trivial if $I_0(H)$ is the identity. If $I(H)$ is compact then so is $I_0(H)$, and the result is again trivial. Assume that $I(H)$ is noncompact and that $I_0(H)$ is not the identity. The result now follows from Proposition 4.10.

PROPOSITION 4.12. *Let $I_0(H)$ be noncompact and act transitively on $H(\infty)$. Then H is isometric either to $\mathbf{R}^n$ with the usual metric or to a rank one symmetric space.*

Remarks. (1) If $H = \mathbf{R}^n$ or a rank one symmetric space, then in either case a maximal compact subgroup K of $I_0(H)$ acts transitively on the unit vectors of $T_p(H)$, where p is a fixed point of K. Hence K acts transitively on $H(\infty)$.

(2) The condition that $I_0(H)$ be noncompact is necessary. There are 2-dimensional spaces H of nonconstant curvature for which $I(H) = I_0(H)$ is compact and acts transitively on $H(\infty)$. See the discussion of *central points* in [4, p. 34].

Proof of the proposition. By Proposition 4.10 either H is flat or $I_0(H)$ is a noncompact semisimple Lie group with trivial center and $L(I_0(H)) = H(\infty)$. Suppose that H is not flat. Let K be a maximal compact subgroup of $I_0(H)$, and let $p \in H$ be a fixed point of K. The orbit $M = I_0(H)(p)$ is a noncompact symmetric space $I_0(H)/K$ relative to the metric induced from H, and the differential maps of K carry $T_p(M)$ into itself. By Theorem 4.5, K is transitive on $H(\infty)$ since $I_0(H)$ is transitive on $H(\infty)$, and therefore the differential maps of K act transitively on the unit vectors of $T_p(H)$. It follows that $H = M$, a symmetric space and also a two point homogeneous space. By [21], H must be a rank one symmetric space.

THEOREM 4.13. *The group $I_0(H)$ is noncompact and acts transitively on $H(\infty)$ under any of the following conditions:*

(1) *$I(H)$ is noncompact and $I_0(H)$ acts minimally on $H(\infty)$.*

(2) *$I(H)$ is noncompact and acts transitively on $H(\infty)$.*

(3) *$I(H)$ is noncompact, acts minimally on $H(\infty)$ and $I(H)/I_0(H)$ is a finite group.*

(4) *H is homogeneous and $I(H)$ acts minimally on $H(\infty)$.*

(5) *$I_0(H)$ satisfies the duality condition and some point of $H(\infty)$ is not a flat point.*

Proof. Case (1). The set $L(I(H))$ is nonempty since $I(H)$ is noncompact and in fact $L(I(H)) = H(\infty)$ since $I_0(H)$ acts minimally and leaves $L(I(H))$ invariant. By the argument of Proposition 4.10, $I_0(H)$ has no fixed point in H and must therefore be noncompact. By Proposition 4.10, either H is isometric to $\mathbf{R}^n$ with the usual metric and we are done, or the orbits in $H(\infty)$ of $I_0(H)$ are all compact. In the latter case $I_0(H)$ also acts transitively on $H(\infty)$ since the orbits of $I_0(H)$ in $H(\infty)$ are all dense by hypothesis.

Case (2). The group $I_0(H)$ is not the identity since $I(H)$ is second countable in the compact-open topology [20, p. 167] and $I(H)$ acts transitively on $H(\infty)$ by hypothesis. By Proposition 4.10, either H is isometric to $\mathbf{R}^n$ with the usual metric and we are done, or $I_0(H)$ is a noncompact semisimple Lie group with trivial center and $L(I_0(H)) = H(\infty)$. In the latter case all orbits of $I_0(H)$ in $H(\infty)$ are compact. However, some orbit of $I_0(H)$ has nonempty interior and must therefore be open since $I(H)/I_0(H)$ is a countable group. Hence $I_0(H)$ acts transitively on $H(\infty)$.

Case (3). By Proposition 4.8, $I_0(H)$ acts minimally on $H(\infty)$, and the result now follows from case (1).

Case (4). This is an immediate corollary of case 3).

