ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 1, Spring 1980

# EXTENSION TO STRICTLY PSEUDOCONVEX DOMAINS OF FUNCTIONS HOLOMORPHIC IN A SUBMANIFOLD IN GENERAL POSITION AND $C^\infty$ UP TO THE BOUNDARY

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## 0. Introduction and notation

Let $D$ be a domain in $\mathbf{C}^n$ and $M$ a complex manifold in $D$. Denote by $H(D)$ (resp. $H(M)$) the space of holomorphic functions in $D$ (resp. $M$).

A classical consequence of Cartan's "Theorem B" asserts that the restriction map $H(D) \to H(M)$ is surjective in the case that $D$ is a domain of holomorphy. This type of extension problem has been studied for different classes of functions and different kind of domains by several authors; see Rudin [11], Bungart [5] and in the most relevant case to this work, Henkin [7]. In this last paper Henkin proved that if $D$ is a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with $C^2$ boundary and $M'$ is a $k$-dimensional complex manifold in a neighborhood $D'$ of $\bar{D}$ that intersects the boundary of $D$ transversally then for $M = D \cap M'$ there exists a continuous linear extension operator

$$L: H^\infty(M) \to H^\infty(D)$$

so that $Lf \in A(D) = H(D) \cap C(\bar{D})$ whenever $f \in A(M) = H(M) \cap C(\bar{M})$.

The purpose of this note is to show that under the same assumptions above stated (but we shall assume $D$ to have $C^\infty$ boundary) every function

$$f \in A^\infty(M) = H(M) \cap C^\infty(\bar{M})$$

is the restriction to $M$ of some function $F \in A^\infty(D)$. It is stated in Henkin's paper that the local version of this extension problem is the case when $D$ is strictly convex and $M$ is a plane section. For this particular case he gives an explicit integral formula for the extension operator. In the first part of this work we use the above mentioned formula plus an integration by parts argument to show that Henkin's extension of any function in $A^\infty(M)$ is in $A^\infty(D)$. In the second part of the paper we use the local result of the first part to obtain the result in the global case. This is done by standard sheaf theory arguments and the main tool we use is an analogue of Cartan's "Theorem B" which is proved in Nagel [10]. We note that in passing from the local to the global case we loose the operator character of our extension.

Finally as an application of our main result we prove an approximation theorem (uniform approximation in all partial derivatives up to a finite order)

Received January 4, 1978.

for functions $f \in A^\infty(D)$ by functions in $H(\Omega)$ where $\Omega$ is a suitable neighborhood of $\bar{D}$. This uses an embedding result of Fornaess [4] and the idea of the proof (embedding-extending-approximating) is classical.

We now fix some notation. We denote by

$$\mathbf{C}^n = \{(z_1, \ldots, z_n) \mid z_j \in \mathbf{C}\}$$

the complex $n$-dimensional Euclidean space. As usual if $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbf{R}$ we put

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right);$$

$d = \partial + \bar{\partial}$ will denote the usual splitting of the exterior differentiation on $\mathbf{C}^n$. If $\alpha = (p, q) \in Z^n \times Z^n$ is any multiindex we set

$$|\alpha| = |p| + |q| = p_1 + \cdots + p_n + q_1 + \cdots + q_n$$

and

$$D_\alpha = \frac{\partial^{|\alpha|}}{\partial z_1^{p_1} \cdots \partial z_n^{p_n} \partial \bar{z}_1^{q_1} \cdots \partial \bar{z}_n^{q_n}}.$$

In $\mathbf{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbf{R}\}$ let

$$D_\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

for any multiindex $\alpha \in Z^n$.

For $z_0 \in \mathbf{C}^n$ and $r > 0$ we define $B(z_0, r) = \{z \in \mathbf{C}^n / |z - z_0| < r\}$.

Let $U$ be a bounded open set in $\mathbf{C}^n$ (or $\mathbf{R}^n$). A function $f \in C^s(U)$ $(1 \leq s \leq \infty)$ will be said to be a $C^s$ function on $\bar{U}$ if all of its partial derivatives up to the order $s$ in $U$ admit a continuous extension to $\bar{U}$. We denote by $C^s(\bar{U})$ the space of all $C^s$ complex valued functions on $\bar{U}$.

For $1 \leq s < \infty$ we provide $C^s(\bar{U})$ with the topology induced by the norm

$$\|u\|_s^U = \sum_{|\alpha| < s} \sup_{z \in U} |D_\alpha u(z)|$$

and we give to $C^\infty(\bar{U})$ the topology induced by the family of semi-norms (actually norms) $\{\| \ \|_l^{\bar{U}}\}_{0 \leq l < \infty}$.

If $M$ is a compact $C^\infty$ real manifold in $\mathbf{C}^n$ we consider the topology induced in $C^s(M)$ $(1 \leq s < \infty)$ by the norm

$$\|u\|_s^M = \sum_{j=1} \|(u \cdot X_j) \circ s_j\|_s^{U_i}$$

where $\{U_j, s_j\}_{1 \leq j \leq l}$ is a $C^\infty$ finite atlas for $M$ and $\{X_j\}_{1 \leq j \leq l}$ is a $C^\infty$ partition of the unity subordinated to the covering $\{U_j\}_{1 \leq j \leq l}$ of $M$. These topologies are independent of the choice of the finite atlas and of the partition of the unity.

