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# SPHERICAL FIBRATIONS IN ALGEBRAIC GEOMETRY

BY

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## Introduction

An important step in Quillen's proof of the Adams conjecture is the use of étale homotopy theory to construct spherical fibrations from vector bundles over schemes. The specific result needed is the following theorem of Friedlander (proved in [9, 5.3] and [8, 3.7]):

(0.1) THEOREM. *Let $V$ be a vector bundle of rank $r$ over a connected scheme $X$ (and $X$ will also denote the zero section of $V$). Let $L$ be a set of primes invertible on $X$.*

(1) *Let $F$ be the homotopy fiber of the map $\{V-X\}^\wedge \to \{X\}^\wedge$ ($\{X\}^\wedge$ is the profinite completion of the étale homotopy type of $X$). Then $(F)_L^\wedge$ ($C_L$ is the class of $L$-primary finite groups and $(\ \ )_L^\wedge$ means $C_L$-completion) is weakly homotopy equivalent to $(S^{2r-1})_L^\wedge$.*

(2) *If $\pi_1(X)^\wedge$ is $C_L$-complete and $r>1$, then the homotopy fiber of $\{V-X\}_L^\wedge \to \{X\}_L^\wedge$ is weakly homotopy equivalent to $(S^{2r-1})_L^\wedge$.*

While the second statement of (0.1) is sufficient to prove the Adams conjecture (see [9, §6] for details), there are still several questions one can ask:

(1) Is the restriction $r>1$ necessary?
(2) Is the restriction that $\pi_1(X)^\wedge$ be $C_L$-complete necessary?

This paper answers these questions completely (see (7.1)). While the answer to the first one is simply "no", the full answer to the second is quite interesting. There is a quotient $\mu_L(X, r)$ of $\pi_1(X)^\wedge$, canonically determined by the roots of unity in $H^0(X, \mathcal{O}_X^*)$, so that the homotopy fiber of $\{V-X\}_L^\wedge \to \{X\}_L^\wedge$ is weakly equivalent to $(S^{2r-1})_L^\wedge$ *if and only if* $\mu_L(X, r)$ is $C_L$-complete.

Another way to formulate this is to ask when the fibration (where $F$ is as in the first statement of (0.1))

$$F \to \{V-X\}^\wedge \to \{X\}^\wedge \tag{0.2}$$

remains a fibration after $C_L$-completion. From the above, we see that this happens precisely when $\mu_L(X, r)$ is $C_L$-complete (see (7.2)).

In many cases it is possible to compute $\mu_L(X, r)$. If $X$ is a variety over any algebraically closed field, $\mu_L(X, r) = 0$. If $X$ is geometrically connected over

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an arbitrary field $k$ and $L$ consists of one prime $l$, then $\mu_l(X, r)$ is $C_l$-complete if and only if $[k(\lambda): k] \mid r$, where $\lambda$ is a primitive $l$-*th* root of unity.

$\mu_L(X, r)$ also describes the action of $\pi_1(X)\hat{}$ on $\pi_{2r-1}((F)\hat{_L})$ coming from (0.2). In (7.5) we see that this has some surprising consequences.

In [8] and [9], Theorem (0.1) is proved by comparing the Leray spectral sequence of $\pi: V-X \to X$ to the Serre spectral sequence for (0.2). Our methods are quite different: we start from the inclusion $V-X \hookrightarrow V$. But before going any further, let us generalize the situation. Instead of the zero section of a vector bundle, we will work with a closed subscheme $Y$ of a connected scheme $X$ which is some over a base $S$. Also, Theorem (0.1) shows that we need to take homotopy fibers with respect to both profinite and $C_L$-completion. To treat both cases in a unified manner, let $L \subseteq K$ be two sets of primes (where the primes in $L$ are invertible on $X$), and let $F$ be the homotopy fiber of the map

$$\{X-Y\}\hat{_K} \to \{X\}\hat{_K}. \tag{0.3}$$

Our goal is to prove things about $(F)\hat{_L}$.

The main tool we use comes from *SGA*4. Assume that $Y$ is smooth over $S$ of relative codimension $c$ in $X$, and let $F$ be a locally constant sheaf of $\mathbf{Z}/n\mathbf{Z}$-modules on $X$ (where $n$ is invertible on $X$). Then

$$\begin{aligned} &\underline{H}^q_Y(X, F(c)) = 0 \qquad \text{for} \quad q \neq 2c, \\ &\underline{H}^{2c}_Y(X, F(c)) \simeq F\,|_Y \quad \text{(canonical isomorphism)} \end{aligned} \tag{0.4}$$

where $F(c) = F \otimes \mu_{n,X}^{\otimes c}$, $\mu_{n,X}$ the sheaf of $n$-*th* roots of unity on $X$. See [1, XVIII and XIX.3] and [7, cycle §2]. From this the spectral sequence $H^p(X, \underline{H}^q_Y(X, F(c)) \Rightarrow H^{p+q}_Y(X, F(c))$ gives us canonical isomorphisms

$$H^{q+2c}_Y(X, F(c)) \simeq H^q(Y, F\,|_Y). \tag{0.5}$$

Also, if $U_X \in H^{2c}_Y(X, \mathbf{Z}/n\mathbf{Z}(c))$ corresponds to $1 \in H^0(Y, \mathbf{Z}/n\mathbf{Z})$, then the cup product structure on the above spectral sequence gives us a commutative diagram

$$\begin{tikzcd} H^q(X, \mathbf{Z}/n\mathbf{Z}) \arrow[r, "\cup U_X"] \arrow[dr] & H^{q+2c}_Y(X, \mathbf{Z}/n\mathbf{Z}(c)) \arrow[d, "\wr"] \\ & H^q(Y, \mathbf{Z}/n\mathbf{Z}). \end{tikzcd} \tag{0.6}$$

The basic idea is that $(F)\hat{_L}$ being a completed sphere is equivalent to $\bigcup U_X$ being an isomorphism (this is the Thom isomorphism theorem and its converse due to Spivak), yet by (0.6), this is equivalent to $H^q(X, \mathbf{Z}/n\mathbf{Z}) \to H^q(Y, \mathbf{Z}/n\mathbf{Z})$ being an isomorphism (and this last condition is certainly satisfied for the zero section of a vector bundle). Of course, the presence of the twisting $\mathbf{Z}/n\mathbf{Z}(c)$ complicates matters and accounts for the role of $\mu_L(X, c)$.

We now give more details as we summarize the paper. §1 relates local cohomology $H^q_Y(X, F)$ to the relative cohomology of the map (0.3). To prove this, we must show how local cohomology is computed using hypercoverings. In §2 we show that $\pi_q((F)_L^\wedge)=1$ for $q \leq 2c-2$. A technical tool used often is the fibration (see (2.9))

$$(F)_L^\wedge \to \lim \{W - W \times_X Y\}_L^\wedge \to \lim \{W\}_L^\wedge = B \tag{0.7}$$

where $W$ runs over all pointed connected $C_K$-torseurs over $X$ (note that $\pi_1(B)=1$, so that the tools of algebraic topology apply easily). (By lim we mean inverse limit and by colim we will mean direct limit.) Then the vanishing of local cohomology in dimensions below $2c$ (*SGA*4) translates (using §1) into the desired result.

§3 defines the $L$-cyclomotic fundamental group $\mu_L(X, c)$, computes it in some cases, and gives criteria (some of which are mentioned above) for it to be $C_L$-complete. From §4 on, $Y$ is assumed to be connected and smooth over $S$. From (0.7) we get an isomorphism

$$\operatorname{Hom}(\pi_{2c-1}((F)_L^\wedge), \mathbf{Z}/n\mathbf{Z}) \simeq \operatorname{colim} H^{2c}_{W\times_X Y}(W, \mathbf{Z}/n\mathbf{Z})$$

and then (0.5), with $F = \mathbf{Z}/n\mathbf{Z}(-c)$, tells us that an isomorphism $\pi_{2c-1}((F)_L^\wedge) \simeq \hat{\mathbf{Z}}_L$ is equivalent to the $W \times_X Y$ being connected and (eventually) trivializing the sheaves $\mathbf{Z}/n\mathbf{Z}(c)$. Here the $L$-cyclotomic fundamental group enters—this last condition means an isomorphism $\mu_L(Y, c) \xrightarrow{\sim} \mu_L(X, c)_K^\wedge$! (4.2) gives the precise result, and, as shown by example (4.7), we must assume $c > 1$. In (4.8) we show that the action of $\pi_1(X)_K^\wedge$ on $\pi_{2c-1}((F)_L^\wedge)$ is determined by the action of $\mu_L(X, c)$ on $\hat{\mathbf{Z}}_L(c)$.

Theorem (5.2) gives *necessary and sufficient* conditions for $(F)_L^\wedge$ to be a completed sphere when $c > 1$. We certainly must have $\mu_L(Y, c) \xrightarrow{\sim} \mu_L(X, c)_K^\wedge$. Then, using the $W$'s of (0.7) in the diagram (0.5), we can ignore the twistings, and we get (using §1) a commutative diagram (all coefficients are $\mathbf{Z}/n\mathbf{Z}$ and $E$ denotes the middle term of (0.7)):

$$\begin{tikzcd} \operatorname{colim} H^q(W) = H^q(B) \arrow[r, "\cup U"] \arrow[dr, "r^*"'] & \operatorname{colim} H^{q+2c}_{W\times_X Y}(W) = H^{q+2c}(B, E) \\ & \operatorname{colim} H^q(W \times_X Y) \arrow[u, "\wr"'] \end{tikzcd} \tag{0.8}$$

Now the Thom isomorphism theorem (and Spivak's converse to it) apply to the top line of (0.8): $\bigcup U$ is an isomorphism if and only if $(F)_L^\wedge$ is a completed sphere. This is equivalent to $r^*$ being an isomorphism, and with a little care, we see that this means that each $H^q(W) \to H^q(W \times_X Y)$ is an isomorphism. This is what we call a strong cohomological $L$-isomorphism (the definition given in (5.1) is in terms of the cohomology of certain sheaves on $X$). §5 ends with some examples and useful special cases of (5.2).

§6 deals with the case $c = 1$, which is much harder because of the fundamental group of $(F)_L^\wedge$. We assume that $K = L$ and give only sufficient conditions for $F$ to be a completed circle. A new phenomena in this case is the existence of locally constant sheaves $F$ on $X-Y$ which don't extend to $X$. The cohomological condition required involves the sheaves $j_*F$ and is stronger than the conditions found in §5 (see (6.1) and (6.3)). It implies that certain branched covers $\bar{V}$ of $X$ (with $Y$ as branch locus) give strong cohomological $L$-isomorphisms $\bar{V}\times_X Y \to \bar{V}$ (see 6.4) and (6.7)). These give us a map

$$\lim\{\bar{V}-\bar{V}\times_X Y\}_L^{\wedge} \to \lim\{\bar{V}\}_L^{\wedge} \tag{0.9}$$

which is weakly equivalent to the map $\tilde{E}\to B$, where $\tilde{E}$ is the universal cover of the middle term of (0.7). The fiber of (0.9) is a simple space, so that §5 implies it is a completed circle. Then the result for $F$ follows easily.

The results of §7 are summarized above.

All schemes that appear are assumed to be locally noetherian, and point always means geometric point. Sheaf means sheaf in the étale topology, and cohomology means étale cohomology.

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## 1. Local cohomology and relative cohomology

If $X$ is a pointed scheme, $HR(X)$ is the category of pointed hypercoverings $V.$ of $X$, where morphisms are homotopy classes of *special* maps $\bar{V}.\to V.$ between them (see [2, §8] and [9, §1] for the definitions of hypercovering and special map). The etale homotopy type of $X$, $\{X\}_{et}$, is the object of Pro-$\mathcal{H}$ ($\mathcal{H}$ is the pointed homotopy category) defined by

$$\{X\}_{et}=\{\pi(V.)\}_{V.\in HR(X)}$$

(where $\pi$ is the connected component functor). A locally constant sheaf $F$ on $X$ gives a local system (denoted $F$) on $\{X\}_{et}$, and there is an isomorphism

$$H^q(X,F)\simeq H^q(\{X\}_{et},F) \tag{1.1}$$

because $H^q(X,F)$ can be computed using hypercoverings (see [2, §§8–10]).