Case (5). Clearly $I_0(H)$ is noncompact and $L(I_0(H)) = H(\infty)$ since $I_0(H)$ satisfies the duality condition. We assert that $I_0(H)$ is semisimple. If it were not semisimple, then it would contain an abelian normal subgroup consisting of Clifford translations by Theorem 2.4, and by Theorem 2.1 H would admit a flat de Rham factor. By the second example of Section 3 every point of $H(\infty)$ is flat if H is a nontrivial Riemannian product manifold. This contradicts our hypothesis and shows that $I_0(H)$ is semisimple. Similarly the center of $I_0(H)$ must contain only Clifford translations by Theorem 2.4 and is therefore also trivial. It follows from Theorem 4.5 that every orbit of $I_0(H)$ in $H(\infty)$ is compact. If $x \in H(\infty)$ is not flat, then by Proposition 4.1, x is an interior point of the orbit $I_0(H)(x)$, which must therefore be open. Hence $I_0(H)(x) = H(\infty)$.

Combining Proposition 4.12 and Theorem 4.13 we obtain characterizations of Euclidean or rank one symmetric spaces. In particular we have:

COROLLARY 4.14. *Let $I_0(H)$ satisfy the duality condition and suppose that some point of $H(\infty)$ is not flat. Then H is isometric to a rank one symmetric space.*

We conclude this section with a characterization of Euclidean space.

THEOREM 4.15. *Let $A \subseteq I(H)$ be a subgroup whose normalizer acts minimally on $H(\infty)$ and has noncompact closure. If $A \neq 1$ and its centralizer $Z \neq 1$, then H is isometric to $\mathbf{R}^n$ with the usual metric and A is a subgroup of translations.*

Remarks. (1) Compare this result to Theorem 2.4.

(2) The result is false if $I(H)$ is compact. See the discussion of *central points* in [4, pp. 33–34].

Proof. Let D be the normalizer of A in $I(H)$. $L(D)$ is nonempty since $\bar{D}$ is noncompact and $L(D) = H(\infty)$ since D acts minimally on $H(\infty)$. The argument of Proposition 4.10 shows that A fixes no point of H and that $L(A) = H(\infty)$. If $\phi \neq 1$ lies in Z, then ϕ fixes every point of $L(A)$ and hence is a Clifford translation by Proposition 2.3. If C denotes the subgroup of $I(H)$ consisting of Clifford translations, then $L(C) = H(\infty)$ since D normalizes C and consequently leaves $L(C)$ invariant. The group C is abelian by Theorem 2.1, and the argument of Proposition 4.10 now shows that H is flat.

We have seen that Z consists of Clifford translations or in this case ordinary translations of $\mathbf{R}^n$. Now D normalizes Z since it normalizes A, and hence D leaves $L(Z)$ invariant. It follows that $L(Z) = H(\infty)$ and the vectors

$\{T(0)/\|T(0)\| : T \in Z\}$ are dense in the unit sphere in $\mathbf{R}^n$. If $a \neq 1$ is an arbitrary element of A it follows by continuity that a commutes with all translations of displacement 1 in $\mathbf{R}^n$, and therefore a must be a translation.

COROLLARY 4.16. *Let $D \subseteq I(H)$ act minimally on $H(\infty)$, and let $\bar{D}$ be noncompact. If $A \neq 1$ is a normal abelian subgroup of D, then H is isometric to $\mathbf{R}^n$ with the usual metric and A is a subgroup of translations.*

5. Applications

The results of this section fall into two categories: (1) the structure of $I(H)$, (2) fundamental groups, geodesic flows and isometry groups of quotient manifolds $M = H/D$. In the second category we assume that $\Omega = SM$; that is, every point of SM is nonwandering relative to the geodesic flow.

The structure of $I(H)$.