In $C^\infty(M)$ we consider the topology induced by the family of semi-norms $\{\| \quad \|_s^M\}_{1\le s<\infty}$. If $u \in C^s(M)$ we will say that all the $s$-order derivatives of $u$ satisfy a Holder condition with exponent $\mu$, $0<\mu\le 1$, if, for some atlas $\{U_j, s_j\}_{1\le j\le l}$ of $M$ and some partition of the unity $\{X_j\}_{1\le j<l}$ subordinated to $\{U_j\}_{1\le j\le l}$, all the functions $D_\alpha[(u \cdot X_j) \circ s_j]$, $|\alpha| = s$, satisfy a Holder condition with exponent $\mu$ in their domain of definition.

If $M$ is now a complex manifold in $\mathbf{C}^n$ and $U$ is a relatively compact open subset of $M$ we define $A^s(U) = H(U) \cap C^s(\bar{U})$; where $H(U)$ stands for the space of holomorphic functions on $U$. The topologies on the spaces $A^s(U)$ will be the ones inherited from the spaces $C^s(\bar{U})$.

## 1. The local case

Throughout this section we let $D$ be a strictly convex domain in $\mathbf{C}^n$ with $C^\infty$ boundary. More precisely $D = \{z \in D'/\rho(z)<0\}$ where $\rho$ is a $C^\infty$ real valued function defined on a neighborhood $D'$ of $\bar{D}$ and the real Hessian of $\rho$ is strictly positive definite at every point $z \in D'$. We further assume that $0 \in D$ and $\bar{D}$ is compact. If $1 \le k \le n$ we identify $\mathbf{C}^k$ with

$$\mathbf{C}^k \times \{0\} \times \cdots \times \{0\} \subset \mathbf{C}^n$$

and we put $M' = \mathbf{C}^k \cap D'$; $M = \mathbf{C}^k \cap D$. Setting $\tilde{\rho} = \rho/M'$ we have that $M = \{z \in M'/\tilde{\rho}(z)<0\}$ is a strictly convex domain with $C^\infty$ boundary. We also observe that $\partial M = \{z \in M'/\tilde{\rho}(z)<0\}$ is a $C^\infty$ compact manifold of real dimension $2k-1$.

1.1. *Henkin's extension formula.* We denote by

$$\operatorname{grad} \tilde{\rho}(\xi) = \left(\frac{\partial \tilde{\rho}}{\partial z_1}(\xi), \ldots, \frac{\partial \tilde{\rho}}{\partial z_k}(\xi)\right)$$

the complex gradient of $\tilde{\rho}$ at $\xi$ and we note that, because of the strict convexity, $\operatorname{grad} \tilde{\rho}(\xi) \neq 0$ for all $\xi \in \partial M$. Following Henkin [7] we consider the differential forms

$$\omega'_k(\varphi) = \sum_{j=1}^{k} (-1)^{j-1} \varphi_j \bar{\partial}\varphi_1 \wedge \cdots \wedge \hat{\bar{\partial}}\varphi_j \wedge \cdots \wedge \bar{\partial}\varphi_k$$

and

$$\omega_k(\varphi) = d\varphi_1 \wedge \cdots \wedge d\varphi_k$$

for $\varphi = (\varphi_1, \ldots, \varphi_k)$ where $\varphi_1, \ldots, \varphi_k$ are $C^\infty$ complex valued functions.

For $\xi \in \partial M$ and $z \in D$ we define

$$\phi(\xi, z) = \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(\xi) \cdot (\xi_j - z_j). \tag{1.1}$$

In Henkin [7] it is shown that for $f \in A(M)$ the equation

$$Lf(z) = \frac{(k-1)!}{(2\pi i)^k} \int_{\xi \in \partial M} f(\xi) \frac{\omega_k'(\operatorname{grad} \tilde{\rho}(\xi) \wedge \omega_k(\xi)}{(\phi(\xi, z))^k} \tag{1.2}$$

defines a bounded linear extension operator $L: A(M) \to A(D)$ where $A(M)$ and $A(D)$ are provided with the topology of the uniform convergence. (The extension property of the operator $L$ follows from the fact that when

$$z = (z_1, \ldots, z_k, 0, \ldots, 0) \in M,$$

formula (1.2) reduced to the Cauchy–Frantappie integral formula for the domain $M$ on $\mathbf{C}^k$ [see Aizenberg [1] or Koppelman [9]].

1.2. *Statement of the local results.* Our main result in §1 is:

THEOREM 1. *Relation* (1.2) *defines a continuous linear operator*

$$L: C^{s+1}(\partial M) \to A^s(\bar{D}), \quad 0 \leq s \leq \infty.$$

*Moreover if* $f \in C^{s+1}(\partial M)$ *and all the* $(s+1)$*-order partial derivatives of* $f$ *satisfy a Holder condition with exponent* $\mu$, $0 \leq \mu \leq 1$, *then* $Lf \in A^{s+1}(\bar{D})$.

*Remark.* (i) For this theorem it is enough to assume that $D$ has $C^{s+2}$ boundary.

(ii) If we take $M = D$ in the above theorem we obtain, for strictly convex domains in $\mathbf{C}^n$, an analogue to a well-known result about the Cauchy integral in one complex variable (see for example Vekua [13, Theorem 1.10, page 21]). The idea of our proof is based on the proof of this theorem.