We want to generalize this to local cohomology. Thus, assume that we have a closed subscheme $Y$ of a scheme $X$, and that $X$ has a geometric point $\xi$ which lies in $U=X-Y$. Let $\mathcal{T}$ be the category of simplicial maps $F.$: $W.\to V.$ between pointed hypercoverings $W.$ of $U$ and $V.$ of $X$, where a morphism of $\mathcal{T}$ is a commutative square

$$\begin{array}{ccc} \bar{W}. & \longrightarrow & W. \\ {\scriptstyle \bar{f}.}\downarrow & & \downarrow{\scriptstyle f.} \\ \bar{V}. & \longrightarrow & V. \end{array}$$

(where the top and bottom maps are special). A simplification of the proof of Lemma 1.4 of [9] yields the following:

(1.2) LEMMA. *Let $J$ be the homotopy category of $\mathcal{T}$.*
(1) *$J$ is filtering.*
(2) *The functor $(f_\bullet\colon W_\bullet \to V_\bullet) \to W_\bullet$ from $J$ to $HR(U)$ is cofinal.*
(3) *The functor $(f_\bullet\colon W_\bullet \to V_\bullet) \to V_\bullet$ from $J$ to $HR(X)$ is cofinal.* □

Given an abelian sheaf $F$ on $X$, we need to associate to each $f_\bullet\colon W_\bullet \to V_\bullet$ in $J$ a complex whose cohomology (in the limit) is $H^q_Y(X, F)$. This is the first step in generalizing (1.1). We start with the mapping cylinder $M(f_\bullet)$ of $f_\bullet$, which is defined by the push-out diagram

(1.3)
$$\begin{tikzcd}
W_\bullet \arrow[r, "\alpha_0"] \arrow[d, "f_\bullet"'] & W_\bullet \times \Delta[1] \arrow[d, "p"] \arrow[dr, "p_1"] & \\
V_\bullet \arrow[r, "i"] \arrow[drr, bend right, "1V_\bullet"'] & M(f_\bullet) \arrow[dr, dashed, "r"] & W_\bullet \arrow[d, "f_\bullet"] \\
 & & V_\bullet
\end{tikzcd}$$

where $\Delta[1]$ is the usual 1-simplex and $\alpha_0$, $\alpha_1$: $W_\bullet \to W_\bullet \times \Delta[1]$ are the two inclusions. The map $r$ of (1.3) exists by the universal property of $M(f_\bullet)$, and it is well known that $r$ is a homotopy inverse of $i$. Let $\beta = p \circ \alpha_1$: $W_\bullet \to M(f_\bullet)$. Then $\beta$ is a monomorphism and we have a commutative diagram

(1.4)
$$\begin{tikzcd}
W_\bullet \arrow[rr, "\beta"] \arrow[dr, "f_\bullet"'] & & M(f_\bullet) \arrow[dl, "r"] \\
 & V_\bullet &
\end{tikzcd}$$

Any étale map $X' \to X$ gives a sheaf $\mathbf{Z}_{X'}$ on $X$ which is the constant sheaf $\mathbf{Z}$ on $X'$ and zero elsewhere (recall that $\operatorname{Ext}^q(\mathbf{Z}_{X'}, F) \simeq H^q(X', F)$). Going from $X'$ to $\mathbf{Z}_{X'}$ is functorial, so that $f_\bullet\colon W_\bullet \to V_\bullet$ in $\mathcal{T}$ gives us simplicial abelian sheaves $\mathbf{Z}_{W_\bullet}$, $\mathbf{Z}_{V_\bullet}$ and $\mathbf{Z}_{M(f_\bullet)}$. Using the map $\beta$ from (1.4), we define another simplicial abelian sheaf

(1.5)
$$K(f_\bullet) = \operatorname{coker}(\mathbf{Z}_\beta\colon \mathbf{Z}_{W_\bullet} \to \mathbf{Z}_{M(f_\bullet)}).$$

Let $F$ be an abelian sheaf on $X$. Then each $f_\bullet$ in $\mathcal{T}$ gives us a cosimplicial abelian group $\operatorname{Hom}(K(f_\bullet), F)$, which becomes a chain complex in the usual way. For $q \geq 0$, the functor $f_\bullet \to H^q(\operatorname{Hom}(K(f_\bullet), F))$ factors through $J$.

The chain complexes $\operatorname{Hom}(K(f_\bullet), F)$, built out of hypercoverings, are what we want:

(1.6) PROPOSITION. *There is a canonical isomorphism*

$$H^q_Y(X, F) \simeq \operatorname*{colim}_{f\in J^0} H^q(\operatorname{Hom}(K(f.), F))$$

*for any abelian sheaf on X.*

*Proof.* Let $\mathbf{Z}_Y$ denote the sheaf $i_*i^*\mathbf{Z}_X$ on $X$ ($i\colon Y \to X$ is the inclusion). Then, for $f.\colon W. \to V.$ in $\mathcal{T}$, (1.4), (1.5) and the injectivity of $\mathbf{Z}_\beta$ give us a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}_{W.} \arrow[r, "\mathbf{Z}_\beta"] \arrow[dd, "a"] \arrow[dr, "\mathbf{Z}_{f.}"'] & \mathbf{Z}_{M(f.)} \arrow[r] \arrow[d, "\mathbf{Z}_r"] & K(f.) \arrow[r] \arrow[dd, dashed, "c"] & 0 \\
& & \mathbf{Z}_{V.} \arrow[d, "b"] & & \\
0 \arrow[r] & \mathbf{Z} \arrow[r] & \mathbf{Z}_X \arrow[r] & \mathbf{Z}_Y \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact (which says that the dotted arrow $c$ exists). The map $a$ is a quasi-isomorphism because $W.$ is a hypercovering of $U$, and similarly for $b$. Also $\mathbf{Z}_r$ is a quasi-isomorphism since $r$ is a homotopy equivalence. Thus, by the 5-lemma, $c$ is a quasi-isomorphism.

Thus $\operatorname{Ext}^q(K(f.), F) \simeq \operatorname{Ext}^q(\mathbf{Z}_Y, F) \simeq H^q_Y(X, F)$ (see [2, § V.6]), so that the usual hyperext spectral sequence becomes a spectral sequence $E_r^{p,q}(f.)$:

$$E_1^{p,q}(f.) = \operatorname{Ext}^q(K(f.)_p, F) \Rightarrow H_Y^{p+q}(X, F).$$

This is functorial with respect to $\mathcal{T}$ and, from $E_2$ on, with respect to $J$.

Since $E_1^{p,0}(f.) = \operatorname{Hom}(K(f.)_p, F)$, we see that

$$E_2^{p,0}(f.) = H^q(\operatorname{Hom}(K(f.), F)).$$

Thus, the proposition reduces to proving that $\operatorname{colim}_{f.\in J^0} E_2^{p,q}(f.) = 0$ for $q > 0$. This will be true if we can prove the following:

(1.7) For $q > 0$ and $x \in E_1^{p,q}(f.)$, there is a map $\bar{f}. \to f.$ in $\mathcal{T}$ such that $x$ goes to 0 in $E_1^{p,q}(\bar{f}.)$.

Let's analyze $K(f.)_p$. Since $(W. \times \Delta[1])_p$ consists of $n+2$ copies of $W_p$, it follows from (1.4) and (1.5) that we have a functorial isomorphism $K(f.)_p \simeq \mathbf{Z}_{V_p} \oplus \sum_1^p \mathbf{Z}_{W_p}$. This gives us isomorphisms (also functorial)

$$\begin{aligned} E_1^{p,q}(f.) = \operatorname{Ext}^q(K(f.)_p, F) &\simeq \operatorname{Ext}^q(\mathbf{Z}_{V_p}, F) \oplus \sum_1^p \operatorname{Ext}^q(\mathbf{Z}_{W_p}, F) \\ &\simeq H^q(V_p, F) \oplus \sum_1^p H^q(W_p, F), \end{aligned}$$

so we need only prove the two analogs of (1.7) in which $E_1^{p,q}$ is replaced first by $H^q(V_p, F)$ and then by $H^q(W_p, F)$.

For $x \in H^q(V_p, F)$, the local vanishing of cohomology and Lemma 2.3 of [9] give us a special map $\bar{V}_\bullet \to V_\bullet$ in $HR(X)$ so that $x$ goes to zero in $H^q(\bar{V}_p, F)$. Then statement 3 of (1.2) gives us the desired map in $\mathcal{T}$. The proof for $x \in H^q(W_p, F)$ is similar. □

The next step in generalizing (1.1) is to interpret $\operatorname{Hom}(K(f_\bullet), F)$, for $F$ locally constant on $X$, as a complex used to compute (again in the limit) the relative cohomology of the map $j_{et}\colon \{U\}_{et} \to \{X\}_{et}$ induced by the inclusion $j\colon U = X - Y \to X$.

We recall the definition of relative cohomology. The crucial fact is that $j_{et}$ is in Pro-$\mathcal{H}_{\text{pairs}}$, i.e., $j_{et}$ is represented by the maps $\pi(f_\bullet)\colon \pi(W_\bullet) \to \pi(V_\bullet)$ indexed by $f_\bullet\colon W \to V_\bullet$ in $J$ (this is true by (1.2)). We form the mapping cylinder of each $\pi(f_\bullet)$, getting a commutative diagram (which is functorial in $f_\bullet$)

$$\begin{tikzcd} \pi(W_\bullet) \arrow[r, "\beta"] \arrow[dr, "\pi(f_\bullet)"'] & M(\pi(f_\bullet)) \arrow[d, "r"] \\ & \pi(V_\bullet) \end{tikzcd}$$

where $\beta$ is an inclusion and $r$ is a homotopy equivalence. Note that $M(\pi(f_\bullet)) = \pi(M(f_\bullet))$ (see (1.4)).

Let $F$ be a locally constant sheaf on $X$, and let $J_F$ be the full subcategory of $J$ consisting of those $f_\bullet\colon W_\bullet \to V_\bullet$ for which $F$ becomes trivial on $V_0$. $J_F$ is cofinal in $J$. Then $F$ gives a local system (called $F$) on $\{X\}_{et}$ which consists of the local systems $\pi(V_\bullet \times F)$ on $\pi(V_\bullet)$ for $f_\bullet\colon W_\bullet \to V_\bullet$ in $J_F$ (see [2, §10]). For relative cohomology, $F$ gives the local system $\pi(M(f_\bullet) \times F)$ on $\pi(M(f_\bullet))$, and relative cohomology itself is defined to be

$$H^q(\{X\}_{et}, \{X-Y\}_{et}; F) = \operatorname*{colim}_{f_\bullet \in J^0} H^q(\pi(M(f_\bullet)), \pi(W_\bullet); \pi(M(f_\bullet) \times F)).$$

Our generalization of (1.1) is

(1.8) Proposition. *For a locally constant abelian sheaf $F$ on $X$, there is a canonical isomorphism $H_Y^q(X, F) \simeq H^q(\{X\}_{et}, \{X-Y\}_{et}; F)$.*

*Proof.* Let $F(f_\bullet)$ denote the local system $\pi(M(f_\bullet) \times F)$. By (1.6), all we have to do is find a canonical isomorphism $H^q(\operatorname{Hom}(K(f_\bullet), F)) \simeq H^q(\pi(M(f_\bullet)), \pi(W_\bullet); F(f_\bullet))$ for $f_\bullet$ in $J_F$.

From [2, §10] we see that the chain complex

$$F(M(f_\bullet)) \simeq \operatorname{Hom}(\mathbf{Z}_{M(f_\bullet)}, F)$$

is isomorphic to the complex $C^*(\pi(M(f_\bullet)), F(f_\bullet))$ used to compute

$$H^q(\pi(M(f_\bullet)), F(f_\bullet)).$$

Similarly $F(W.) \simeq \mathrm{Hom}\,(\mathbf{Z}_{W.}, F) \simeq C^*(\pi(W.), F(f.)|_{\pi(W.)})$. The complex used to compute $H^q(\pi(M(f.)), \pi(W.); F(f.))$ is the kernel of the map

$$C^*(\pi(M(f.)), F(f.)) \to C^*(\pi(W.), F(f.)|_{\pi(W.)})$$

(see [11, 4.2]) which by the above is the kernel of the map

$$\mathrm{Hom}\,(\mathbf{Z}_{M(f.)}, F) \to \mathrm{Hom}\,(\mathbf{Z}_{W.}, F).$$

By (1.5), this kernel is $\mathrm{Hom}\,(K(f.), F)$. □

This proposition is true in other contexts as well. For example, if $Y$ is a closed subset of a topological space $X$ and $F$ is a locally constant sheaf on $X$, then there is an isomorphism

$$H^q(X, X-Y; F) \simeq H^q_Y(X, F)$$

(see [3, II.12]). There is also a version of (1.8) for simplicial schemes:

(1.9) PROPOSITION. *Let* $Y \subseteq X$ *be as in* (1.8), *and let* $V.$ *be a simplicial scheme in* $X_{\mathrm{et}}$. *Then, for any locally constant sheaf* $F.$ *on* $V.$, *there is a canonical isomorphism*

$$H^q_{V.\times_X Y}(V., F.) \simeq H^q(\{V.\}_{\mathrm{et}}, \{V. - V.\times_X Y\}_{\mathrm{et}}; F.). \qquad \square$$

The relevant definitions can be found in [6, Chapters I–II], and the proof is identical to the proof of (1.8). We use (1.9) often in [5].

## 2. The connectivity of the fiber

We need some notation for the various kinds of completions we deal with. If $K$ is a set of primes, $C_K$ denotes the class of finite $K$-groups, and $(\ )\hat{}_K$ is the $C_K$-completion of a group or space. When $X$ is a scheme, $(\{X\}_{\mathrm{et}})\hat{}_K$ is written $\{X\}\hat{}_K$ (we also write $\pi_1(X, \xi)$ instead of $\pi_1(\{X\}_{\mathrm{et}}, \xi)$).

The basic situation we consider is a commutative diagram

(2.1)
$$\begin{tikzcd} Y \arrow[rr, hook, "i"] \arrow[dr] & & X \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

where $X$ is pointed and connected, $i$ is a closed immersion (where the point $\xi$ of $X$ misses $Y$), and $f$ is smooth. $Y$ has relative codimension $\geq c$ in $X$ if for every $s \in S$, $Y_s$ has codimension $\geq c$ in $X_s$.

Our main object of study is the homotopy fiber $F$ of the map

$$\{X - Y\}\hat{}_K \to \{X\}\hat{}_K.$$

In this section we translate the following facts about (2.1) into information about $F$:

(2.2) If the relative codimension of $Y$ in $X$ is $\geq 1$ (resp. $>1$), then the map $\pi_1(X-Y, \xi) \to \pi_1(X, \xi)$ is surjective (resp. bijective).

(2.3) If $Y$ has relative codimension $\geq c$ in $X$ and $n$ is invertible on $X$, then $H^q_Y(X, \mathbf{Z}/n\mathbf{Z}) = 0$ for $q < 2c$.

Statement (2.2) follows easily from [1, XVI 3.2.1 and 3.3], and (2.3) follows from [1, XVIII 3.1.7] or [7, Cycle 2.2].

The result we get is the following:

(2.4) PROPOSITION. *Let $Y \subseteq X$ be as in* (2.1), *where $Y$ has relative codimension $\geq c$ in $X$. Let $L \subseteq K$ be two sets of primes, and let $F$ denote the homotopy fiber of the map $\{X-Y\}\hat{_K} \to \{X\}\hat{_K}$. Then $\pi_q((F)\hat{_L}) = 0$ for $q \leq 2c-2$ if every prime of $L$ is invertible on $X$.*

*Proof.* We first step up some machinery to avoid $\pi_1(X)$ (which also will be useful later on). Recall that a map $X_1 \to X_2$ in Pro-$\mathscr{H}_0$ is a $\#$-isomorphism if the induced maps $\pi_q(X_1) \to \pi_q(X_2)$ are isomorphisms for all $q \geq 0$.