THEOREM 5.1. *Let $D \subseteq I(H)$ satisfy the duality condition, and let S be a solvable subgroup of D of finite index. Then H is isometric to $\mathbf{R}^n$ with the usual metric.*

Proof. Let D be the union of the cosets $\{d_i S : 1 \le i \le k\}$ and let $S_i = d_i S d_i^{-1}$. Then $S^* = \bigcap_{i=1}^k S_i$ is a normal solvable subgroup of D of finite index in D. Define $C_0 = S^*$ and $C_i = [C_{i-1}, C_{i-1}]$, the commutator. The fact that S^* is solvable means that $C_{n+1} = \{1\}$ and C_n is an abelian group for some n. The subgroups C_i are invariant under any automorphism of S^* and in particular under any inner automorphism of D. By Theorem 2.4, C_n consists of Clifford translations. By Theorem 2.1, H is a Riemannian product $H_1 \times H_2$, where H_1 is a flat Euclidean space and H_2 has no flat factor. We shall assume that H_2 has positive dimension and obtain a contradiction. Every isometry of D can be written as $\phi = \phi_1 \times \phi_2$, where $\phi_i = \pi_i(\phi) \in I(H_i)$, $i = 1, 2$. It is easy to see that $D_2 = \pi_2(D) \subseteq I(H_2)$ satisfies the duality condition and that $S_2 = \pi_2(S)$ is a solvable subgroup of D_2 of finite index. Repeating the argument above we find that H_2 has a flat de Rham factor, contrary to the hypothesis on H_2.

Remark. Our proof is similar to that of Wolf in [30] for the case that S is nilpotent.

THEOREM 5.2. *Let $D \subseteq I(H)$ be a subgroup satisfying the duality condition and suppose that $L(I_0(H)) = H(\infty)$. Then either H is flat and D has a solvable subgroup of finite index or D contains a nonabelian free group.*

Proof. Let $\mathfrak{G}$ denote the Lie algebra of $I_0(H)$, the space of Killing vector fields on H. If G denotes $I(H)$, then $\mathrm{Ad}: G \to \mathrm{Aut}\,(\mathfrak{G})$ is a homomorphism whose kernel Z consists of the centralizer of $I_0(H)$ in G. By Proposition 2.3, an element $\phi \neq 1$ of Z fixes every point of $L(I_0(H)) = H(\infty)$ and must therefore be a Clifford translation of H. By Theorem 2.1 the Clifford translations of H are an abelian subgroup of $I(H)$, and therefore Z is abelian. By a theorem of Tits [28] the matrix group Ad $(D) \subseteq$ Aut $(\mathfrak{G})$ contains either a solvable subgroup S of finite index or a nonabelian free group F. Suppose that the first possibility occurs, and let $S^* = \text{Ad}^{-1}(S) \subseteq D$. If

$$S_i = [S_{i-1}, S_{i-1}] \quad \text{and} \quad S_i^* = [S_{i-1}^*, S_{i-1}^*]$$

are the derived series for S, S^* respectively, then Ad $(S_i^*) = S_i$ for every i. By hypothesis $S_n = \{1\}$ for some n and hence $S_n^* \subseteq \ker \text{Ad} = Z$. It follows that $S_{n+1}^* = \{1\}$ and S^* is solvable since Z is abelian. Moreover, H is flat by the preceding result since S^* has finite index in D. Suppose next that Ad (D) contains a nonabelian free group F. We may assume by passing to a subgroup of F that F is generated by two elements x and y. Let x^*, y^* be elements of D so that Ad $(x^*) = x$ and Ad $(y^*) = y$. The group F^* generated by x^* and y^* is a nonabelian free subgroup of D, for any word relations between x^* and y^* must also exist between x and y.

COROLLARY 5.3. *Let $I(H)$ be nondiscrete and act minimally on $H(\infty)$. Let $D \subseteq I(H)$ be a subgroup that satisfies the duality condition. Then either H is flat and D contains a solvable subgroup of finite index or D contains a nonabelian free group.*

Proof. By the argument of Proposition 4.10, $I_0(H)$ cannot have a fixed point in H and must therefore be noncompact. Hence the limit set of $I_0(H)$ is $H(\infty)$ since it is nonempty, closed and invariant under $I(H)$. The result now follows from Theorem 5.2.