An immediate consequence of Theorem 1 is:

COROLLARY 1. *Relation* (1.2) *defines a continuous linear extension operator*

$$L: A^\infty(M) \to A^\infty(D).$$

An application of Corollary 1 gives:

THEOREM 2. *If* $f \in A^\infty(D)$ *and* $f/M \equiv 0$ *then there exist functions*

$$h_{k+1}, \ldots, h_n \in A^\infty(D)$$

*so that* $f(z) = z_{k+1} \cdot h_{k+1}(z) + \cdots + z_n \cdot h_n(z)$ *for all* $z \in D$.

This last result will be used later to make an identification of sheaves that allows us to pass from the local case to the global case.

1.3. *Some necessary estimates.* In this section we state some results that will be used in the integration by parts. The proofs are very similar to the proofs of the corresponding lemmas in [7] and [8] and they will be omitted here.

LEMMA 1.1. *Let C, h and $\mu$ be constants satisfying $0<c\le 1$, $0<h\le 1$, $0<\mu\le 1$. Set*

$$I=\int_{0\le t_2+\cdots+t_{2k}{}^2\le h^2}\frac{(c+t_2^2+\cdots+t_{2k}^2)^{\mu/2}}{((c+t_2^2+\cdots+t_{2k}^2)^2+t_2^2)^{k/2}}\,dt_2\cdots dt_{2k}.$$

*Then*

$$I\le\frac{A_k}{\mu}\cdot h^{\mu}\quad \textit{if}\quad k\ge 2, \tag{1.3}$$

$$I\le\frac{A_1}{\mu}(C^{\mu/2}+h^{\mu/2})\quad \textit{if}\quad k=1, \tag{1.4}$$

*where the constants $A_k=1,2,\ldots$ depend on k only.*

We now introduce the following notation: For $\sigma>0$

$$(\partial M)_\sigma=\{z\in\bar D/d(z,\partial M)<\sigma\}.$$

Using Lemma 1.1 and a change of variables due to Henkins one can show:

LEMMA 1.2. *There exist constants $\sigma$, $B>0$, $\sigma<1$, so that for any $\delta<\sigma$,*

$$\int_{\xi\in\partial M\cap B(z,\delta)}\frac{|\xi-z|^{\mu}}{|\phi(\xi,z)|^k}\,m(d\xi)\le\frac{B}{\mu}\cdot\delta^{\mu/2}\quad \textit{for all}\quad z\in(\partial M)_\sigma$$

*where $0<\mu\le 1$ and $m(d\xi)$ is the measure induced by Lebesgue measure on $\mathbf{C}^k$.*

Lemma 1.2 and the compactness of $\bar D\times\bar D$ imply:

LEMMA 1.3. *There exists a constant $C>0$ so that*

$$I(z)=\int_{\xi\in\partial M}\frac{|\xi-z|}{|\phi(\xi,z)|^k}\,m(d\xi)<C\quad \textit{for all}\quad z\in\bar D.$$

Finally Lemma 1.3 and Henkin's result about extension of holomorphic functions continuous up to the boundary imply via standard arguments.

LEMMA 1.4. *Let $\varphi:\bar D\times\bar D\to\mathbf{C}$ be a continuous function that satisfies a Holder condition of the form*

$$|\varphi(\xi,z)-\varphi(\xi',z|\le K\cdot|\xi-\xi'|^{\mu}$$

*for some $0<\mu\le 1$. Then the function*

$$F(z)=\begin{cases}\dfrac{(k-1)!}{(2\pi i)^k}\displaystyle\int_{\xi\in\partial M}\varphi(\xi,z)\frac{\omega_k'(\operatorname{grad}\tilde\rho(\xi))\wedge\omega_k(\xi)}{(\phi(\xi,z))^k} & \textit{if}\quad z\in\bar D\backslash\partial M\\ \varphi(z,z)+\dfrac{(k-1)!}{(2\pi i)^k}\displaystyle\int_{\xi\in\partial M}(\varphi(\xi,z)-\varphi(z,z)\frac{\omega_k'(\operatorname{grad}\tilde\rho(\xi))\wedge\omega_k(\xi)}{(\phi(\xi,t))^k} & \textit{if}\quad z\in\partial M\end{cases}$$

*is continuous on $\bar D$.*

1.4 *The integration by parts.* We denote by $H(\xi)$ the complex Hessian matrix,

$$\left[\frac{\partial \rho}{\partial z_i \partial \bar{z}_j}(\xi)\right]_{1\le i,j\le k},$$

of $\tilde{\rho}$ at the point $\xi$. Since $M$ is strictly convex $H(\xi)$ is a strictly positive definite $(k\times k)$-matrix over $\mathbf{C}$. Lemma 1.5 below can be implicitly found in Aizenberg [1].