Let $J$ be the inverse system of pointed connected $G$-torseurs $W \to X$, as $G$ ranges over $C_K$ (thus $J$ is the category of surjective maps $\phi\colon \pi_1(X) \to G$—see [2, §10]). Given such a $G$-torseur $W \to X$, Lemma 4.4 of [9] shows that $\{W\}_{\text{et}}$ is $\#$-isomorphic to the covering space of $\{X\}_{\text{et}}$ determined by $\ker(\phi) \subseteq \pi_1(X)$. Since $G$ is in $C_K$, [2,4.11] says that $\{W\}\hat{_K}$ is $\#$-isomorphic to the covering space of $\{X\}\hat{_K}$ determined by $\ker(\phi\hat{_K})$, i.e., we have a fibration

$$\{W\}\hat{_K} \longrightarrow \{X\}\hat{_K} \xrightarrow{S(\phi)} K(G, 1) \tag{2.5}$$

where $S(\phi)$ is the map determined by $\phi\hat{_K}\colon \pi_1(X)\hat{_K} \to G$. Since the maps $S(\phi)$ describe the map $\{X\}\hat{_K} \to K(\pi_1(X)\hat{_K}, 1)$, we see that $\lim_J \{W\}\hat{_K}$ is $\#$-isomorphic to the universal cover of $\{X\}\hat{_K}$ ($\lim_J \{W\}\hat{_K}$ means organizing all of the inverse systems $\{W\}\hat{_K}$ into one large inverse system—see [2, A.4.4] for details).

Given $\phi\colon \pi_1(X) \to G$ in $J$, we see by (2.2) that the composition

$$\pi_1(X-Y) \to \pi_1(X) \to G$$

(which classifies the $G$-torseur $W - W \times_X Y$ over $X-Y$) is surjective. The argument used for (2.5) then gives us a fibration

$$\{W - W \times_X Y\}\hat{_K} \to \{X-Y\}\hat{_K} \to K(G, 1). \tag{2.6}$$

From (2.5) and (2.6) one shows (using the $3 \times 3$ lemma of [2, §1]) that we have a fibration

$$F \to \{W - W \times_X Y\}\hat{_K} \to \{W\}\hat{_K}. \tag{2.7}$$

Since this is true for any $W \to X$ in $J$, we can put all of these together to get a fibration

$$F \to \lim_J \{W - W \times_X Y\}_K^{\wedge} \to \lim_J \{W\}_K^{\wedge}, \tag{2.8}$$

and since $\lim_J \{W\}_K^{\wedge}$ is simply connected, (2.8) remains a fibration after $C_L$-completion (see [8, 4.1]). Thus we get a fibration

$$(F)_L^{\wedge} \to \lim_J \{W - W \times_X Y\}_L^{\wedge} \to \lim_J \{W\}_L^{\wedge}$$

which we will write as

$$(F)_L^{\wedge} \to E \to B. \tag{2.9}$$

The relative cohomology of the map $E \to B$ is easy to compute. Let $G$ be an abelian group in $C_L$. Then

$$\begin{aligned} H^q(B, E; G) &= \operatorname*{colim}_{J^0} H^q(\{W\}_L^{\wedge}, \{W - W \times_X Y\}_L^{\wedge}; G) \\ &\simeq \operatorname*{colim}_{J^0} H^q(\{W\}_{\mathrm{et}}, \{W - W \times_X Y\}_{\mathrm{et}}; G) \quad (\text{since } G \in C_L) \\ &\simeq \operatorname*{colim}_{J^0} H^q_{W \times_X Y}(W, G) \quad (\text{by } (1.8)). \end{aligned} \tag{2.10}$$

Since each $W \to X$ is étale, $W$ is smooth over $S$ and $W \times_X Y$ has relative codimension $\geq c$ in $W$. Thus, by (2.2), $H^q(B, E; G) = 0$ for $q \leq 2c - 1$. We can relate this to $(F)_L^{\wedge}$ by the relative Serre spectral sequence

$$E_2^{p,q} = H^{p,q} = H^p(B, \mathcal{H}^q(\bar{F}, (F)_L^{\wedge}; G)) \Rightarrow H^{p+q}(B, E; G) \tag{2.11}$$

(where $\bar{F}$ is the cone over $(F)_L^{\wedge}$). Since $\pi_1(B) = 1$, the local system $\mathcal{H}^q(\bar{F}, (F)_L^{\wedge}; G)$ is just the constant group $H^{q-1}((F)_L^{\wedge}, G)$. Then the vanishing of $H^q(B, E; G)$ for $q \leq 2c - 1$ and (2.11) tell us that

$$\begin{aligned} H^q((F)_L^{\wedge}, G) &= 0 \quad \text{for} \quad q \leq 2c - 2, \\ H^{2c-1}((F)_L^{\wedge}, G) &\simeq H^{2c}(B, E; G). \end{aligned} \tag{2.12}$$

Now (2.4) is easy. The case $c = 1$ follows from (2.12). When $c > 1$, (2.2) tells us that $\pi_1(E) \simeq \pi_1(B) \cong 1$, so that $\pi_1((F)_L^{\wedge})$ is abelian (being a quotient of $\pi_2(B)$). Thus, to show $\pi_1((F)_L^{\wedge}) = 0$, we need only show that $\operatorname{Hom}(\pi_1((F)_L^{\wedge}), G) \simeq H^1((F)_L^{\wedge}, G)$ is zero for any abelian group $G \in C_L$. This is true by (2.12).

Assume that $\pi_1((F)_L^{\wedge}) = \cdots = \pi_{q-1}((F)_L^{\wedge}) = 0$ for some $q \leq 2c - 2$. Then the Hurewicz theorem and the universal coefficient theorem imply that for $G$ in $C_L$,

$$\operatorname{Hom}(\pi_q((F)_L^{\wedge}), G) \simeq H^q((F)_L^{\wedge}, G) \tag{2.13}$$

and the latter group is zero by (2.12). Since $\pi_q((F)_L^{\wedge})$ is $C_L$-complete, it must be zero. □

### 3. The $L$-cyclotomic fundamental group

Let $\xi$: Spec $(\Omega) \to X$ be a point of a connected scheme $X$. If $n$ is invertible on $X$, the locally constant sheaf $\mathbf{Z}/n\mathbf{Z}(c)$ is described by an action of $\pi_1(X, \xi)$ on $\mu_n(\Omega)^{\otimes c}$.

Let $L$ be a set of primes invertible on $X$. A number $n$ is called an $L$-number if every prime divisor of $n$ lies in $L$, and let $\hat{\mathbf{Z}}_L(c)$ denote the pro-group $\{\mu_n(\Omega)^{\otimes c}\}_{n \text{ an } L\text{-number}}$. Then we get an action of $\pi_1(X, \xi)$ on $\hat{\mathbf{Z}}_L(c)$, i.e., a map

$$\pi_1(X, \xi) \to \operatorname{Aut}(\hat{\mathbf{Z}}_L(c)) \simeq \hat{\mathbf{Z}}_L^*. \tag{3.1}$$

(3.2) Definition. $\mu_L(X, c)$, the $L$-cyclotomic fundamental group of $X$, is the pro-group which is the image of the map $\pi_1(X, \xi) \to \mathbf{Z}_L^*$.

One sees easily that $\mu_L(X, c)$ is just the image of $\mu_L(X, 1)$ under the map

$$\hat{\mathbf{Z}}_L^* \xrightarrow{c^{\text{th}}\text{ power}} \hat{\mathbf{Z}}_L^*.$$

Also $\mu_L(X, 1)$ is trivial if and only if $H^0(X, \mathcal{O}_X^*)$ has a primitive $n^{\text{th}}$ root of unity for every $L$-number $n$.

$\mu_L(X, c)$ gets its name from the following result.

(3.3) Lemma. *Let $X$ be a pointed scheme which is geometrically connected over a field $k$. The point $\xi$:* Spec $(\Omega) \to X$ *gives an embedding $k \to \Omega$, and we set $K = k$(all $n^{\text{th}}$ roots of unity in $\Omega$, $n$ any $L$-number). Then $\mu_L(X, 1) \simeq$* Gal $(K/k)$ *(and so $\mu_L(X, c)$ depends only on $k$).*

*Proof.* The sheaf $\mu_{n,k}$ is classified by the obvious map

$$\pi_1(\operatorname{Spec}(k), \xi) = \operatorname{Gal}(\Omega/k) \to \operatorname{Gal}(k(\mu_n(\Omega))/k) \subseteq \operatorname{Aut}(\mu_n(\Omega)).$$

Set $G_n = \operatorname{Gal}(k(\mu_n(\Omega))/k)$. Since $\mu_{n,X}$ is the pullback of $\mu_{n,k}$ via the map $X \to \operatorname{Spec}(k)$, $\mu_{n,X}$ is classified by the composed map

$$\phi_n : \pi_1(X, \xi) \twoheadrightarrow G_n \operatorname{Aut}(\mu_n(\Omega)).$$

We need only show that each $\phi_n$ is onto.

Spec $(k(\mu_n(\Omega))$ is the $G_n$-torseur over Spec $(k)$ classified by the above map Gal $(\Omega/k) \to G_n$. Thus, $X \times_k k(\mu_n(\Omega))$ is the $G_n$-torseur classified by $\phi_n$. Since $X$ is geometrically connected, $X \times_k k(\mu_n(\Omega))$ is connected and $\phi_n$ is onto. □

Later on we will need to know when $\mu_L(X, c)$ is $C_L$-complete. Here are some cases when this is true:

(3.4) Lemma. (1) *Suppose that $(l-1)/(l-1, c)$ is an $L$-number for every $l \in L$. Then $\mu_L(X, c)$ is $C_L$-complete.*

(2) *Suppose that $X$ is geometrically connected over a field $k$ and that $L$ consists of one prime $l$. Then $\mu_L(X, c)$ is $C_L$-complete if and only if $[k(\lambda): k] \mid c$, where $\lambda$ is a primitive $l^{\text{th}}$ root of unity.*

*Proof.* If $l$ is an odd prime, then we have

$$\mu_l(X, 1) \hookrightarrow \hat{\mathbf{Z}}_l^* \simeq \hat{\mathbf{Z}}_l \oplus \mathbf{Z}/(l-1)\mathbf{Z}. \tag{3.5}$$

Since $\mu_l(X, c)$ is the image of this under multiplication by $c$, $\mu_l(X, c) \hookrightarrow \hat{\mathbf{Z}}_l \oplus$ (the subgroup of $\mathbf{Z}/(l-1)\mathbf{Z}$ generated by $c$), and this cyclic group has order $(l-1)/(l-1, c)$. Then the first statement follows from the inclusion $\mu_L(X, c) \to \Pi_{l \in L}\, \mu_l(X, c)$.

To prove the second statement, note that $\mu_l(X, 1) \simeq \mathrm{Gal}\,(K/k)$, $K = k$ (all $l^n$-*th* roots of 1) by (3.3). Then Galois theory says that the image of $\mu_l(X, 1)$ in $\mathbf{Z}/(l-1)\mathbf{Z}$ in (3.5) is $\mathrm{Gal}\,(k(\lambda)/k)$. This is a cyclic group of order $[k(\lambda)\colon k]$. Then $\mu_l(X, c)$ is $C_l$-complete if and only if multiplication by $c$ kills this, i.e., $[k(\lambda)\colon k] \mid c$. □

If $L = \{l\}$, then the first condition reduces to $(l-1) \mid c$. Also, if $l-1$ is an $L$-number for every $l \in L$, then $\mu_L(X, c)$ is $C_L$-complete for every $c$ (for example, $L = \{2, 3, 7\}$). Some other examples where the first condition is satisfied are $L = \{3, 7\}$, $c = 2$; $L = \{3, 5, 7\}$, $c = 4$.

## 4. The first non-vanishing homotopy group

To get results stronger than Proposition (2.4), we must strengthen our hypotheses. For the rest of the paper we will deal with a commutative diagram

$$\begin{tikzcd} Y \arrow[rr, hook, "i"] \arrow[dr] & & X \arrow[dl] \\ & S & \end{tikzcd} \tag{4.1}$$

where now both $X$ and $Y$ are connected and smooth over $S$ (we say that $Y \to X$ is a smooth connected couple). We also assume that $X$ has a point $\xi$, missing $Y$, which specializes to a point $\eta$ of $Y$. Thus, we have pointed immersions $i\colon Y \to X$ and $j\colon X - Y \to X$.

Now we can say something about $\pi_{2c-1}((F)_L^\wedge)$:

(4.2) Proposition. *Let $Y \to X$ be a smooth connected couple of relative codimension $c > 1$, and let $L \subseteq K$ be two sets of primes where every prime in $L$ is invertible on $X$. If $F$ is the homotopy fiber of the map $\{X - Y\}_K^\wedge \to \{X\}_K^\wedge$, then the following are equivalent*:

(1) *There is an isomorphism $\pi_{2c-1}((F)_L^\wedge) \simeq \hat{\mathbf{Z}}_L(c)$.*

(2) *$\pi_1(Y)_K^\wedge \to \pi_1(X)_K^\wedge$ is surjective and the map $\mu_L(Y, c) \to \mu_L(X, c)$ induces an isomorphism $\mu_L(Y, c) \xrightarrow{\sim} \mu_L(X, c)_K^\wedge$.*

*Proof.* The second statement can be interpreted as follows. To say that $\pi_1(Y)_K^\wedge \to \pi_1(X)_K^\wedge$ is surjective means that for every connected $C_K$-torseur

$W$ over $X$, $W\times_X Y$ is also connected. And the isomorphism $\mu_L(Y,c)\xrightarrow{\sim}\mu_L(X,c)\hat{}_K$ means that for every $L$-number $n$, there is a connected $C_K$-torseur $W$ over $X$ such that $\mathbf{Z}/n\mathbf{Z}(c)$ becomes trivial over $W\times_X Y$.