THEOREM 5.4. *Let H be homogeneous. Then $I(H)$ satisfies the duality condition if and only if H is the Riemannian product of a flat Euclidean space and a symmetric space of noncompact type (either factor may be trivial).*

Proof. If H is $\mathbf{R}^n$ with the usual metric, then $I(H)$ satisfies the duality condition since any geodesic of H is translated by some Euclidean translation. If H is a symmetric space, then let there be given a geodesic γ with $p = \gamma(0)$. Let s be the geodesic symmetry fixing p and s_n the geodesic symmetry fixing $\gamma(n)$ for any positive integer n. If $\phi_n = s_n \circ s$ then we have $\phi_n(p) = \gamma(2n)$ and $\phi_n^{-1}(p) = \gamma(-2n)$, which shows that the points $\gamma(\infty)$ and $\gamma(-\infty)$ are dual relative to $I(H)$. If H_1 is a flat Euclidean space and H_2 is a symmetric space, then $I(H) = I(H_1) \times I(H_2)$ where H is the Riemannian product $H_1 \times H_2$, and it is easy to verify that $I(H)$ satisfies the duality condition since both of its factors do.

Conversely suppose that H is a homogeneous space such that $I(H)$ satisfies the duality condition. If $I_0(H)$ is semisimple, then H is a symmetric space and we are done. If $I_0(H)$ is not semisimple, then there exists an abelian subgroup $A \subseteq I_0(H)$ that is invariant under all continuous automorphisms of $I_0(H)$ and in particular is normal in $I(H)$. By Theorem 2.4, A must consist of Clifford translations, and by Theorem 2.1 we can write H as a Riemannian product

$H_1 \times H_2$, where H_1 is a flat Euclidean space of positive dimension and H_2 has no flat de Rham factor. We may assume that H_2 has positive dimension for otherwise we are done. Now $I(H) = I(H_1) \times I(H_2)$, where $I(H_1) = I(H_1) \times \{1\}$ and $I(H_2) = \{1\} \times I(H_2)$. Moreover H_2 is homogeneous since both H and H_1 are homogeneous, and $I(H_2)$ satisfies the duality condition since $I(H)$ satisfies it. If $I_0(H_2)$ were not semisimple, then by repeating the argument above we would be able to produce a flat de Rham factor for H_2, a contradiction. Therefore $I_0(H_2)$ is semisimple and H_2 is a symmetric space.

THEOREM 5.5. *Let $D \subseteq I(H)$ be a subgroup that satisfies the duality condition. If $H(\infty)$ contains no flat points, then D acts minimally on $H(\infty)$.*

Proof. Let $x \in H(\infty)$ be arbitrary, and let $A = \overline{D(x)}$. The set A has empty boundary by Theorem 3.2 since $H(\infty)$ has no flat points, and therefore $A = H(\infty)$.

COROLLARY 5.6. *Let H admit a point p such that all sectional curvatures at p are negative. Let $D \subseteq I(H)$ be a subgroup that satisfies the duality condition. Then D acts minimally on $H(\infty)$.*

THEOREM 5.7. *Let $D \subseteq I(H)$ be a subgroup such that $\bar{D}$ is noncompact. Then D does not act minimally on $H(\infty)$ under either of the following conditions:*

(1) *D is a direct product of nontrivial subgroups D_1 and D_2.*
(2) *H is the Riemannian product of two manifolds of positive dimension.*

Proof. Suppose (1) holds and D acts minimally on $H(\infty)$. If $A = D_1$ then D_2 centralizes A while D normalizes A and has noncompact closure in $I(H)$. By Theorem 4.15, H is isometric to $\mathbf{R}^n$, and D_1 is a subgroup of translations. A similar argument shows that D_2 and hence D is a subgroup of translations. Therefore D fixes every point of $H(\infty)$, a contradiction.

Now suppose that (2) holds and let $H = H_0 \times H_1 \times \cdots \times H_k$ be the de Rham decomposition of H, which by hypothesis has at least two factors.

$$G = I(H_0) \times I(H_1) \times \cdots \times I(H_k)$$

is the subgroup of $I(H)$ that leaves invariant the foliations $\mathfrak{M}_i$ of H induced by the tangent spaces of the factors H_i, $0 \leq i \leq k$. G has finite index in $I(H)$ since each isometry of H permutes the foliations $\mathfrak{M}_i$ [23, p. 192]. If D acted minimally on $H(\infty)$, then $I(H)$ would also act minimally. By Proposition 4.8, G would act minimally on $H(\infty)$, contradicting part (1) of this result.