LEMMA 1.5. *Let $U\subset \mathbf{C}^k$ be open and assume that, for some $1\le m\le k$,*

$$\frac{\partial \tilde{\rho}}{\partial z_m}(\xi)\neq 0 \quad \textit{for all} \quad \xi\in\partial M\cap U.$$

*Then*

$$\omega_k'(\operatorname{grad}\tilde{\rho}(\xi))\wedge\omega_k(\xi)=\frac{K(\xi)}{\partial\tilde{\rho}/\partial z_m(\xi)}\,d\bar{\xi}_1\wedge\cdots\wedge d\hat{\bar{\xi}}_m\wedge\cdots\wedge d\bar{\xi}_k\wedge d\xi_1\cdots\wedge d\xi_k$$

*as differential forms over $\partial M$ and where*

$$K(\xi)=\det\left[\begin{array}{c|c} 0 & \dfrac{\partial\tilde{\rho}}{\partial z_1}(\xi)\cdots\dfrac{\partial\tilde{\rho}}{\partial z_k}(\xi) \\ \hline \dfrac{\partial\tilde{\rho}}{\partial \bar{z}_1}(\xi) & \\ & H(\xi) \\ \dfrac{\partial\tilde{\rho}}{\partial \bar{z}_k}(\xi) & \end{array}\right]. \tag{1.5}$$

*Proof.* The proof is just algebra and uses the fact that in $\partial M$ the equation

$$\frac{\partial\tilde{\rho}}{\partial z_1}(\xi)\,d\xi_1+\cdots+\frac{\partial\tilde{\rho}}{\partial z_k}(\xi)\,d\xi_k+\frac{\partial\tilde{\rho}}{\partial \bar{z}_1}(\xi)\,d\bar{\xi}_1+\cdots+\frac{\partial\tilde{\rho}}{\partial \bar{z}_k}(\xi)\,d\bar{\xi}_k=0$$

holds. We now show:

LEMMA 1.6. *If $K(\xi)$ is defined as in* (1.5), *then $K(\xi)\neq 0$ for all $\xi\in\partial M$.*

*Proof.* Expanding the right side of (1.5) by minors of order $(k-1)\times(k-1)$ with respect to the first row and the first column we obtain

$$K(\xi)=\overline{(\operatorname{grad}\tilde{\rho}(\xi)^T)}\cdot H(\xi)\cdot(\operatorname{grad}\tilde{\rho}(\xi))$$

which is not 0 because $H(\xi)$ is strictly positive definite and

$$\operatorname{grad}\tilde{\rho}(\xi)\neq 0 \quad \textit{for all} \quad \xi\in\partial M.$$

So the lemma is proved.

We need now the following observation: Let $z^* \in \partial M$ and assume

$$\frac{\partial \rho}{\partial \bar{z}_1}(z^*) \neq 0.$$

We can find $\sigma_0$, $c > 0$ so that

$$\left|\frac{\partial \rho}{\partial z_1}(\xi)\right| > 2c \quad \text{for all} \quad \xi \in B(z^*, \sigma_0).$$

Since the relation

$$\frac{\partial \rho}{\partial z_1}(\xi)\, d\xi_1 + \cdots + \frac{\partial \rho}{\partial z_k}(\xi)\, d\xi_k + \frac{\partial \rho}{\partial \bar{z}_1}(\xi)\, d\bar{\xi}_1 + \cdots + \frac{\partial \rho}{\partial \bar{z}_k}(\xi)\, d\bar{\xi}_k = 0$$

holds in $\partial M$ we have, for $\xi \in \partial M \cap B(z^*, \sigma_0)$,

$$\text{(1.6)} \qquad d_\xi\left[\frac{1}{(\phi(\xi, z))^k}\right] \equiv \frac{-k\left[\frac{\partial \rho}{\partial z_1}(\xi) + h(\xi, z)\right]}{(\phi(\xi, z))^{k+1}}\, d\xi_1 \qquad (\text{mod } d\bar{\xi}_2, \ldots, d\bar{\xi}_k, d\xi_2, \ldots, d\xi_k)$$

as differential forms over $\partial M$, where

$$\text{(1.7)} \qquad h(\xi, z) = \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial z_1\, \partial z_j}(\xi)(\xi_j - z_j) - \frac{\dfrac{\partial \rho}{\partial z_1}(\xi)}{\dfrac{\partial \rho}{\partial \bar{z}_1}(\xi)} \sum_{j=1}^{n} \frac{\partial^2 \rho}{\partial \bar{z}_1\, \partial z_j}(\xi)(\xi_j - z_j).$$

We observe that $h(\xi, z) = 0(|\xi - z|)$ for $\xi$, $z$ near $z^*$. Consequently, *under the assumption*

$$\frac{\partial \rho}{\partial z_1}(z^*) \neq 0$$

*we can find* $\sigma$, $c > 0$ *so that*

$$\left|\frac{\partial \rho}{\partial z_1}(\xi)\right| > c > 0,$$

$K(\xi) \neq 0$ and

$$\left|\frac{\partial \rho}{\partial z_1}(\xi) + h(\xi, z)\right| > c > 0 \quad \textit{for all} \quad (\xi, z) \in B(z^*, \sigma) \times B(z^*, \sigma).$$

Now we can show:

LEMMA 1.7 (integration by parts). *Let* $\varphi(\xi, z)\colon \bar{D} \times \bar{D} \to \mathbf{C}$ *and let* $z^* \in \delta M$. *Assume that*

$$\frac{\partial \rho}{\partial z_1}(z^*) \neq 0$$

*and*

(i) $\varphi \in C^{s+1}(\bar{D}\times\bar{D})$ $(1\le s\le\infty)$,
(ii) supp $\varphi(\xi, z)$ *is compactly contained in* $B(z^*, \sigma)\times B(z^*, \sigma)$ *where* $\sigma>0$ *is constructed as in the previous observation.*

*Define*

$$G(z)=\frac{(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}\varphi(\xi, z)\frac{\omega_k'(\text{grad }\tilde{\rho}(\xi))\wedge\omega_k(\xi)}{(\phi(\xi, z))^k}. \tag{1.8}$$