We will use the notation of the proof of (2.4), specifically the category $J$ and (2.9)–(2.13). Let $n$ be an $L$-number. Then we have isomorphisms

$$\begin{aligned}\operatorname{Hom}(\pi_{2c-1}((F)\hat{}_L),\mathbf{Z}/n\mathbf{Z}) &\simeq H^{2c-1}((F)\hat{}_L\mathbf{Z}/n\mathbf{Z}) && \text{by (2.13)}\\ &\simeq H^{2c}(B,E;\mathbf{Z}/n\mathbf{Z}) && \text{by (2.12)}\\ &\simeq \operatorname*{colim}_{J^0} H^{2c}_{W\times_X Y}(W,\mathbf{Z}/n\mathbf{Z}). && \text{by (2.10)}\end{aligned}$$

For $W$ in $J$, $W\times_X Y\to W$ is a smooth couple over $S$, so that by (0.5) (with $F=\mathbf{Z}/n\mathbf{Z}(-c)$), we have an isomorphism

$$H^{2c}_{W\times_X Y}(W,\mathbf{Z}/n\mathbf{Z})\simeq H^0(W\times_X Y,\mathbf{Z}/n\mathbf{Z}(-c)). \tag{4.3}$$

Thus we have a canonical isomorphism

$$\operatorname{Hom}(\pi_{2c-1}((F)\hat{}_L),\mathbf{Z}/n\mathbf{Z})\simeq\operatorname*{colim}_{J^0} H^0(W\times_X Y,\mathbf{Z}/n\mathbf{Z}(-c)). \tag{4.4}$$

Now we prove (1) implies (2). If $\pi_{2c-1}((F)\hat{}_L)\simeq\hat{\mathbf{Z}}_L(c)$ ($\simeq\hat{\mathbf{Z}}_L$, non-canonically), then the right hand side of (4.4) has $n$ elements. Then one can find a cofinal family $\{W'\}$ of $J$ for which each

$$H^0(W'\times_X Y,\mathbf{Z}/n\mathbf{Z}(-c))$$

has $n$ elements.

Since $W'\times_X Y$ is a torseur over $W'$, all of its connected components are isomorphic. Thus the order $n$ of $H^0(W'\times_X Y,\ \mathbf{Z}/n\mathbf{Z}(-c))$ is some number raised to the power $\#\pi(W'\times_X Y)$. If we let $n$ be a prime in $L$, this says that $W'\times_X Y$ must be connected. Since the $W'$ are cofinal in $J$, all of the $W\times_X Y$ are connected. Thus $\pi_1(Y)\hat{}_K\to\pi_1(X)\hat{}_K$ is onto.

But then, since $W'\times_X Y$ is connected and $\mathbf{Z}/n\mathbf{Z}(-c)$ is locally constant, $H^0(W'\times_X Y,\mathbf{Z}/n\mathbf{Z}(-c))$ can have $n$ elements only when $\mathbf{Z}/n\mathbf{Z}(-c)$ (and hence $\mathbf{Z}/n\mathbf{Z}(c)$) is trivial on $W'\times_X Y$. Thus we have found a $C_K$-torseur $X$ on $X$ with $\mathbf{Z}/n\mathbf{Z}(c)$ trivial on $W\times_X Y$.

Next, we prove (2) implies (1). The specialization $\xi\to\eta$ says for $h\colon W\to X$ in $J$, there is a bijection $h^{-1}(\eta)\xrightarrow{\sim}h^{-1}(\xi)$. Since $W$ has a point lying over $\xi$ (it's a pointed torseur), $W$ acquires a point $\eta'$ lying over $\eta$, and there is an isomorphism between the stalk of $\mathbf{Z}/n\mathbf{Z}(-c)$ on $W\times_X Y$ at $\eta'$ and $\operatorname{Hom}(\mu_n(\Omega)^{\otimes c},\mathbf{Z}/n\mathbf{Z})$. This gives us a homomorphism

$$H^0(W\times_X Y,\mathbf{Z}/n\mathbf{Z}(-c))\to\operatorname{Hom}(\mu_n(\Omega)^{\otimes c},\mathbf{Z}/n\mathbf{Z}) \tag{4.5}$$

which is compatible with morphisms in $J$. Thus from (4.4) we get a homomorphism

$$\begin{aligned}&\operatorname{Hom}(\pi_{2c-1}((F)\hat{}_L),\mathbf{Z}/n\mathbf{Z})\\ &\qquad\to\operatorname{Hom}(\mu_n(\Omega)^{\otimes c},\mathbf{Z}/n\mathbf{Z})\simeq\operatorname{Hom}(\hat{\mathbf{Z}}_L(c),\mathbf{Z}/n\mathbf{Z})\end{aligned} \tag{4.6}$$

Our hypothesis tells us that the $W\times_X Y$ are connected and $\mathbf{Z}/n\mathbf{Z}(-c)$ trivializes on some $W\times_X Y$. Thus, (4.5) is an isomorphism for a cofinal family of $W$'s, so that (4.6) is an isomorphism. Since $c>1$, $\pi_{2c-1}((F)_L^\wedge)$ is abelian, and so (4.6) implies that $\pi_{2c-1}((F)_L^\wedge)\simeq\hat{\mathbf{Z}}_L(c)$. □

The proposition is not true when $c=1$. Here is an example.

(4.7) *Example.* Let $C$ be the curve in $\mathbf{P}^2(\mathbf{C})$ defined by

$$(x^2+y^2)^3+(y^3+z^3)^2=0.$$

Zariski (12) has shown that $\pi_1(\mathbf{P}^2-C)\simeq(\mathbf{Z}/2\mathbf{Z})*(\mathbf{Z}/3\mathbf{Z})$ (this is the topological fundamental group). Let $C_0$ be the singular points of $C$, and set $X=\mathbf{P}^2-C_0$, $Y=C-C_0$. Since $\pi_1(X)=1$, $YX$ satisfies the conditions of (4.2) for $K=L=\{\text{all primes}\}$. Since $\pi_1(X-Y)^\wedge\simeq((\mathbf{Z}/2\mathbf{Z})*(\mathbf{Z}/3\mathbf{Z}))^\wedge$ is a non-abelian quotient of $\pi_1(F)^\wedge$, we cannot have $\pi_1(F)^\wedge\simeq\hat{\mathbf{Z}}$.

When $\pi_{2c-1}((F)_L^\wedge)\simeq\hat{\mathbf{Z}}_L(c)$, $\mu_L(X,c)$ has a further role to play: it describes how $\pi_1(X)_K^\wedge$ acts on homotopy. Recall that the fibration

$$F\to\{X-Y\}_K^\wedge\to\{X\}_K^\wedge$$

gives an action of $\pi_1(X)_K^\wedge$ on the fiber $F$, so that by functorality it acts on $(F)_L^\wedge$ and hence on $\pi_{2c-1}((F)_L^\wedge)$. But $\pi_1(X)_K^\wedge$ has a canonical action on $\hat{\mathbf{Z}}_L(c)$, described as follows. Since $\mu_L(X,c)$ is abelian, we have a splitting $\mu_L(X,c)\simeq\mu_L(X,c)_K^\wedge\oplus\mu_L(X,c)_{K'}^\wedge$ (where $K'=\{\text{all primes not in } K\}$). Then the action of $\mu_L(X,c)$ on $\hat{\mathbf{Z}}_L(c)$ gives an action of $\mu_L(X,c)_K^\wedge$ (and hence of $\pi_1(X)_K^\wedge$) on $\hat{\mathbf{Z}}_L(c)$.

These actions are the same:

(4.8) Proposition. *With the same hypotheses as* (4.2), *assume that*

$$\pi_{2c-1}((F)_L^\wedge)\simeq\hat{\mathbf{Z}}_L(c).$$

*Then the two actions described above agree via the isomorphism constructed in* (4.6).

*Proof.* It suffices to show that the two actions on

$$\operatorname{Hom}(\pi_{2c-1}((F)_L^\wedge),\mathbf{Z}/n\mathbf{Z})\simeq\operatorname{Hom}(\hat{\mathbf{Z}}_L(c),\mathbf{Z}/n\mathbf{Z}) \tag{4.9}$$

agree for every $L$-number $n$. The left hand side is $H^{2c-1}((F)_L^\wedge,\mathbf{Z}/n\mathbf{Z})$ by (2.13), which is isomorphic to $H^{2c-1}(F,\mathbf{Z}/n\mathbf{Z})$. The action of $\pi_1(X)_K^\wedge$ is the usual one from the Serre spectral sequence.

Let $W$ be the torseur over $X$ whose group $G$ is the $C_K$ completion of the image of the map $\pi_1(X)\to\operatorname{Aut}(\mu_n(\Omega)^{\otimes c})$ which describes $\mathbf{Z}/n\mathbf{Z}(c)$ (see §3). Since $\mu_L(Y,c)\twoheadrightarrow\mu_L(X,c)_K^\wedge$ by (4.2), $\mathbf{Z}/n\mathbf{Z}(c)$ becomes trivial on $W\times_X Y$ (which is connected). Applying the spectral sequence (2.11) to the fibration (2.7) gives us an isomorphism

$$H^0(\{W\}_K^\wedge,\mathcal{H}^{2c}(\bar{F},F;\mathbf{Z}/n\mathbf{Z}))\simeq H^{2c}(\{W\}_K^\wedge,\{W-W\times_X Y\}_K^\wedge;\mathbf{Z}/n\mathbf{Z})$$

which we can rewrite as

$$H^{2c-1}(F,\mathbf{Z}/n\mathbf{Z})^{\pi_1(W)_K} \simeq H^{2c}_{W\times_X Y}(W,\mathbf{Z}/n\mathbf{Z}) \tag{4.10}$$

where we use the action induced by $\pi_1(W)_K^\wedge \subseteq \pi_1(X)_K^\wedge$. From (4.3) and the construction of $W$ we see that the right hand side of (4.10) has $n$ elements, so that $\pi_1(W)_K^\wedge$ acts trivially on $H^{2c-1}(F,\mathbf{Z}/n\mathbf{Z})$. Thus, the action of $\pi_1(X)_K^\wedge$ factors through $G$.

Furthermore, the action of $G$ and

$$H^{2c-1}(F,\mathbf{Z}/n\mathbf{Z}) \simeq H^{2c}_{W\times_X Y}(W,\mathbf{Z}/n\mathbf{Z})$$

gives the action of $G$ (on $H^{2c}_{W\times_X Y}(W,\mathbf{Z}/n\mathbf{Z})$) induced by the action of $G$ on $W$. Under the isomorphisms (4.3) and (4.5), this gives the action of $G$ on $\operatorname{Hom}(\hat{\mathbf{Z}}_L(c),\mathbf{Z}/n\mathbf{Z})$ induced by $\mu_L(X,c)_K^\wedge$ (this was how $W$ was defined). Thus, the two actions of $\pi_1(X)_K^\wedge$ on (4.9) agree. □

This proposition has some unexpected consequences—see (7.5).

## 5. When the fiber is a sphere

Below we give necessary and sufficient conditions for $(F)_L^\wedge$ to be a completed sphere. This happens quite rarely—it requires a very strong cohomological condition on the embedding $Y \to X$. To make this precise, we need a definition.

(5.1) DEFINITION. A map $h: X_1 \to X_2$ in Pro-$\mathscr{H}_0$ is a strong cohomological $L$-isomorphism if for every local system $F$ on $X_2$ whose fiber $M$ is in $C_L$, the map $H^q(X_2, F) \to H^q(X_1, h^*F)$ is an isomorphism for $q \geq 0$.

Note that the map $\pi_1(X_2) \to \operatorname{Aut}(M)$ describing $F$ need not factor through $\pi_1(X_2)_L^\wedge$, so that $F$ might not come from a local system on $(X_2)_L^\wedge$. This is why we use the word "strong" in (5.1). See (5.6) for more information.

Now, one of the main results of this paper:

(5.2) THEOREM. *Let $Y \subseteq X$ be a connected smooth couple of relative codimension $c > 1$ (4.1), and let $L \subseteq K$ be two sets of primes where every prime in $L$ is invertible on $X$. If $F$ is the homotopy fiber of the map $\{X - Y\}_K^\wedge \to \{X\}_K^\wedge$, then the following are equivalent:*

(1) *There is a $\#$-isomorphism $(F)_L^\wedge \xrightarrow{\sim} (S^{2c-1})_L^\wedge$.*

(2) *$\{Y\}_K^\wedge \to \{X\}_K^\wedge$ is a strong cohomological $L$-isomorphism and the map*

$$\mu_L(Y,c) \to \mu_L(X,c)$$

*induces an isomorphism $\mu_L(Y,c) \xrightarrow{\sim} \mu_L(X,c)_K^\wedge$.*

*Remark.* We will see in (5.8) that a strong cohomological $L$-isomorphism $\{Y\}_K^\wedge \to \{X\}_K^\wedge$ induces a surjection $\pi_1(Y)_K^\wedge \to \pi_1(X)_K^\wedge$. Thus the hypotheses of (4.2) and (5.2) fit together nicely. And when $(F)^{\wedge}_L \simeq (S^{2c-1})^{\wedge}_L$, we can use (4.8) to describe the action of $\pi_1(X)^{\wedge}_K$ on $(F)^{\wedge}_L$.

*Proof of* (5.2). We will need an "untwisted" version of (0.6). Assume that we have a trivialization $\alpha: \mathbf{Z}/n\mathbf{Z}(c) \simeq \mathbf{Z}/n\mathbf{Z}$ on $Y$. Then we get isomorphisms

$$H_Y^{q+2c}(X, \mathbf{Z}/n\mathbf{Z}(c)) \simeq H^q(Y, \mathbf{Z}/n\mathbf{Z}) \simeq H^q(Y, \mathbf{Z}/n\mathbf{Z}(-c)) \simeq H_Y^{q+2c}(X, \mathbf{Z}/n\mathbf{Z})$$

(where the middle isomorphism is induced by $\alpha$). The element

$$U_X \in H_Y^{2c}(X, \mathbf{Z}/n\mathbf{Z}(c))$$

of (0.6) gives us an element $\bar{U}_X \in H_Y^{2c}(X, \mathbf{Z}/n\mathbf{Z})$, and from (0.6) we get a commutative diagram

$$\begin{tikzcd} H^q(X, \mathbf{Z}/n\mathbf{Z}) \arrow[rr, "\cup \bar{U}_X"] \arrow[dr] & & H_Y^{q+2c}(X, \mathbf{Z}/n\mathbf{Z}) \\ & H^q(Y, \mathbf{Z}/n\mathbf{Z}) \arrow[ur, "\wr"'] & \end{tikzcd} \tag{5.3}$$

We now prove (1) implies (2). If $(F)^{\wedge}_L$ is a completed sphere, then from (4.1) we can conclude that $\mu_L(Y, c) \xrightarrow{\sim} \mu_L(X, c)^{\wedge}_K$. Let $n$ be an $L$-number, and let $W$ be the $C_K$-torseur constructed in the proof of (4.8). Then $\mathbf{Z}/n\mathbf{Z}(c)$ is trivial on $W \times_X Y$ and $\pi_1(W)^{\wedge}_K \subseteq \pi_1(X)^{\wedge}_K$ acts trivially on $H^{2c-1}(F, \mathbf{Z}/n\mathbf{Z})$. An argument similar to the proof of (0.6) (this time using the spectral sequence (2.11) for the fibration (2.7)) gives us a Thom isomorphism

$$\cup U: H^q(\{W\}^{\wedge}_K, \mathbf{Z}/n\mathbf{Z}) \xrightarrow{\sim} H^{q+2c}(\{W\}^{\wedge}_K, \{W - W \times_X Y\}^{\wedge}_K; \mathbf{Z}/n\mathbf{Z}) \tag{5.4}$$

where $U \in H^{2c}(\{W\}^{\wedge}_L, \{W - W \times_X Y\}^{\wedge}_K; \mathbf{Z}/n\mathbf{Z} = \mathbf{Z}/n\mathbf{Z}$ is any generator. Using (1.1) and (1.8) we can identify (5.4) as the top line of the commuative diagram

$$\begin{tikzcd} H^q(W, \mathbf{Z}/n\mathbf{Z}) \arrow[rr, "\cup U_W"] \arrow[dr] & & H_{W \times_X Y}^{q+2c}(W, \mathbf{Z}/n\mathbf{Z}) \\ & H^q(W \times_X Y, \mathbf{Z}/n\mathbf{Z}) \arrow[ur, "\wr"'] & \end{tikzcd}$$

that we get from (5.3). Thus, we see that the map

$$H^q(W, \mathbf{Z}/n\mathbf{Z}) \to H^q(W \times_X Y, \mathbf{Z}/n\mathbf{Z}) \tag{5.5}$$

is an isomorphism for all $q \geq 0$. Note that if $W$ is any connected $C_K$-torseur over $X$ for which $\mathbf{Z}/n\mathbf{Z}(c)$ trivializes on $W \times_X Y$, (5.5) is still an isomorphism.