THEOREM 5.8. *Let $M = H/D$ be a Visibility manifold with $\Omega = SM$. Then either $I(H)$ is discrete or H is a rank one symmetric space.*

COROLLARY 5.9. *Let $M = H/D$ be a complete Riemannian manifold with sectional curvature $K \leq c < 0$ and finite volume. Then either $I(H)$ is discrete or H is a rank one symmetric space.*

Remark. The corollary includes a result of Heintze [16] while the theorem preceding it includes part of Theorem 4.1 of [8].

Proof of the Theorem. By Theorem 6.3 of [12] there is a vector $v \in SM$ whose orbit under the geodesic flow is dense in SM. Then D acts minimally on $H(\infty)$ by Theorem 4.14 of [12]. As we observed at the beginning of Section 3 the Visibility axiom implies that $H(\infty)$ has no flat points. Assume now that $I(H)$ is not discrete. By Proposition 4.10, $L(I_0(H)) = H(\infty)$. The set of points in $H(\infty)$ fixed by $I_0(H)$ is closed and invariant under D, hence empty since $I_0(H)$ contains no Clifford translations. By Proposition 2.6 of [11], $I_0(H)$ satisfies the duality condition. Now apply Corollary 4.14.

Quotient manifolds. In all of the results of this subsection we assume that $M = H/D$ has nonpositive sectional curvature and satisfies no further conditions such as the Visibility axiom unless explicitly stated.

THEOREM 5.10. *Let $\Omega = SM$ and let $A \neq 1$ be a normal abelian subgroup of $\pi_1(M)$. Then there exists a regular finite covering $M' \to M$ of M, where $\pi_1(M')$ has center of rank $k \geq 1$. Both M and M' are foliated by flat totally geodesic tori of dimension k.*

Remark. This result and its proof extends the center theorem of [24] to manifolds M with $\Omega = SM$ and fundamental group having nontrivial center.

Proof. Expressing M as a quotient manifold H/D it follows from Proposition 4.9 of [12] that the deckgroup D satisfies the duality condition. Let $A \neq 1$ be a normal abelian subgroup of $D \approx \pi_1(M)$. By Theorem 2.4, A consists of Clifford translations. If Z is the centralizer in D of A, then Z is a normal subgroup of D and Z has finite index in D by the argument of Lemma 3 of [32]. If $M' = H/Z$, then M' is a regular finite covering of M, and A is contained in the center of $\pi_1(M')$. The remaining assertions follow as in [29] or [24].

THEOREM 5.11. *Let $\Omega = SM$ and let M have solvable fundamental group. Then M is compact and flat.*

Remark. This result has been proved in [15] and [32] under the hypothesis that M be compact.

Proof. Expressing M as a quotient manifold H/D the deckgroup D is solvable by hypothesis and satisfies the duality condition since $\Omega = SM$. By Theorem 5.1, both M and H are flat. By Theorem 1 of [15], D leaves invariant a flat totally geodesic submanifold E of H such that E/D is compact. It follows that $E = H$ since $L(D) = H(\infty)$.

THEOREM 5.12. *Let $M = H/D$ be locally homogeneous with $\Omega = SM$. Then either M is a compact flat manifold or $\pi_1(M)$ contains a nonabelian free group.*

Proof. D satisfies the duality condition since $\Omega = SM$ and $L(I_0(H)) = H(\infty)$ since $I_0(H)$ is transitive on H. Therefore by Theorem 5.2 either M and H are flat and $D \approx \pi_1(M)$ contains a solvable subgroup of finite index or D contains a nonabelian free group. In the former case M is compact by the argument of the preceding result.