*Then for every first order complex partial derivative*

$$D_\alpha=\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n},\frac{\partial}{\partial \bar{z}_1},\ldots,\frac{\partial}{\partial \bar{z}_n}$$

*we have*

$$D_\alpha G(z)=\frac{(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}\varphi_{D_\alpha}(\xi, z)\frac{\omega_k'(\text{grad }\tilde{\rho}(\xi))\wedge\omega_k(\xi)}{(\phi(\xi, z))^k} \tag{1.9}$$

*where* $\varphi_{D_\alpha}\in C^s(\bar{D}\times\bar{D})$ *satisfies* (ii). *Moreover there exist constants* $K_t$, $0\le t<\infty$, *independent of* $\varphi$, *so that*

$$\|\varphi_{D_\alpha}(\xi, z)\|_t\le K_t\,\|\varphi(\xi, z)\|_{t+1}\quad \textit{for}\quad 0\le t\le s. \tag{1.10}$$

*In addition if all the partial derivatives of* $\varphi$ *of order* $s+1$ *satisfy a Holder condition with exponent* $\mu$, $0<\mu\le 1$, *then the same holds for all the partial derivatives of order* $s$ *of* $\varphi_{D_\alpha}$.

*Proof.* Since the case $D_\alpha=\partial/\partial\bar{z}_i$ $(1\le i\le n)$ is immediate we assume $D_\alpha=\partial/\partial z_j$.

Differentiating formula (1.8) under the integral and using Lemma 1.5 we get

$$\frac{\partial G}{\partial z_j}(z)=G_1(z)+G_2(z) \tag{1.11}$$

where

$$G_1(z)=\frac{(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}\frac{\partial\varphi}{\partial z_j}(\xi, z)\frac{\omega_k'(\text{grad }\tilde{\rho}(\xi))\wedge\omega_k(\xi)}{(\phi(\xi, z))^k} \tag{1.12}$$

and

$$G_2(z)=\frac{k!}{(2\pi i)^k}\int_{\xi\in\partial M}g(\xi, z)\frac{d\bar{\xi}_2\wedge\cdots\wedge d\bar{\xi}_k\wedge d\xi_1\wedge\cdots\wedge d\xi_k}{(\phi(\xi, z))^{k+1}} \tag{1.13}$$

with

$$g(\xi, z)=\frac{\varphi(\xi, z)\cdot\dfrac{\partial\rho}{\partial z_j}(\xi)\cdot K(\xi)}{\dfrac{\partial\rho}{\partial\bar{z}_1}(\xi)}$$

that extended as the zero function outside $B(z^*, \sigma)\times B(z^*, \sigma)$ belongs to $C^{s+1}(\bar{D}\times\bar{D})$.

Using relation (1.6) and Stoke's theorem on the $C^\infty$ manifold $\partial M$ we get, from (1.13),

$$G_2(z)=\frac{(-1)^{k-1}k!}{(2\pi i)^k}\int_{\xi\in\partial M}(g(\xi,z)\,d\bar{\xi}_2\wedge\cdots\wedge d\bar{\xi}_k\wedge d\xi_2\wedge\cdots\wedge d\xi_k\wedge\frac{d\xi_1}{(\phi(\xi,z))^{k+1}}$$

$$=\frac{(-1)^k(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}\frac{g(\xi,z)}{\dfrac{\partial\rho}{\partial z_1}(\xi)+h(\xi,z)}$$

$$\times\, d\bar{\xi}_2\wedge\cdots\wedge d\bar{\xi}_k\wedge d\xi_2\wedge\cdots\wedge d\xi_k\wedge d_\xi\left(\frac{1}{(\phi(\xi,k))^k}\right)$$

$$=\frac{(-1)^{k+1}(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}\frac{d_\xi(k(\xi,z))}{(\phi(\xi,z))^k}\wedge d\bar{\xi}_2\wedge\cdots\wedge d\bar{\xi}_k\wedge d\xi_2\wedge\cdots\wedge d\xi_k$$

where

$$k(\xi,z)=\frac{g(\xi,z)}{\dfrac{\partial\rho}{\partial z_1}(\xi)+h(\xi,z)}\in C^{s+1}(\bar{D}\times\bar{D})$$

and (ii) is satisfied.

But for $(\xi,z)\in B(z^*,\sigma)\times B(z^*,\sigma)$ with $\xi\in\partial M$ we have

$$d_\xi(k(\xi,z))\equiv\frac{\partial k}{\partial\xi_1}(\xi,z)\,d\xi_1+\frac{\partial k}{\partial\bar{\xi}_1}(\xi,z)\,d\bar{\xi}_1$$

$$\equiv\frac{\dfrac{\partial\rho}{\partial z_1}(\xi)\cdot\dfrac{\partial k}{\partial\xi_1}(\xi,z)-\dfrac{\partial\rho}{\partial z_1}(\xi)\cdot\dfrac{\partial k}{\partial\xi_1}(\xi,z)}{\dfrac{\partial\rho}{\partial\bar{z}_1}(\xi)}\,d\xi_1$$