Now, let $F$ be a local system on $\{X\}^{\wedge}_K$ whose stalk $M$ is in $C_L$. Then $F$ is a locally constant sheaf on $X$, which becomes the constant sheaf $M$ on some connected $C_K$-torseur $W$ over $X$. We can also assume that the sheaves $\mathbf{Z}/n_i\mathbf{Z}(c)$ are trivial on $W \times_X Y$, where $M = \oplus \mathbf{Z}/n_i\mathbf{Z}$. Then (5.5) gives us isomorphisms

$$H^q(W, M) \to H^q(W \times_X Y, M) \quad \text{for} \quad q \geq 0.$$

If $G$ is the group of $W$ over $X$, then we have a map of spectral sequences

$$\begin{array}{ccc} H^p(G, H^q(W, M)) & \to & H^p(G, H^q(W \times_X Y, M)) \\ \Downarrow & & \Downarrow \\ H^{p+q}(X, F) & \to & H^{p+q}(Y, F|_Y). \end{array}$$

Since we have an isomorphism on the $E_2$-level, the map $H^q(X, F) \to H^q(Y, F|_Y)$ is an isomorphism for $q \geq 0$. Thus, $\{Y\}_K^\wedge \to \{X\}_K^\wedge$ is a strong cohomological $L$-isomorphism.

To prove (2) implies (1), we first need:

(5.6) LEMMA. *Let $h: X_1 \to X_2$ in* Pro-$\mathcal{H}_0$ *is a strong cohomological $L$-isomorphism.*

(1) *$\pi_1(X_1)^\wedge \to \pi_2(X_2)^\wedge$ is surjective.*

(2) *Let $W_2$ be a finite regular covering space of $X_2$ and let $W_1$ be the induced covering space of $X_1$. Then the map $W_1 \to W_2$ is a strong cohomological $L$-isomorphism.*

*Proof.* To prove the first statement, we must show that the composition

$$\pi_1(X_1) \to \pi_1(X_2) \to G$$

is onto whenever we have a surjection $\pi_1(X_2) \to G$ with $G$ finite. Let $M$ be any non-zero group in $C_L$. Then $\operatorname{Hom}(G, M)$ is in $C_L$ and has an obvious $G$ action. This gives us a local system $F$ on $X_2$ with

$$H^0(X_2, F) \simeq \operatorname{Hom}(G, M)^G = \{\text{all constant functions from } G \text{ to } M\}.$$

But then $H^0(X_1, h^*F) \simeq \operatorname{Hom}(G, M)^{\pi_1(X_1)} = \{\text{all functions } G \to M \text{ which are constant on the cosets of } \operatorname{im}(\pi_1(X_1) \to G)\}$. Since the two groups are isomorphic, $\pi_1(X_1) \to G$ must be surjective.

The map $u: W_2 \to X_2$ gives us the commutative diagram

(5.7)
$$\begin{array}{ccc} W_1 & \xrightarrow{h'} & W_2 \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ X_1 & \xrightarrow{h} & X_2. \end{array}$$

If $F$ is a local system on $W_2$ given by a representation of $\pi_1(W_2)$ on $M$ in $C_L$, then $u_*F$ is the local system on $X_2$ obtained from the representation induced by $\pi_1(W_2) \subseteq \pi_1(X_2)$ (note that the stalk of $u_*F$ is in $C_L$). Since $\pi_1(X_1)^\wedge \to \pi_1(X_2)^\wedge$ is onto, one sees easily that the natural map

$$h^*u_*F \to u'_*h'^*F$$

is an isomorphism. Thus, from (5.7) we get a commutative diagram

$$\begin{array}{ccccc} H^q(W_2, F) & & \longrightarrow & & H^q(W_1, h'^*F) \\ \wr\wr & & & & \wr\wr \\ H^q(X_2, u_*F) & \xrightarrow{\approx} & H^q(X_1, h^*u_*F) & \xrightarrow{\approx} & H^q(X_1, u'_*h'^*F) \end{array}$$

which proves that $W_1 \to W_2$ is a strong cohomological $L$-isomorphism. □

We will use the notation of the proof of (2.4) (especially the category $J$ and (2.9)). Let $n$ be an $L$-number. Since $\mu_L(Y, c) \xrightarrow{\approx} \mu_L(X, c)\hat{_K}$, we get $W_0$ in $J$ which trivializes $\mathbf{Z}/n\mathbf{Z}(c)$ on $W_0 \times_X Y$. Pick a specific isomorphism $\alpha: \mathbf{Z}/n\mathbf{Z}(c) \to \mathbf{Z}/n\mathbf{Z}$ on $W_0 \times_X Y$. Set $J' = J/W_0$, and note that for any $W$ in $J'$, $\alpha$ induces a trivialization of $\mathbf{Z}/n\mathbf{Z}(c)$ on $W \times_X Y$.

From (5.3) we get elements $\bar{U}_W \in H^{2c}_{W \times_X Y}(W, \mathbf{Z}/n\mathbf{Z})$, which fit together to give an element $\bar{U} \in \operatorname{colim}_{J'_0} H^{2c}_{W \times_X Y}(W, \mathbf{Z}/n\mathbf{Z}) \simeq H^{2c}(B, E; \mathbf{Z}/n\mathbf{Z})$ (see (2.10)). Then from (5.3) we get a commutative diagram

$$\begin{tikzcd} \operatorname{colim} H^q(W, \mathbf{Z}/n\mathbf{Z}) \arrow[rr, "\smile \bar{U}"] \arrow[dr, "r"'] & & \operatorname{colim} H^{q+2c}_{W \times_X Y}(W, \mathbf{Z}/n\mathbf{Z}) \arrow[dl, "\simeq"] \\ & \operatorname{colim} H^q(W \times_X Y, \mathbf{Z}/n\mathbf{Z}) & \end{tikzcd}$$

where the map $r$ is an isomorphism by (5.6). Thus (using 2.10)) we have an isomorphism

$$\smile \bar{U}: H^q(B, \mathbf{Z}/n\mathbf{Z}) \to H^{q+2c}(B, E; \mathbf{Z}/n\mathbf{Z}) \quad \text{for all } q.$$

Since $B$ is simply connected, Spivak's converse to the Thom isomorphism theorem (see [4, I.4.3]) says that $H^q((F)\hat{_L}, \mathbf{Z}/n\mathbf{Z}) = 0$ for $q \neq 0, 2c-1$. Since $(F)\hat{_L}$ is simply connected $(c > 1)$ and $\pi_{2c-1}((F)\hat{_L}) \simeq \hat{\mathbf{Z}}_L$ (by (5.6) and (4.8)), one easily proves that $(F)\hat{_L}$ is #-isomorphic to $(S^{2c-1})\hat{_L}$ (see [2,4.15]). □

In §6 we will treat the case $c = 1$.

Most applications of (5.2) have either $K = \{\text{all primes}\}$ or $K = L$. An example of the former is:

(5.8) Corollary. *Let $Y \subseteq X$ be a smooth connected couple of relative codimension $c > 1$, and let $L$ be a set of primes invertible on $X$. If $\{Y\}\hat{} \xrightarrow{\approx} \{X\}\hat{}$ and $F$ is the homotopy fiber of the map $\{X - Y\}\hat{} \to \{X\}\hat{}$, then $(F)\hat{_L}$ is # isomorphic to $(S^{2c-1})\hat{_L}$.* □

Over a field $k$, here is an application of the case $K = L$ which uses (3.3) and (3.4):

(5.9) Corollary. *Let $Y \subseteq X$ be a smooth couple of relative codimension $c > 1$, and assume $X$ and $Y$ are geometrically connected over $k$. Let $L$ be a set of primes invertible on $X$, and suppose that $\{Y\}\hat{_L} \to \{X\}\hat{_L}$ is a #-isomorphism. Then the homotopy fiber of the map $\{X - Y\}\hat{_L} \to \{X\}\hat{_L}$ is #-isomorphic to $(S^{2c-1})\hat{_L}$ if and only if $\mu_L(X, c)$ is $C_L$-complete. In particular, if $L = \{l\}$, this happens if and only if $[k(\lambda): k] \mid c$, where $\lambda$ is a primitive $l^{\text{th}}$ root of unity.* □

Finally, we give some examples where $(F)^{\wedge}_{L}$ is not a completed sphere. Let $X=\mathbf{A}^n_k$, $n>1$ (where $k$ is a field of characteristic $p>0$), and let $Y$ be the origin. Set $L=\{\text{all primes but } p\}$.

(5.10) *Example.* If $k$ is finite, then the homotopy fiber of $\{X-Y\}^{\wedge}_{L}\to\{X\}^{\wedge}_{L}$ is not a completed sphere. This is because $\mu_L(X,1)\simeq\mathrm{Gal}(\bar{k}/k)\simeq\hat{Z}$ (by (3.3)) so that $\mu_L(X,n)$ is not $C_L$-complete (see (5.9).

(5.13) *Example.* Now assume that $k$ is algebraically closed. Then the homotopy fiber of $\{X-Y\}^{\wedge}_{L}\to\{X\}^{\wedge}_{L}$ is a completed sphere by (5.9). But the homotopy fiber of $\{X-Y\}_{\mathrm{et}}\to\{X\}_{\mathrm{et}}$ is not a completed sphere. The reason is that $\{Y\}_{\mathrm{et}}\to\{X\}_{\mathrm{et}}$ cannot be a strong cohomological $L$-isomorphism because $\pi_1(Y)\to\pi_1(X)$ is not surjective—$\pi_1(Y)=1$ but $\pi_1(X)$ is non-trivial by Artin-Schrier theory. (We used (5.2) and (5.6).)

## 6. The codimension one case

When $Y$ has relative codimension one in $X$, showing that $(F)^{\wedge}_{L}\simeq(S^1)^{\wedge}_{L}$ is fairly difficult. We only give sufficient conditions for this to happen, and we treat only the case $K=L$. As in (5.2) we will put a strong condition on the embedding $i\colon Y\to X$, but now it will involve the sheaves $j_*F$ ($j\colon X-Y\to X$ is the inclusion) where $F$ is $C_L$-complete, defined as follows.

(6.1) DEFINITION. Let $L$ be a set of primes. A sheaf $F$ on $X-Y$ is called $C_L$-complete if $F$ is locally constant, its fiber $M$ is in $C_L$, and the map $\pi_1(X-Y)\to\mathrm{Aut}(M)$ (which determines $F$) factors through $\pi_1(X-Y)^{\wedge}_{L}$.

This is equivalent to a local system on $\{X-Y\}^{\wedge}_{L}$. Note that if $Y$ has relative codimension $\geq 2$ in $X$ (and $X$ is smooth over $S$ as in (2.1)) then (6.1) is equivalent to a local system on $\{X\}^{\wedge}_{L}$.

Now we state the codimension one version of Theorem (5.2).

(6.2) THEOREM. *Let $Y\subseteq X$ be a smooth, connected couple of relative codimension one* (4.1), *and let L be a set of primes invertible on X. Assume that the following condition is satisfied.*

(6.3) *For every abelian $C_L$-complete sheaf $F$ on $X-Y$, the map $H^q(X,j_*F)\to H^q(Y,i^*j_*F)$ is an isomorphism for $q\geq 0$.*

*Then the following are equivalent*:

(1) *The homotopy fiber $F$ of $\{X-Y\}^{\wedge}_{L}\to\{X\}^{\wedge}_{L}$ is #-isomorphic to $(S^1)^{\wedge}_{L}$.*

(2) $\mu_L(Y,1)\twoheadrightarrow\mu_L(X,1)^{\wedge}_{L}$.

*Proof.* That (1) implies (2) follows immediately from the proof of (4.2). But before we can start proving (2) implies (1), we need to develop some machinery. Recall from §4 that we have points $\xi$ of $X-Y$ and $\eta$ of $Y$, and that $\xi$ specializes to $\eta$. Let $X_\eta$ (resp. $Y_\eta$) be the strict henselization of $X$ (resp. $Y$) at $\eta$. Then $X_\eta-Y_\eta$ is the inverse image of $X-Y$ in $X_\eta$, and the specialization $\xi\to\eta$ gives us a point of $X_\eta-Y_\eta$.