THEOREM 5.13. *Let the geodesic flow of $M = H/D$ have a dense orbit in SM. Then either $I(H)$ is discrete or $\pi_1(M)$ contains a nonabelian free group.*

Proof. We know that $\Omega = SM$ since all vectors tangent to a dense orbit of the geodesic flow lie in Ω. Therefore D acts minimally on $H(\infty)$ by Theorem 4.14 of [12]. If $I_0(H) \neq 1$, then $L(I_0(H)) = H(\infty)$ since $L(I_0(H))$ is invariant under D. We now apply Theorem 5.2. If H were flat, then M would be compact by the argument above, and the Bieberbach group D would contain a normal subgroup of finite index consisting of translations of the Euclidean space H. The orbits of D in $H(\infty)$ would then be finite, contradicting the minimality of D, since a translation fixes all points of $H(\infty)$. Therefore H is not flat and $D \approx \pi_1(M)$ contains a nonabelian free group by Theorem 5.2.

THEOREM 5.14. *Let $M = H/D$ be locally homogeneous with $\Omega = SM$. Then H is a Riemannian product $H_1 \times H_2$, where H is isometric to a Euclidean space with the usual metric and H_2 is a symmetric space.*

Remark. This result has been proved in [16] for the case that M has finite volume and strictly negative sectional curvature and in [3, p. 28] for the case that M has finite volume and nonpositive sectional curvature.

Proof. The deckgroup D and hence $I(H)$ satisfy the duality condition since $\Omega = SM$. The result now follows from Theorem 5.4.

THEOREM 5.15. *Let $\Omega = SM$. If $H(\infty)$ has no flat points, then the geodesic flow has a dense orbit in SM.*

COROLLARY 5.16. *Let $\Omega = SM$ and suppose that all sectional curvatures at some point p of M are negative. Then the geodesic flow has a dense orbit in SM.*

Proof of the Theorem. D satisfies the duality condition, where $M = H/D$, since $\Omega = SM$. By Theorem 5.5, D acts minimally on $H(\infty)$ and by Theorem 4.14 of [12] the geodesic flow has a dense orbit in SM.

Remark. Both Theorem 5.15 and its corollary generalize the well known result that if M is a compact Riemannian manifold with negative sectional curvature, then the geodesic flow has a dense orbit in SM [1]. This result remains true if M is a Visibility manifold with $\Omega = SM$ [12], and this extension itself is a corollary of Theorem 5.15. It is reasonable to ask if the converse to Theorem 5.15 is true.

THEOREM 5.17. *Let $M = H/D$ be a complete manifold of nonpositive sectional curvature. Then the geodesic flow in SM has no dense orbit in SM under either of the following conditions:*

(1) *$\pi_1(M)$ is a direct product of nontrivial subgroups.*
(2) *H is the Riemannian product $H_1 \times H_2$ of two manifolds of positive dimension.*

Proof. If the geodesic flow has a dense orbit in SM, then $\Omega = SM$ and by Theorem 4.14 of [12] D acts minimally on $H(\infty)$. However, D cannot act minimally under either condition by Theorem 5.7 above.

Remark. This result is a generalization of a theorem of Preissmann which says that if $M = M_1 \times M_2$ is a compact C^∞ product manifold, then M admits no metric of negative sectional curvature. One can derive Preissmann's result from ours and the remark following Corollary 5.16.

The structure of $I(M)$.
THEOREM 5.18. *Let $\Omega = SM$. If $I(M)$ is not discrete, then every Killing vector field on M is a parallel vector field on M.*

Remark. In the case that M is compact this result is the starting point for Theorem 3 of [24], which says that the center of $\pi_1(M)$ has rank k if and only if M admits k linearly independent parallel vector fields. This result has been extended to compact manifolds without focal points in [26] and to manifolds with finitely generated fundamental group and finite volume in [24]. Is the result still valid if M has nonpositive sectional curvature and $\Omega = SM$?

Proof. Suppose that $I(M)$ is not discrete and let $\{\phi_t\} \subseteq I(M)$ be a nontrivial one parameter subgroup. Let X be the Killing vector on M with flow transformations $\{\phi_t\}$, and let $\tilde{X}$ be the lift of X to H such that $\pi_* \tilde{X}(p) = X(\pi p)$, where $\pi: H \to M$ is the projection. If $\{\tilde{\phi}_t\}$ are the flow transformations of $\tilde{X}$ it follows that $\pi \circ \tilde{\phi}_t = \phi_t \circ \pi$ and

$$\pi \circ (\delta \circ \tilde{\phi}_t \circ \delta^{-1}) = \phi_t \circ \pi$$

for all isometries δ in D, where $M = H/D$. We conclude that $\{\tilde{\phi}_t\}$ are isometries of H that centralize the deckgroup D of M. By Propositions 4.2 and 2.2 the isometries $\{\tilde{\phi}_t\}$ are Clifford translations and $\tilde{X}$ is a parallel vector field in H of the form $p \to V(p, x)$ for some $x \in H(\infty)$ fixed by D. Therefore X is parallel in H.