$$(\text{mod } d\bar{\xi},\ldots,d\bar{\xi}_k,d\xi,\ldots,d\xi_k).$$

So using Lemma 1.5 and Lemma 1.6 we get

$$G(z)=\frac{(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}\psi(\xi,z)\frac{\omega_k'(\operatorname{grad}\tilde{\rho}(\xi))\wedge\omega_k(\xi)}{(\phi(\xi,z)^k}\tag{1.14}$$

where

$$\psi(\xi,z)=\frac{\dfrac{\partial\rho}{\partial\bar{z}_1}(\xi)\cdot\dfrac{\partial k}{\partial\xi_1}(\xi,z)-\dfrac{\partial\rho}{\partial z_1}(\xi)\cdot\dfrac{\partial k}{\partial\bar{\xi}_1}(\xi,z)}{K(\xi)}$$

belongs to $C^s(\bar{D}\times\bar{D})$ and satisfies (ii). Now (1.9) follows from (1.11), (1.12), and (1.13) and relation (1.10) can be obtained from the explicit expression for $\varphi_{D_\alpha}(\xi,z)$. The lemma is proved.

We now use induction and Lemma 1.7 to obtain:

LEMMA 1.8. *With the same hypothesis and definitions of Lemma 1.7 we have $G(z)\in C^s(\bar{D})$ and there exist constants $K_s$, $0\le s<\infty$, independent of $\varphi$ so that*

$$(1.15)\qquad |D_\alpha G(z)|\le K_s\|\varphi(\xi,z)\|_{s+1}\quad \text{for all}\quad z\in D\quad \text{and all}\quad |\alpha|\le s.$$

*Moreover if all the partial derivatives of $\varphi$ of order $s+1$, satisfy a Holder condition with exponent $0<\mu\le 1$, then $G(z)\in C^{s+1}(\bar{D})$.*

*Proof.* An induction based on Lemma 1.7 allows us to construct, for every $|\alpha|\le s+1$, a function $\varphi_{D_\alpha}(\xi,z)\in C^{s+1-|\alpha|}(\bar{D}\times\bar{D})$ and a constant $K'_s$, independent of $\varphi$, so that

$$D_\alpha G(z)=\frac{(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}\varphi_{D_\alpha}(\xi,z)\frac{\omega'_k(\operatorname{grad}\tilde{\rho}(\xi))\wedge\omega_k(\xi)}{(\phi(\xi,z))^k}$$

and

$$\|\varphi_{D_\alpha}(\xi,z)\|_1\le K'_s\|\varphi(\xi,z)\|_{s+1}\quad \text{for}\quad |\alpha|\le s.$$

Moreover if all the partial derivatives of order $s+1$ of $\varphi$ satisfy a Holder condition with exponent $0<\mu\le 1$, so do all the $\varphi_{D_\alpha}(\xi,z)$ with $|\alpha|=s+1$.

For $z\in D$ we set

$$h(z)=\frac{(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}1\,\frac{\omega'_k(\operatorname{grad}\tilde{\rho}(\xi))\wedge\omega_k(\xi)}{(\phi(\xi,z))^k}$$

and, by Henkin's result [7], we note that $h\in A(D)$. Now we can write

$$D_\alpha G(z)=\frac{(k-1)!}{(2\pi i)^k}\int_{\xi\in\partial M}(\varphi_{D_\alpha}(\xi,z)-\varphi_{D_\alpha}(z,z))\frac{\omega'_k(\operatorname{grad}\tilde{\rho}(\xi))\wedge\omega_k(\xi)}{(\phi(\xi,z))^k}+\varphi_{D_\alpha}(z,z)\cdot h(z).$$

Thus

$$|D_\alpha G(z)|\le C_1\|\varphi_{D_\alpha}(\xi,z)\|_1\int_{\xi\in\partial M}\frac{|\xi-z|}{|\phi(\xi,z)|^k}m(d\xi)+C_2\|\varphi_{D_\alpha}(\xi,z)\|_0$$

or, using Lemma 1.3,

$$|D_\alpha G(z)|\le C_3\|\varphi_{D_\alpha}(\xi,z)\|_1\le K_s\|\varphi(\xi,z)\|_{s+1}.$$

This proves 1.15. The fact that $G(z)\in C^{s-1}(\bar{D})$ follows immediately from 1.15 and since all the $\varphi_{D_\alpha}(\xi,z)$ with $|\alpha|=s$ satisfy a Holder condition with exponent 1 we get $G(z)\in C^s(\bar{D})$ as a consequence of Lemma 1.4. The last assertion of the statement also follows from Lemma 1.4.

1.5 *Proof of Theorem* 1. We are to show that

$$(1.3)\qquad Lf(z)=\frac{(k-1)!}{(2\pi i)^k}\int_{\xi\in M}f(\xi)\frac{\omega'_k(\operatorname{grad}\tilde{\rho}(\xi))\wedge\omega_k(\xi)}{(\phi(\xi,z))^k}$$

defines a continuous linear operator $L: C^{s+1}(\partial M) \to A^s(\bar{D})$ and $Lf \in A^{s+1}(\bar{D})$ whenever all the partial derivatives of order $s+1$ of $f$ satisfy a Holder condition with exponent $\mu$, $0 < \mu \leq 1$.