Here is a global version of the relative Abhyankar lemma (in this and the other lemmas below, $Y \subseteq X$ is as in (6.2)):

(6.4) LEMMA. *Let $f: V \to X - Y$ be a torseur classified by a map $\pi_1(X - Y) \to G$, where the order of $G$ is invertible on $X$. Then there is a commutative diagram*

(6.5)
$$\begin{tikzcd} V \arrow[r] \arrow[d, "f"'] & \bar{V} \arrow[d, "\bar{f}"] \\ X - Y \arrow[r, "i"] & X \end{tikzcd}$$

*where $\bar{f}$ is flat and finite and $\bar{V}$ is smooth over $S$.*

*Furthermore, if the composition $\pi_1(X_\eta - Y_\eta) \to \pi_1(X - Y) \to G$ is surjective, then $\bar{V} \times_X Y \to Y$ is radical and has a section.*

*Proof.* If $\Phi$: $W \to X$ is any map, set $W' = \Phi^{-1}(X - Y)$.

Let $\gamma$ be a point of $Y$, and let $X_\gamma$ and $Y_\gamma$ be the corresponding strict henselizations. Then $V(\gamma) = V \times_{X-Y} X'_\gamma$ is a $G$-torseur over $X'_\gamma$. Let $W$ be a connected component of $V(\gamma)$, and let $H$ be its stabilizer subgroup. Then $W$ is an $H$-torseur and $V(\gamma) \simeq W \times_H G$. It follows easily from the relative Abhyankar lemma [1, XVI 3.5] that $H \simeq \mathbf{Z}/n\mathbf{Z}$ and $W \simeq \operatorname{Spec}(\mathcal{O}_{X_\gamma'}[z]/(z^n - t))$, where $Y_\gamma$ is defined by $t = 0$ in $X_\gamma$.

By descent we get a collection $\mathcal{U}$ of etale neighborhoods of $X$ with $Y \subseteq \bigcup_{U \in \mathcal{U}} \operatorname{im}(U \to X)$, and each $U \in \mathcal{U}$ has the following property: there is a subgroup $H \subseteq G$ with $H \simeq \mathbf{Z}/n\mathbf{Z}$ such that

$$V \times_{X-Y} U' \simeq \operatorname{Spec}(\mathcal{O}_{U'}[z]/(z^n - t)) \times_H G,$$

where $U \times_X Y$ is defined by $t = 0$ in $U$, and $H$ acts via powers of a primitive $n^{\text{th}}$ root of unity $\lambda$. For $U \in \mathcal{U}$, set $V|_{U'} = V \times_{X-Y} U'$, and then define

$$\bar{V}|_U = \operatorname{Spec}(\mathcal{O}_U[z]/(z^n - t)) \times_H G.$$

Note that $\bar{V}|_U$ is finite and flat over $U$, and smooth over $S$. If we can construct descent data for the $\bar{V}|_U$ compatible with the obvious descent data for the $\bar{V}|_{U'} = V|_{U'}$, then we will be done by [10, VIII 2.7].

Let $U_1$ and $U_2$ be in $\mathcal{U}$. Then there is a unit $u$ on $U_1 \times_X U_2$ so that $t_1 = ut_2$ (both $t_1$ and $t_2$ define the inverse image of $Y$). Let $U \to U_1 \times_X U_2$ be an etale map where $u$ has an $n_1$-*th* root $v$ and $U$ is connected. If we can show that every $G$-torseur map

$$\phi: \bar{V}|_{U_1|U'} \to \bar{V}|_{U_2|U'}$$

extends uniquely to a map $\bar{\phi}$: $\bar{V}|_{U_1|U} \to \bar{V}|_{U_2|U}$, then we easily get the desired descent data. Set $W_i = \operatorname{Spec}(\mathcal{O}_U[z]/(z^{n_i} - t_i))$, $i = 1, 2$, so that

$$\bar{V}|_{U_i|U} = W_i \times_{H_i} G.$$

Note that $W_i'$ is connected because $U$ is (see [10, XIII 5.4]). Since $\phi$ gives an isomorphism $\phi\colon W_1'\times_{H_1}G\to W_2'\times_{H_2}G$, $\phi(W_1')$ and $W_2'$ are connected components of $V|_{U'}$, so that there is $g\in G$ with $g\phi(W_1')=W_2'$. We need only show that $g\phi\colon W_1'\to W_2'$ extends to an equivariant map $g\bar{\phi}\colon W_1\to W_2$. Since $gH_1g^{-1}=H_2$, we see that $n_1=n_2$, and the only equivariant maps between $W_1'$ and $W_2'$ are those which send $z$ to $\lambda v z$, where $\lambda$ is an $n_1$-*th* root of unity. These maps obviously extend.

Finally, if $\pi_1(X_\eta-Y_\eta)\to\pi_1(X-Y)\to G$ is onto, then $V(\eta)$ is connected. This, plus the fact that $Y$ is connected, easily imply that for $U\in\mathcal{U}$, $H=G\simeq\mathbf{Z}/n\mathbf{Z}$ and $\bar{V}|_U=\operatorname{Spec}(\mathcal{O}_U[z]/(z^n-t))$. Then

$$\bar{V}|_{U\times_X Y}=\operatorname{Spec}(\mathcal{O}_{U\times_X Y}[z]/(z^n))$$

(this shows that $\bar{V}\times_X Y$ is radical over $Y$) and the subscheme defined by $z=0$ is just $U\times_X Y$. These subschemes patch to give the desired section of $\bar{V}\times_X Y\to Y$. □

Our first application of (6.4) deals with the fundamental group:

(6.6) LEMMA. *Let $L$ be a set of primes invertible on $X$. Assume that $\bar{V}\times_X Y$ is connected whenever $V$ is a connected $C_L$-torseur over $X-Y$. Then we have an exact sequence*

$$\pi_1(X_\eta-Y_\eta)_L^\wedge\to\pi_1(X-Y)_L^\wedge\to\pi_1(X)_L^\wedge\to 1.$$

*Proof.* There is an exact sequence of pointed sets

$$1\to H^1(X,G)\to H^1(X-Y,G)\to H^0(X,R^1j_*G)$$

for any group $G$ (see [1, XII 3.2]). But if $G$ is in $C_L$, $R^1j_*G$ is locally constant on $Y$ [1, XVI 3.6], so that

$$H^0(X,R^1j_*G)\to(R^1j_*G)_\eta=H^1(X_\eta-Y_\eta,G)$$

is injective because $Y$ is connected. Then we easily get an exact sequence

$$1\to\operatorname{Hom}(\pi_1(X)_L^\wedge,G)\to\operatorname{Hom}(\pi_1(X-Y)_L^\wedge,G)\to\operatorname{Hom}(\pi_1(X_\eta-Y_\eta)_L^\wedge,G).$$

So we need only prove that $\operatorname{im}(\pi_1(X_\eta-Y_\eta)_L^\wedge\to\pi_1(X-Y)_L^\wedge)$ is a normal subgroup of $\pi_1(X-Y)_L^\wedge$. This means showing that for any finite quotient $G$ of $\pi_1(X-Y)_L^\wedge$, the image of $\pi_1(X_\eta-Y_\eta)_L^\wedge$, which we call $G_\eta$, is a normal subgroup of $G$.

The map $\pi_1(X-Y)_L^\wedge$ classifies a connected $G$ torseur $V$. Let $\gamma$ be a point of $\bar{V}\times_X Y$; it lies above a point $\gamma'$ of $Y$. Since

$$\pi(\bar{V}\times_X X_{\gamma'})=\pi(V\times_X X_{\gamma'})$$

(see the proof of (6.4)), $\gamma$ gives us a connected component of $V\times_X X_{\gamma'}$. Let $G(\gamma)$ be its stabilizer. Let $\delta$ be a point of $V$ which specializes to $\gamma$. Then $\delta$ lies above a point $\delta'$ of $X-Y$, and $G(\gamma)$ is the image of the map

$$\pi_1(X_{\gamma'}-Y_{\gamma'},\delta')_L^\wedge\to\pi_1(X-Y,\delta')_L^\wedge\xrightarrow{\phi}G$$

(where $\phi$ classifies $V$ with the point $\delta$). From this, we see two things: first, if

$\gamma_1$ specializes to $\gamma_2$ in $\bar{V}\times_X Y$, then $G(\gamma_1)=G(\gamma_2)$, and second, there is a point $\gamma$ of $\bar{V}\times_X Y$ with $G(\gamma)=G_\eta$.

Since $\bar{V}\times_X Y$ is connected, $G(\gamma)=G_\eta$ for any point of $\bar{V}\times_X Y$. Thus, for $g\in G$, $G_\eta=G(\gamma)=G(g\gamma)=gG(\gamma)g^{-1}=gG_\eta g^{-1}$, so that $G_\eta$ is normal in $G$.

Next we show how condition (6.3) relates to the $\bar{V}$'s. This will show us how to translate (6.3) into information about $\pi_1$ via (6.6).

(6.7) LEMMA. *Condition* (6.3) *holds for* $Y\subseteq X$ *if and only if for every connected* $C_L$*-torseur* $V$ *over* $X-Y$ *and every abelian group* $M$ *in* $C_L$, *the map*

$$H^q(\bar{V},M)\to H^q(\bar{V}\times_X Y,M) \tag{6.8}$$

*is an isomorphism for* $q\ge 0$.

*Furthermore, if* $Y\subseteq X$ *satisfies condition* (6.3), *then so do the embeddings* $\bar{V}\times_X Y\to\bar{V}$ *for* $V$ *any connected* $C_L$*-torseur over* $X-Y$.

*Proof.* Let $V$ be a connected $C_L$-torseur over $X-Y$. Then we have a diagram

$$\begin{tikzcd} V \arrow[r, "j'"] \arrow[d, "f"] & \bar{V} \arrow[d, "\bar{f}"] & V\times_X Y \arrow[l, "i'"'] \arrow[d, "f'"] \\ X-Y \arrow[r, "j"] & X & Y. \arrow[l, "i"'] \end{tikzcd}$$

For any torsion sheaf $F$ on $V$, the finiteness of $\bar{f}$ (see (6.4)) gives us isomorphisms

$$H^q(X,j_*f_*F)\simeq H^q(X,\bar{f}_*j'_*F)\simeq H^q(\bar{V},j'_*F) \tag{6.9}$$
$$H^q(Y,i^*j_*f_*F)\simeq H^q(Y,i^*\bar{f}_*j'_*F)\simeq H^q(Y,f'_*i'^*j'_*F)\simeq H^q(\bar{V}\times_X Y,i'^*j'_*F).$$

Assume that (6.3) holds. An abelian group $M$ in $C_L$ gives a constant sheaf on $V$ where $f_*M$ is $C_L$-complete and $J'_*M\simeq M$, so that (6.8) follows from (6.9) (with $F=M$). Also, if $F$ is $C_L$-complete on $V$, then $f_*F$ is $C_L$-complete on $X-Y$. So (6.9) also shows that $\bar{V}\times_X Y\to\bar{V}$ satisfies condition (6.3).

Next, assume that (6.8) holds for every connected $C_L$-torseur $f\colon V\to X-Y$. Then (6.9) shows that we have isomorphisms

$$H^q(X,j_*f_*M)\to H^q(Y,i^*j_*f_*M)\quad\text{for}\quad q\ge 0 \tag{6.10}$$

for any $M\in C_L$. Given any $C_L$-complete sheaf $F$ on $X-Y$, choose $f\colon V\to X-Y$ as above so that $f^*F$ is the constant sheaf $M$. Set

$$\bar{V}_p=\bar{V}\times_X\bar{V}\times\cdots\times_X\bar{V}\quad(p+1\text{ times})$$

and let $\bar{f}_p\colon\bar{V}_p\to X$ be the projection; similarly define $V_p$ and $f_p\colon V_p\to X-Y$. Then we get a map of spectral sequences

$$\begin{array}{ccc} E_1^{p,q}=H^q(X,\bar{f}_{p*}\bar{f}_p^*j_*F) & \to & E_1^{p,q}=H^q(Y,i^*\bar{f}_{p*}\bar{f}_p^*j_*F) \\ \Downarrow & & \Downarrow \\ H^{p+q}(X,j_*F) & \longrightarrow & H^{p+q}(Y,i^*j_*F). \end{array}$$

We need to show that the map on the $E_1$-level is an isomorphism.

The Cartesian diagram

$$\begin{tikzcd} V_p \arrow[r, "j_p"] \arrow[d, "f_p"'] & \bar{V}_p \arrow[d, "\bar{f}_p"] \\ X-Y \arrow[r, "j"] & X \end{tikzcd}$$

gives us a base change map $\bar{f}_p^* j_* F \to j_{p*} f_p^* F$ which is an isomorphism over $V_p$. Applying $\bar{f}_{p*}$ to this map, and using the equality $\bar{f}_{p*} j_{p*} = j_* f_{p*}$, we get an exact sequence of sheaves on $X$

$$(6.11) \qquad 0 \to i_* A \to \bar{f}_{p*} \bar{f}_p^* j_* F \to j_* f_{p*} f_{p*} F \to i_* B \to 0$$

where $A$ and $B$ are sheaves supported on $Y$. Now $V_p$ is a disjoint sum of copies of $V$, so that $f_{p*} f_p^* F$ is a direct sum of copies of $f_* M$. Then (6.10) shows that the map $H^q(X, j_* f_{p*} f_p^* F) \to H^q(Y, i^* j_* f_{p*} f_p^* F)$ is an isomorphism. From this and (6.11) we easily get the desired isomorphism on $E_1$. $\square$

Now we prove (2) implies (1) for (6.2). Condition (6.3) says then in particular $\{Y\}_L\hat{} \to \{X\}_L\hat{}$ is a strong cohomological $L$-isomorphism (since $j_* j^* F = F$ for $F$ locally constant on $X$). Thus, for every $L$-number $n$, we have isomorphisms

$$\begin{aligned} & \operatorname{Hom}(\pi_1(F), \mathbf{Z}/n\mathbf{Z}) \simeq \operatorname{Hom}(\hat{\mathbf{Z}}_L(1), \mathbf{Z}/n\mathbf{Z}), \\ (4.6) \qquad & H^q(F, \mathbf{Z}/n\mathbf{Z}) \simeq \mathbf{Z}/n\mathbf{Z} \quad \text{for} \quad q = 0, 1, \qquad \text{(the proof of (5.2))}. \\ & \phantom{H^q(F, \mathbf{Z}/n\mathbf{Z})} \simeq 0 \qquad\quad \text{for} \quad q > 1 \end{aligned}$$

If we can show that $\pi_1(F)$ is abelian, the first line of (6.12) will show that $\pi_1(F) \simeq \hat{\mathbf{Z}}_L(1)$.