This result could also have been proved by using the facts that $f = \|X\|$ is a continuous convex function on M [4, Proposition 5.5] and that all continuous convex functions on a manifold M with $\Omega = SM$ are constant.

COROLLARY 5.19. *Let $\Omega = SM$. Then $I(M)$ is discrete under any of the following conditions:*

(1) *H has no flat de Rham factor.*
(2) *Some geodesic γ of M does not bound an immersed flat half plane.*
(3) *The geodesic flow has a dense orbit in SM.*
(4) *M has negative definite Ricci Tensor.*

Remark. See also Corollary 3 of Theorem 4 of [24].

Proof. The flow transformations of a parallel vector field on H are Clifford translations. If H has no flat de Rham factor, then H admits no parallel vector fields by Theorem 2.1 and this establishes case (1).

Condition (2) holds if and only if some point of $H(\infty)$ is not flat. It follows by Example 2 of Section 3 that H is not a Riemannian product of manifolds and in particular has no flat de Rham factor. The result now follows from (1).

Condition (3) implies that the deckgroup D of M acts minimally on $H(\infty)$ by Theorem 4.14 of [12]. Suppose that $I(M)$ is not discrete. Then H admits a parallel vector field, and by Theorem 2.1 H is a Riemannian product $H_1 \times H_2$, where H_1 is flat and H_2 has no flat de Rham factor. If the factor H_2 were trivial, then by Corollary 3.3.4 of [31] and our Theorem 2.4, D would admit a subgroup of Euclidean translations of finite index in D. This would imply that any orbit of D in $H(\infty)$ is finite, contradicting the fact that D acts minimally on $H(\infty)$. Therefore H_2 is nontrivial and every isometry ϕ of D can be written $\phi = \phi_1 \times \phi_2$, where $\phi_i \in I(H_i)$, $i = 1, 2$. The group $D_2 = \{\phi_2 : \phi \in D\}$ satisfies the duality condition in H_2 since D satisfies the duality condition in H $(\Omega = SM)$. In particular D_2 is a nontrivial subgroup of $I(H)$, identifying ϕ with $\{1\} \times \phi$. The group D_2 is normalized by D and centralized by $D_1 \approx D_1 \times \{1\}$, and by Theorem 4.15, H must be flat, contradicting the fact that H_2 is nontrivial. Therefore $I(M)$ is discrete.

Condition (4) implies that M admits no parallel vector fields. If X is a parallel vector field and p any point of M, then the sectional curvature of any 2-plane of $T_p(M)$ that contains $X(p)$ is zero. Therefore $X(p)$ has Ricci curvature zero for any p, which contradicts our hypothesis.

6. Problems

We conclude with some problems and questions. Two obvious problems of a general nature deserve further attention. The first is to describe the structure of flat points in $H(\infty)$ under various conditions on H or $I(H)$. The second is to understand in more depth the structure of the orbits in $H(\infty)$ of an isometry group $D \subseteq I(H)$ that satisfies the duality condition.

The discussion in Section 3 completely settles the first problem in symmetric spaces; either all points of $H(\infty)$ are flat (rank ≥ 2) or none of them are (rank 1). As a next step it is reasonable to ask what one can say about flat points in $H(\infty)$ if H is homogeneous. A homogeneous Hadamard manifold H has a simply transitive solvable group of isometries [17], [29] and hence H can be viewed as a connected solvable Lie group G with a left invariant metric. One can then give necessary and sufficient algebraic conditions on the Lie algebra of G for the left invariant metric on G to have strictly negative sectional curvature [2], [3], [17]. Using the methods of [2], [3], [17] can one find analogous algebraic criteria that describe $H(\infty)$ and determine which points of $H(\infty)$ are flat?