Let $z^*$ be any point in $\partial M$. Since grad $\tilde{\rho}(z^*) \neq 0$, we can assume without loss of the generality that

$$\frac{\partial \rho}{\partial z_1}(z^*) \neq 0.$$

Let $\sigma > 0$ be chosen as in Lemma 1.7. Define

$$C_0^{s+1}(B(z^*, \sigma)) = \{f \in C^{s+1}(\mathbf{C}^k)/\text{supp} f \Subset B(z^*, \sigma)\}.$$

We first show:

LEMMA 1.9. (1.3) *defines a continuous linear operator*

$$L: C_0^{s+1}(B(z^*, \sigma/3)) \to A^s(D).$$

*Moreover* $Lf \in A^{s+1}(D)$ *whenever all the partial derivatives of order* $s+1$ *of* $f$ *satisfy a Holder condition with exponent* $\mu$, $0 < \mu \leq 1$.

*Proof.* It is easy to check that $Lf \in H(D)$. It remains to show that $Lf \in C^s(\bar{D})$ and that $L$ is continuous. To do this fix a $C^\infty$ function $X_1$ so that $X_1 \equiv 1$ in $B(z^*, \frac{2}{3}\sigma)$ and $X_1 \equiv 0$ off $B(z^*, \sigma)$. Put $X_2 = 1 - X_1$.

Now we can write

$$Lf(z) = L_1 f(z) + L_2 f(z) \tag{1.16}$$

where

$$L_i f(z) = \frac{(k-1)!}{(2\pi i)^k} \int_{\xi \in \partial M} f(\xi) \cdot X_i(z) \frac{\omega_k'(\text{grad } \tilde{\rho}(\xi)) \wedge \omega_k(\xi)}{(\phi(\xi, z))^k} \quad \text{for} \quad i = 1, 2.$$

Since $X_2 = 0$ on $B(z^*, 2/3\sigma)$ and $\phi(\xi, z)$ is bounded away from zero for $\xi \in B(z^*, \sigma/3)$ and $z \notin B(z^*, 2/3\sigma)$ $(z \in D)$ we can check, by differentiating under the integral that $L_2 f \in C^\infty(\bar{D})$ and that $L_2$ is a continuous operator.

As for the operator $L_1$; we observe that the map

$$f(\xi) \to \varphi(\xi, z) = f(\xi) \cdot X_1(z)$$

is continuous and so the result follows from Lemma 1.8.

In order to finish the proof of Theorem 1 we pick $z_1^*, \ldots, z_q^* \in \partial M$ in such a way that $\partial M \subset \bigcup_{i=1}^q B(z_i^*, \sigma/6)$ where $\sigma$ is chosen so small that Lemma 1.9 holds and for each $i = 1, 2, \ldots, q$ there exists a $C^\infty$ change of variables

$$\eta_i: B(z_i^*, \sigma) \to V_i \subset \mathbf{R}^{2n}$$

so that $\eta_i(B(z_i^*, \sigma) \cap \partial M) = V_i \cap \mathbf{R}^{2k-1}$.

Let $\{X_i\}_{i=1}^q$ be a $C^\infty$ partition of the unity for $\partial M$ subordinated to $\{B(z_i^*, \sigma/6)\}_{i=1}^q$; let $p: \mathbf{R}^{2n} \to \mathbf{R}^{2k-1}$ be the canonical projection and let $\{s_i\}_{i=1}^q$ be $C^\infty$ functions on $\mathbf{C}^n$ so that $s_i \equiv 1$ on $B(z_i^*, \sigma/6)$ and supp $s_i \Subset B(z_i^*, \sigma/3)$.

We define operators $H_i\colon C^{s+1}(\partial M) \to C_0^{s+1}(B(z_i^*, \sigma/3))$ by

$$H_i f(z) = \begin{cases} [(X_i \cdot f) \circ \eta_i^{-1} \circ p \circ \eta_i(z)] \cdot s_i(z) & \text{if defined} \\ 0 & \text{otherwise.} \end{cases}$$

Then the operators $H_i$, $1 \le i \le q$, are continuous and if all of the partial derivatives of order $s+1$ of $f$ satisfy a Holder condition with exponent $\mu$ then the same holds for $H_i f$. Moreover for any $f \in C^{s+1}(\partial M)$ we have $f = \sum_{i=1}^q (H_i f)/\partial M$.

Finally since $Lf = \sum_{i=1}^q L \circ H_i(f)$, Theorem 1 is proved.

1.5. *Proof of Theorem* 2. We split the proof into two cases.

*Case* $k = n-1$. We are to show that if $f \in A^\infty(D)$ and $f/M \equiv 0$ then there exists $h \in A^\infty(D)$ so that $f(z) = z_n \cdot h(z)$ for all $z \in D$. A well known consequence of Cartan's "Theorem B" [6, Theorem 18, page 245] shows that there exists $h$, holomorphic in $D$, so that $f(z) = z_n \cdot h(z)$. Our task now is to show that $h$ is $C^\infty$ on $\bar{D}$. Since $f \in A^\infty(D)$ it can be easily seen that $h$ is $C^\infty$ in a relative neighborhood of $z$ in $\bar{D}$ for all $z \in \bar{D} - \partial M$. So we only have to show that for every $z^* \in \partial M$, $h$ is $C^\infty$ in a relative neighborhood of $z^*$. To do this let

$$z^* = (z_1^*, \ldots, z_{n-1}^*, 0) \in \partial M.$$

Since $\operatorname{grad} \tilde{\rho}(z^*) \neq 0$ we can assume, without loss of generality, that, for some $\sigma_0 > 0$,

$$\left| \frac{\partial \rho}{\partial z_1}(z) \right| > c > 0 \quad \text{for all} \quad z \in B(z^*, \sigma_0).$$