We use the notation of the proof of (2.4), especially the category $J$ (with $K = L$) and the fibration (2.9). Each $W$ in $J$ has a point over $\xi$, so that the specialization $\xi \to \eta$ gives us a point $\eta'$ of $W$ over $\eta$. Thus we get a pointed map $X_\eta \to W$ which carries $X_\eta - Y_\eta$ to $W - W \times_X Y$. These are compatible with the maps of $J$, so that we get a map of fibrations

$$\begin{tikzcd} \{X_\eta - Y_\eta\}_L\hat{} = \{X_\eta - Y_\eta\}_L\hat{} \arrow[d, "a"'] & \arrow[l, phantom] \{X_\eta - Y_\eta\}_L\hat{} \arrow[r] \arrow[d, "b"] & \{X_\eta\}_L\hat{} = * \arrow[d] \\ F \arrow[r, "i"] & E \arrow[r] & B. \end{tikzcd}$$

Then any abelian group $G$ in $C_L$ gives us a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}(\pi_1(F), G) \arrow[rr, dashed, shift left, "\check{a}_*"] & & \operatorname{Hom}(\pi_1(X_\eta - Y_\eta)_L\hat{}, G) \arrow[ll, dashed, shift left, "s"] \\ & \operatorname{Hom}(\pi_1(E), G) \arrow[ul, "\check{i}_*"] \arrow[ur, "\check{b}_*"'] & \end{tikzcd}$$

where the top line is an isomorphism because each one is isomorphic to $\mathrm{Hom}\,(\hat{\mathbf{Z}}_L(1), G)$ by (6.12) (and the isomorphism (6.12) is functorial with respect to the map of couples $(Y_\eta \to X_\eta) \to (Y \to X)$). Since $\pi_1(X_\eta - Y_\eta)\hat{}_L$ is isomorphic to $\hat{\mathbf{Z}}_L(1)$ (see [10, XIII 5.3]), it is abelian. If we let $G$ run over the finite quotients of $\pi_1(X_\eta - Y_\eta)\hat{}_L$, the isomorphism $a_*$ of (6.13) gives us a map $s\colon \pi_1(F) \to \pi_1(X_\eta - Y_\eta)\hat{}_L$ where $s \circ a_* = 1$, and we have

(6.14) $$i_*^\vee = s^\vee \circ b_*^\vee.$$

Since condition (6.3) holds for $Y \subseteq X$, (6.7) says that it also holds for the embeddings $W \times_X Y \to W$ for $W \in J$ (because $\overline{W - W \times_X Y} = W$). Then (6.8) (for $q = 0$) and (6.6) gives us an exact sequence

$$\pi_1(X_\eta - Y_\eta)\hat{}_L \to \lim_J \pi_1(W - W \times_X Y)\hat{}_L \to \lim_J \pi_1(W)\hat{}_L = 1.$$

(using the fact that $W_{\eta'} = X_\eta$). Thus $\pi_1(E)$, being a quotient of $\pi_1(X_\eta - Y_\eta)\hat{}_L$, is abelian, so that from (6.13) and (6.14) we conclude that

$$i_* = b_* \circ s\colon \pi_1(F) \to \pi_1(E).$$

In particular, we see that $\ker s \subseteq \ker i_*$. Since $\ker i_*$ lies in the center of $\pi_1(F)$, so does $\ker s$. Thus we have a map $s\colon \pi_1(F) \to \pi_1(X_\eta - Y_\eta)\hat{}_L$ which has a section $a_*$ and whose kernel is central. From this we easily see that $\pi_1(F)$ is abelian.

But knowing $\pi_1(F) \simeq \hat{\mathbf{Z}}_L(1)$ is not enough to prove that $F \simeq (S^1)\hat{}_L$.

We need to construct the universal cover of $E = \lim_J \{W - W \times_X Y\}\hat{}_L$. Let $H$ be the category of pairs $(W, V)$ where $W$ is in $J$ and $V \to W - W \times_X Y$ is a connected $G$-torseur classified by $\phi\colon \pi_1(W - W \times_X Y) \to G$ where $G$ is in $C_L$ and the composition

$$\pi_1(E) \to \pi_1(W - W \times_X Y) \to G$$

is surjective. In the proof of (2.4) we showed that $\lim_J \{W\}\hat{}_K$ is the universal cover of $\{X\}\hat{}_K$; a similar proof, slightly more complicated, shows that $\tilde{E} = \lim_h \{V\}\hat{}_L$ is $\#$ isomorphic to the universal cover of $E$. If $\tilde{F}$ is the homotopy fiber of $\tilde{E} \to B$, then the diagram

$$\begin{array}{ccccc}
\tilde{F} & \longrightarrow & \tilde{E} & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & E & \longrightarrow & B \\
\downarrow & & \downarrow & & \downarrow \\
K(\pi_1(E), 1) & = & K(\pi_1(E), 1) & \longrightarrow & *
\end{array}$$

and the $3 \times 3$ lemma (see [2, §1]) give us a fibration

$$\tilde{F} \to F \to K(\pi_1(E), 1). \tag{6.15}$$

Since $\pi_1(\tilde{F}) \subseteq \pi_1(F)$, we see that $\pi_1(F) \simeq \hat{\mathbf{Z}}_{L'}(1)$ for some set of primes $L' \subseteq L$ ($L'$ could be empty).

Each $(W, V)$ in $H$ gives us a map $\bar{V} \to W$ by (6.4). This construction is functorial, so that we can set $\tilde{B} = \lim \{\bar{V}\}_L\hat{}$, and then we get a commutative diagram

$$\begin{CD} \tilde{E} = \lim_H \{V\}_L\hat{} @>>> \tilde{B} = \lim_H \{\bar{V}\}_L\hat{} \\ @VVV @VVV \\ E @>>> B = \lim_J \{W\}_L\hat{}. \end{CD} \tag{6.16}$$

We claim that the map $\tilde{B} \to B$ is a $\#$-isomorphism. Note that $\pi_1(\tilde{E}) \to \pi_1(\tilde{B})$ is surjective (this follows from (2.2)), so that $\pi_1(\tilde{B}) = \pi_1(B) = 1$. Thus, we need only compare cohomology with constant coefficients.

Take $(W, V)$ in $H$. As we have already observed, $W \times_X Y \to W$ satisfies condition (6.3), so that from (6.7) we have isomorphisms

$$H^q(W, M) \simeq H^q(W \times_X Y, M), \qquad H^q(\bar{V}, M) \simeq H^q(\bar{V} \times_X Y, M). \tag{6.17}$$

The definition of $H$ says that the map

$$\pi_1(E) \to \pi_1(W - W \times_X Y) \xrightarrow{\phi} G$$

is onto (where $\phi$ classifies $W$). In proving that $\pi_1(F)$ was abelian we used the fact that $\pi_1(X_\eta - Y_\eta)_L\hat{} = \pi_1(W_{\eta'} - W_{\eta'} \times_X Y)_L\hat{} \to \pi_1(E)$ is onto. Combining these shows that the last condition of (6.4) is fulfilled, so that

$$\bar{V} \times_X Y \to W \times_X Y$$

is radical (and has a section). Thus we have an isomorphism

$$H^q(W \times_X Y, M) \to H^q(\bar{V} \times_X Y, M),$$

and then from (6.17) we conclude that $H^q(W, M) \to H^q(\bar{V}, M)$ is an isomorphism. Thus the map $\tilde{B} \to B$ is a $\#$-isomorphism, so that $\tilde{F}$ is the homotopy fiber of the top line of (6.16) (up to $\#$-isomorphism).

$\tilde{F}$ is easy to analyze. For $(W, V)$ in $H$, the section

$$s\colon W \times_X Y \to \bar{V} \times_X Y$$

noted above gives us a smooth connected couple $W \times_X Y \to \bar{V}$ of relative codimension one. By (6.7) we see that $\{W \times_X Y\}_L\hat{} \to \{\bar{V}\}_L\hat{}$ is a strong cohomological $L$-isomorphism, and the condition on the $L$-cyclotomic fundamental group is satisfied. Thus, the proof of Theorem (5.2) implies that $H^q(F_V, M) = 0$ for $q > 1$ and $M \in C_L$, where $F_V$ is the homotopy fiber of

$\{V\}_L^\wedge \to \{\bar{V}\}_L^\wedge$. Since $\tilde{F} \simeq \lim_H F_V$, $H^q(\tilde{F}, M) = 0$ for $q > 1$. Note that $\tilde{F}$ is a simple space because the action of $\pi_1(\tilde{F})$ on $\pi_n(\tilde{F})$ factors through $\pi_1(\tilde{E}) = 1$ (see [8, Appendix]). Thus the map $\tilde{F} \to K(\hat{\mathbf{Z}}_{L'}(1), 1)$ (coming from $\pi_1(\tilde{F}) \simeq \mathbf{Z}'_L(1)$) between simple spaces induces isomorphisms on the fundamental group and constant cohomology. Thus $\tilde{F}$ is $\#$-isomorphic to $K(\hat{\mathbf{Z}}_{L'}(1), 1)$.

The homotopy sequence for the fibration (6.15) then shows that $\pi_n(F) = 0$ for $n \geq 2$. Thus, we have $F \simeq K(\pi_1(F), 1) \simeq K(\hat{\mathbf{Z}}_L, L) \simeq K(\mathbf{Z}, 1)_L^\wedge \simeq (S^1)_L^\wedge$ (since $\mathbf{Z}$ is $C_L$-good—see [2, §6]). □

When $L$ consists of one prime $l$, the relation between a strong cohomological $L$-isomorphism and condition (6.3) is easy to state.

(6.18) Proposition. *Let $Y \to X$ be a smooth connected couple of relative codimension one, and let $l$ be a prime invertible on $X$. The following are equivalent*:

(1) $\{Y\}_l^\wedge \to \{X\}_l^\wedge$ *is a strong cohomological $l$-isomorphism and there is an exact sequence*

$$(6.19) \qquad \pi_1(X_\eta - Y_\eta)_l^\wedge \to \pi_1(X - Y)_l^\wedge \to \pi_1(X)_l^\wedge \to 1.$$

(2) $\{Y\}_l^\wedge \to \{X\}_l^\wedge$ *is a strong cohomological $l$-isomorphism and for every abelian $C_l$-complete sheaf $F$ on $X - Y$, $H^0(X, j_*F) \to H^0(Y, i^*j_*F)$.*

(3) $Y \to X$ *satisfies* (6.3) *for* $L = \{l\}$.

*Proof.* (3) implies (2) is trivially, and (2) implies (1) by (6.6) and (6.9). So we need only prove (1) implies (3).

Let $V \to X - Y$ be a connected $G$-torseur classified by $\phi$: $\pi_1(X - Y) \to G$, $G \in C_l$. By (6.7), we need only show that the map

$$H^q(\bar{V}, \mathbf{Z}/l\mathbf{Z}) \to H^q(\bar{V} \times_X Y, \mathbf{Z}/l\mathbf{Z})$$

is an isomorphism for all $q$.

*Case* 1. Assume that the composition $\pi_1(X_\eta - Y_\eta) \to \pi_1(X - Y) \to G$ is onto. By (6.4), $\bar{V} \times_X Y \to Y$ is radical and has a section, so that we need only prove that $H^q(X, \mathbf{Z}/l\mathbf{Z}) \to H^q(\bar{V}, \mathbf{Z}/l\mathbf{Z})$ is an isomorphism for $q \geq 0$ ($H^q(X, \mathbf{Z}/l\mathbf{Z}) \to H^q(Y, \mathbf{Z}/l\mathbf{Z})$ is an isomorphism by assumption).

From $\bar{f}$: $\bar{V} \to X$ we get a map of sheaves on $X$, $\mathbf{Z}/l\mathbf{Z} \to \bar{f}_*\bar{f}^*\mathbf{Z}/l\mathbf{Z}$, which is injective since $\bar{f}$ is onto. Taking the cokernel gives a short exact sequence

$$(6.20) \qquad 0 \to \mathbf{Z}/l\mathbf{Z} \to \bar{f}_*\mathbf{Z}/l\mathbf{Z} \to Q \to 0.$$

If $\gamma$ is a point of $Y$, there is only one point of $\bar{V}$ lying above it, so that $(\bar{f}_*\mathbf{Z}/l\mathbf{Z})_\gamma \simeq \mathbf{Z}/l\mathbf{Z}$. This shows that $Q_\gamma = 0$, so that $Q = j_!j^*Q$. On $X - Y$, $j^*Q$ is described by the exact sequence

$$(6.21) \qquad 0 \to \mathbf{Z}/l\mathbf{Z} \to f_*\mathbf{Z}/l\mathbf{Z} \to j^*Q \to 0.$$

Let $R = \mathbf{Z}/l\mathbf{Z}[G]$ be the group ring of $G$ over $\mathbf{Z}/l\mathbf{Z}$. There is an equivalence of categories between $R$ modules of finite length and locally constant sheaves $F$ on $X-Y$ whose fiber $M$ is a finite $\mathbf{Z}/l\mathbf{Z}$ module, and where the map $\pi_1(X-Y)\to \mathrm{Aut}(M)$ describing $F$ factors through the above map $\phi\colon \pi_1(X-Y)\to G$. This equivalence preserves exact sequences, and from (6.21), we see that $j^*Q$ is such a sheaf. Since $G$ is an $l$-group, $R$ is an Artinian local ring, so that every finite $R$ module has a composition series whose successive quotients are $\mathbf{Z}/l\mathbf{Z}$ with trivial $G$ action. Translating this into sheaves, $j^*Q$ has a filtration whose successive quotients are the constant sheaf $\mathbf{Z}/l\mathbf{Z}$.

Since $j_!$ is exact, $Q=j_!j^*Q$ has a filtration whose successive quotients are $j_!\mathbf{Z}/l\mathbf{Z}$. The exact sequence $0\to j_!\mathbf{Z}/l\mathbf{Z}\to \mathbf{Z}/l\mathbf{Z}\to i_*i^*\mathbf{Z}/l\mathbf{Z}\to 0$ on $X$ and our assumption that $H^q(X,\mathbf{Z}/l\mathbf{Z})\xrightarrow{\sim} H^q(Y,\mathbf{Z}/l\mathbf{Z})$ for $q\geq 0$ show that $H^q(X, j_!\mathbf{Z}/l\mathbf{Z})=0$ for $q\geq 0$. Then induction on the length of the filtration of $Q$ shows that $H^q(X,Q)=0$ for $q\geq 0$. This, together with (6.20), shows that $H^q(X,\mathbf{Z}/l\mathbf{Z})\simeq H^q(X,\bar{f}_*\mathbf{Z}/l\mathbf{Z})$, which becomes $H^q(X,\mathbf{Z}/l\mathbf{Z})\xrightarrow{\sim} H^q(\bar{V},\mathbf{Z}/l\mathbf{Z})$ because $\bar{f}$ is finite.