A starting point for the second problem seems less clear. For example, what can one say beyond Propositions 4.4 and 4.5 in the symmetric space case? For a locally symmetric space $M = H/D$ that is compact or has finite volume Mautner has described the ergodic parts of the geodesic flow in SM in terms of the orbit structure of $I_0(H)$ acting in SH by differential maps [25]. If $\Omega = SM$, where $M = H/D$, then it should be possible to transfer information about the orbit structure of D in $H(\infty)$ into information about the geodesic flow on SM.

We also list in fairly random order a number of specific questions in the form of assertions to be proved or disproved.

(1) Let H be a symmetric space of noncompact type, and let $D \subseteq I_0(H)$ be a subgroup. Then the duality condition and the Selberg property (S) relative to $I_0(H)$ are equivalent conditions on D.

(2) If H is not the Riemannian product of two manifolds of positive dimension and if $I(H)$ satisfies the duality condition, then either $I(H)$ is discrete or H is a symmetric space of noncompact type.

(3) Let $I_0(H)$ satisfy the duality condition. Then H is a Riemannian product $H_1 \times H_2$, where H_1 is a Euclidean space with the standard flat metric and H_2 is a symmetric space of noncompact type (this would follow from 2). Compare also Corollary 4.14 and Theorem 5.4).

(4) If $I(H)$ is noncompact and acts minimally on $H(\infty)$, then either H is a Euclidean space with the standard flat metric or $H(\infty)$ has no flat points (this would prove the converse of Theorem 5.15).

(5) If $I(H)$ is noncompact and acts minimally on $H(\infty)$, then either $I(H)$ is discrete or H is a Euclidean space with the standard flat metric or H is a rank one symmetric space (compare with Proposition 4.10).

(6) Let $\Omega = SM$ and suppose that M admits exactly k linearly independent parallel vector fields. Then the center of $\pi_1(M)$ is a free abelian group of rank k.

(7) Let H_1 and H_2 be Hadamard manifolds with different Riemannian metrics but the same underlying C^∞ manifold H. Let D be a group of diffeomorphisms of H such that the topological quotient space H/D is compact and D is a noncompact subgroup of both $I(H_1)$ and $I(H_2)$. If D acts minimally on $H_1(\infty)$, then it also acts minimally on $H_2(\infty)$. (Compare Theorem 5.1 of [10]. In view of Theorem 4.14 of [12] an affirmative answer to this question would imply that if a compact C^∞ manifold M has a dense geodesic flow orbit in SM_g for some nonpositive curvature metric g, then there is a dense geodesic flow orbit in SM_g for any nonpositive curvature metric g.)

Added in proof. The dissertation of Werner Ballmann, University of Bonn, 1978, strengthens certain results of this paper. Ballmann has proved the following:

THEOREM. *Suppose that some point of $H(\infty)$ is not flat, and let $D \subseteq I(H)$ be a subgroup satisfying the duality condition. Then* (1) *D acts minimally on $H(\infty)$;* (2) *the geodesic flow in $T_1 H$ is topologically mixing modulo D; that is, given open sets O, U in SH we can choose a number $T = T(O, U) > 0$ such that for each number t with $|t| \geq T$ we can find $\phi = \phi(t) \in D$ with the property that $(\phi)_* T_t(O) \cap U$ is nonempty.*

Ballmann's theorem implies immediately that if $I(H)$ satisfies the duality condition and if some point of $H(\infty)$ is not flat, then every point of $H(\infty)$ is not flat. Proposition 3.1 and Theorem 3.2 of this paper now follow directly. Moreover, Proposition 4.9 and Theorems 5.5, 5.8 and 5.15 of this paper remain true under the weaker hypothesis that some point of $H(\infty)$ is not flat. (We remark that some point of $H(\infty)$ is not flat if and only if some geodesic of H does not bound an imbedded flat half plane.) In Theorem 5.15 and Corollary 5.16, it is also true by Ballmann's result that the geodesic flow in SM is topologically mixing.

Added in proof. We have proved assertion (5) above. Details will appear elsewhere.

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