*Case* 2. The general case. Let $G_\eta$ be the image of $\pi_1(X_\eta-Y_\eta)$ in $G$. The exactness of (6.19) shows that $G_\eta$ is normal in $G$ and that $V/G_\eta$ extends to a $G/G_\eta$ torseur $W=\overline{V/G_\eta}$ over $X$. Then $V$ is a $G_\eta$-torseur over $W-W\times_X Y$, $\pi_1(W_{\eta'}-W_{\eta'}\times_X Y)=\pi_1(X_\eta-Y_\eta)\to G_\eta$ is onto (as usual, $W$ acquires a point $\eta'$ above $\eta$), and, by (5.6), $\{W\times_X Y\}_l^\wedge\to\{W\}_l^\wedge$ is a strong cohomological $l$-isomorphism. Thus, Case 1 applies, and we are done. □

## 7. Applications to vector bundles

Vector bundles provide the most natural application of our results. They fit into our framework as follows. If $V$ is a vector bundle of rank $r$ over a scheme $X$, we also use $X$ to denote the zero section of $V\to X$ (so that $V-X$ is $V$ minus the zero section). Then $X\hookrightarrow V$ is a smooth couple over $X$ of relative codimension $r$.

(7.1) THEOREM. *Let $V$ be a vector bundle of rank $r$ over a connected scheme $X$, and let $L$ be a set of primes invertible on $X$.*

(1) *If $F$ is the homotopy fiber of the map $\{V-X\}^\wedge\to\{X\}^\wedge$, then $(F)_L^\wedge$ is $\#$-isomorphic to $(S^{2r-1})_L^\wedge$.*

(2) *The homotopy fiber of the map $\{V-X\}_L^\wedge\to\{X\}_L^\wedge$ is $\#$-isomorphic to $(S^{2r-1})_L^\wedge$ if and only if $\mu_L(X,r)$ is $C_L$-complete.*

*Furthermore, for the fiber $F$ of $\{V-X\}^\wedge\to\{X\}^\wedge$, there is a canonical isomorphism $\pi_{2r-1}((F)_L^\wedge)\simeq\hat{\mathbf{Z}}_L(r)$ so that the action of $\pi_1(X)^\wedge$ on $\pi_{2r-1}((F)_L^\wedge)$ (see §4) is just the usual action of $\mu_L(X,r)$ on $\mathbf{Z}_L(r)$.*

*Proof.* We first prove statement (2). Since we have maps $i\colon X\to V$ and $\pi\colon V\to X$ with $\pi\circ i=1_X$, we get an isomorphism $\mu_L(X,r)\xrightarrow{\sim}\mu_L(V,r)$. Thus, the condition that $\mu_L(X,r)\to\mu_L(V,r)_L^\wedge$ be an isomorphism reduces to $\mu_L(X,r)$ be $C_L$-complete.

[1, XV 2.2] shows that the projection map induces a #-isomorphism $\{V\}_L^\wedge \Rightarrow \{X\}_L^\wedge$. Then $F$, defined above, is #-isomorphic to the homotopy fiber of $\{V-X\}_L^\wedge \to \{V\}_L^\wedge$, and $X \to V$ induces a strong cohomological $L$-isomorphism $\{X\}_L^\wedge \Rightarrow \{V\}_L^\wedge$. Thus, when $r>1$, the second statement follows from Theorems (5.2) and (4.8) (applied to $X \to V$). And the case $r=1$ follows from Theorem (6.2) once we verify that $X \to V$ satisfies condition (6.3).

First, assume that $X = \operatorname{Spec}(A)$, where $A$ is a strict hensel local ring. Then $V$, being a line bundle over $X$, must be $\operatorname{Spec}(A[T])$. The only connected $C_L$-torseurs over $V-X$ are the maps $f_n\colon W_n = V-X \to V-X$ which take $T$ to $T^n$ (here $n$ is any $L$-number). This assertion follows easily from the relative Abhyankar lemma [1, XVI 3.5] and the fact that $\pi_1(V)_L^\wedge = 1$ (see XV 2.2]). Since $\bar{W}_n = V$ (where $\bar{f}_n\colon \bar{W}_n \to V$ takes $T$ to $T^n$),

$$H^q(\bar{W}_n, M) \simeq H^q(\bar{W}_n \times_V X, M),$$

again by [1, XV 2.2]. Then, if $F$ is any $C_L$-complete sheaf on $V-X$, (6.7) tells us that $H^0(V, j_*F) \simeq H^0(X, i^*j_*F)$ and $H^q(V, j_*F) = 0$ for $q>0$ ($X$ is strict hensel).

Now, let $\pi\colon V \to X$ be a line bundle over any connected $X$. If $F$ is a $C_L$-complete sheaf on $V-X$ and $\gamma$ is a point of $X$, then

$$(R^q\pi_*(j_*F))_\gamma \simeq H^q(V \times_X X_\gamma, j_*F).$$

The above calculation shows that $(R^q\pi_*(j_*F))_\gamma = 0$ for $q>0$ and

$$(\pi_*j_*F)_\gamma \simeq (i^*j_*F)_\gamma,$$

so that $R^q\pi_*(j_*F) = 0$ for $q>0$ and $\pi_*j_*F \simeq i^*j_*F$. The Leray spectral sequence then shows that $H^q(V, j_*F) \simeq H^q(X, i^*j_*F)$ for all $q \geq 0$. Thus condition (6.3) is satisfied, and so for $r=1$, we are done by Theorem (6.2).

Now, we prove the first statement. Since $V \to X$ has a section, the map

$$\pi_1(V)^\wedge \to \pi_1(X)^\wedge$$

is onto. Then (2.2) shows that the map $\pi_1(V-X)^\wedge \to \pi_1(X)^\wedge$ is also onto. Let $J$ be the inverse system of pointed connected finite torseurs $W \to X$. Then the argument used in the proof of (2.4) shows that $F$ is #-isomorphic to the homotopy fiber of the map $\operatorname{colim}_J \{V \times_X W - W\}^\wedge \to \operatorname{colim}_J \{W\}^\wedge$, and we also see that $\lim_J \{W\}^\wedge$ is #-isomorphic to the universal cover of $\{X\}^\wedge$. Thus we can complete (see [8, 4.1]) to get a fibration

$$(F)_L^\wedge \to \lim_J \{V \times_X W - W\}_L^\wedge \to \lim_J \{W\}_L^\wedge.$$

Since $W$ ranges over *all* finite connected torseurs of $X$, we see that

$$\lim_J \mu_L(W, r) = 0$$

(so in particular, it is $C_L$-complete). This is not quite the situation of the second statement of (7.1) (which we proved above), but a close look at the proofs of (5.2) and (6.2) shows that here we do have a #-isomorphism $(F)_L^\wedge \simeq (S^{2r-1})_L^\wedge$. Then an easy modification of the proof of (4.8) shows that the action of $\pi_1(X)_L^\wedge$ on $\pi_{2r-1}((F)_L^\wedge)$ is as claimed. □

This theorem is much stronger than Theorem (0.1) discussed in the introduction. We can exploit the extra strength of (7.1) in several ways. First, we analyze the relation between completion and taking homotopy fiber.

(7.2) THEOREM. *Let $V$, $X$ and $L$ be as in* (7.1), *and let $F$ be the homotopy fiber of* $\{V-X\}^\wedge \to \{X\}^\wedge$. *The following are equivalent*:

(1) *The $C_L$-completion of the fibration*

$$F \to \{V-X\}^\wedge \to \{X\}^\wedge \tag{7.3}$$

*is also a fibration* (*up to # isomorphism*).

(2) *The action of $\pi_1(X)^\wedge$ on $\pi_{2r-1}((F)_L^\wedge)$ factors through $\pi_1(X)_L^\wedge$.*

(3) *$\mu_L(X, r)$ is $C_L$-complete.*

*Proof.* It is obvious that (1) implies (2), and (2) implies (3) by the description of the action of $\pi_1(X)^\wedge$ given in (7.1). To prove that (3) implies (1), note that by (7.1), the homotopy fiber $\bar{F}$ of $\{V-X\}_L^\wedge \to \{X\}_L^\wedge$ is #-isomorphic to $(S^{2r-1})_L^\wedge$. Furthermore, the isomorphisms (see (4.9))

$$\operatorname{Hom}(\pi_{2r-1}(\bar{F}), \mathbf{Z}/n\mathbf{Z}) \simeq \operatorname{Hom}(\hat{\mathbf{Z}}_L(r), \mathbf{Z}/n\mathbf{Z}),$$
$$\operatorname{Hom}(\pi_{2r-1}((F)_L^\wedge), \mathbf{Z}/n\mathbf{Z}) \simeq \operatorname{Hom}(\hat{\mathbf{Z}}_L(r), \mathbf{Z}/n\mathbf{Z})$$

are canonical, which shows that the natural map $\pi_{2r-1}((F)_L^\wedge) \to \pi_{2r-1}(\bar{F})$ is an isomorphism. Thus the map $(F)_L^\wedge \to \bar{F}$ is a #-isomorphism. □

The naturality referred to above has other uses. For example, one can easily prove the following result (a special case of [8, 3.7]):

(7.4) THEOREM. *Let $V$, $X$ and $L$ be as in* (7.1), *and let $F$ be as in* (7.3). *For any geometric point $\eta$ of $X$, let $V'_\eta$ be the fiber of $V-X \to X$ above $\eta$. Then there are #-isomorphisms $(V'_\eta)_L^\wedge \simeq (F)_L^\wedge \simeq (S^{2r-1})_L^\wedge$.*

A second instance of the power of Theorem (7.1) is the following unusual example:

(7.5) *Example.* Let $X$ be a geometrically connected scheme over $\mathbf{Q}$, and let $V$ be a vector bundle of rank $r$ over $X$. By (7.1) we get a completed spherical fibration

$$(S^{2r-1})^\wedge \to \{V-X\}^\wedge \to \{X\}^\wedge. \tag{7.6}$$

However, there is no spherical fibration over $\{X\}^\wedge$ whose completion is (7.6). For if (7.6) came from a spherical fibration, the action of $\pi_1(X)^\wedge$ on the cohomology of the fiber would factor through $\mathbf{Z}/2\mathbf{Z}$ (since the cohomology is $\mathbf{Z}$ before completing). By (3.3), $\mu_L(X, r)$ ($L = \{\text{all primes}\}$) is much larger than $\mathbf{Z}/2\mathbf{Z}$, so by (7.1) the action of $\pi_1(X)^\wedge$ cannot factor through $\mathbf{Z}/2\mathbf{Z}$.

For varieties over a field, (3.3) and (3.4) tell us a lot about $\mu_L(X, r)$. From (7.1) and (7.2) we get:

(7.7) COROLLARY. *Let $X$ be a geometrically connected scheme over a field $k$, and let $L$ be a set of primes not including the characteristic of $k$. For any vector bundle $V$ of rank $r$ over $X$, let $F$ be the homotopy fiber of $\{V-X\}^\wedge \to \{X\}^\wedge$, so that we get a fibration (where $(F)_L^\wedge \simeq (S^{2r-1})_L^\wedge$)*

$$F \to \{V-X\}^\wedge \to \{X\}^\wedge. \tag{7.8}$$

(1) *If $k$ is algebraically closed, then* (7.8) *remains a fibration after $C_L$-completion.*

(2) *If $L$ consists of one prime $l$, then the following are equivalent:*

(A) *The homotopy fiber of $\{V-X\}_l^\wedge \to \{X\}_l^\wedge$ is #-isomorphic to $(S^{2r-1})_l^\wedge$.*

(B) (7.8) *remains a fibration after $C_l$-completion.*

(C) *$[k(\lambda): k] \mid r$, where $\lambda$ is a primitive $l$-th root of unity.* □

In (7.7), when $L$ consists of more than one prime, remember that $\mu_L(X, r)$ depends only on $k$, $L$ and $r$.

Finally, recall that in (3.4) we give conditions on $L$ and $r$ which insure that $\mu_L(X, r)$ is always $C_L$-complete, independent of $X$.

## REFERENCES

1. M. ARTIN, A. GROTHENDIECK, and J. L. VERDIER, *Théorie des topos et cohomologie étale des schémas* (*SGA* 4), Lecture Notes in Math., vols. 269, 270, and 305, Springer, New York, 1970.
2. M. ARTIN and B. MAZUR, *Etale homotopy theory*, Lecture Notes in Math., vol. 100, Springer, New York, 1969.
3. G. BREDON, *Sheaf theory*, McGraw-Hill, Hightstown, N.J., 1967.
4. W. BROWDER, *Surgery on simply connected manifolds*, Erg. Math., vol. 65, Springer, New York, 1972.
5. D. COX, *Algebraic tubular neighborhoods II*, Math. Scand., vol. 42 (1978), pp. 229–242.
6. ———, *Homotopy theory of simplicial schemes*, Compositio Math, to appear.
7. P. DELIGNE and others, *Cohomologie étale* (*SGA* 4 1/2), Lecture Notes in Math., vol. 569, Springer, New York, 1977.
8. E. FRIEDLANDER, *The étale homotopy theory of a geometric fibration*, Manuscripta Math., vol. 10 (1973), pp. 209–244.
9. ———, *Fibrations in étale homotopy theory*, Publ. Math. I.H.E.S., vol. 42 (1972), pp. 5–46.
10. A. GROTHENDIECK, *Revêtements étales et groupe fondamental* (*SGA* 1), Lecture Notes in Math., vol. 224, Springer, New York, 1971.
11. K. LAMOTKE, *Semisimpliziale algebraische topologie*, Grund. Math. Wiss., vol. 147, Springer, New York, 1968.
12. O. ZARISKI, *On the problem of existence of algebraic functions of two variables possessing a given branch curve*, Amer. J. Math., vol. 51 (1929), pp. 305–328.

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