Extend $\rho$ to a $C^\infty$ function on $\mathbf{C}^n$ and consider the map $F\colon \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$ defined by

$$F(z, \xi) = (\xi - z_1) \cdot \frac{\partial \rho}{\partial z_1}(\xi, z_2, \ldots, z_{n-1}, 0) - z_n \frac{\partial \rho}{\partial z_n}(\xi, z_2, \ldots, z_{n-1}, 0).$$

Then $F$ is a $C^\infty$ function, $F(z^*, z_1^*) = 0$ and

$$D_\xi F(z^*, z_1^*) = \frac{\partial \rho}{\partial z_1}(z^*)\, d\xi$$

is a linear homeomorphism $\mathbf{C} \to \mathbf{C}$. Thus by the implicit function theorem there exists $\sigma > 0$ and a $C^\infty$ map $\xi^*(\cdot)\colon B(z^*, \sigma) \to \mathbf{C}$ so that $\xi^*(z^*) = z_1^*$ and $F(z, \xi^*(z)) = 0$ for all $z \in B(z^*, \sigma)$. Setting

$$\eta(z) = (\xi^*(z), z_2, \ldots, z_{n-1}, 0)$$

and making $\sigma$ smaller if necessary, we have

$$\left| \frac{\partial \rho}{\partial z_1}(\eta(z)) \right| > c > 0$$

and

$$\sum_{i=1}^{n} \frac{\partial\rho}{\partial z_i}(\eta(z)) \cdot (\eta_i(z) - z_i) = 0 \quad \text{for} \quad z \in B(z^*, \sigma). \tag{1.17}$$

Relation (1.17) means that $z$ lies in the complex tangent plane at $\eta(z)$ to the corresponding level curve of $\rho$. This together with the strict convexity of $\rho$ implies that $\eta(z) \in M$ whenever $z \in D \cap B(z^*, \sigma)$.

Now for $z \in B(z^*, \sigma) \cap D$ we consider the map

$$\lambda(\cdot)\colon t \to t \cdot z + (1-t)\eta(z) \quad \text{for} \quad 0 \leq t \leq 1.$$

By the convexity of $D$ we have $\lambda(t) \in D$ for $0 \leq t \leq 1$. Consequently we can write

$$\begin{aligned} f(z) &= f(z) - f(\eta(z)) \\ &= (z_1 - \xi^*(z)) \cdot \int_0^1 \frac{\partial\rho}{\partial z_1}(\lambda(t))\,dt + z_n \cdot \int_0^1 \frac{\partial f}{\partial z_n}(\lambda(t))\,dt. \end{aligned}$$

So setting

$$g_i(z) = \int_0^1 \frac{\partial f}{\partial z_i}(\lambda(t))\,dt \quad \text{for} \quad i = 1, n$$

and using (1.17) we get

$$f(z) = z_n \cdot h_1(z) \quad \text{for all} \quad z \in D \cap B(z^*, \sigma)$$

where

$$h_1(z) = g_n(z) - \frac{\dfrac{\partial\rho}{\partial z_n}(\eta(z))}{\dfrac{\partial\rho}{\partial z_1}(\eta(z))}\, g_1(z).$$

Since $f \in A^\infty(D)$ and $\xi^*(z)$ is $C^\infty$ in $B(z^*, \sigma)$ we can show, by differentiating under the integral, that $g_i(z)$ is a $C^\infty$ function on $D \cap B(z^*, \sigma/2)$ for $i = 1, n$; and moreover all of its partial derivatives are bounded in $D \cap B(z^*, \sigma/2)$. Therefore the same holds for $h_1$. It is well known that this implies $h_1$ is $C^\infty$ on $\overline{D \cap B(z^*, \sigma/2)}$. Finally, since for $z \in D \cap B(z^*, \sigma/2)$ with $z_n \neq 0$ we have $h(z) = h_1(z)$, we get $h(z) = h_1(z)$ for all $z \in D \cap B(z^*, \sigma/2)$ because of the continuity of $h$ and $h_1$. This ends the proof of the case $k = n-1$.

*General case.* The proof is done by reverse induction on $k$. We first introduce the notation $M_k = \mathbf{C}^k \cap D$ for $k = 1, \ldots, n-1$. $M_k$ is strictly convex with $C^\infty$ boundary. Assume that the conclusion holds for $k+1$. We are to show it for $k$. Let $f \in A^\infty(D)$ be so that $f/M_k \equiv 0$. Define the auxiliary function

$$l(z_1, \ldots, z_{k+1}) = f(z_1, \ldots, z_{k+1}, 0, \ldots, 0) \quad \text{on} \quad M_{k+1};$$

then $l \in A^\infty(M_{k+1})$ and $l/M_k \equiv 0$. So by the preceding case there exists $g \in A^\infty(M_{k+1})$ so that

$$l(z_1, \ldots, z_{k+1}) = z_{k+1} \cdot g(z_1, \ldots, z_{k+1}).$$

By Corollary 1 there exists $h_{k+1} \in A^\infty(D)$ so that $h_{k+1}/M_{k+1} = g$. Now the function $f(z) - z_{k+1}h_{k+1}(z)$ defined on $D$ belongs to $A^\infty(D)$ and vanishes on $M_{k+1}$. So by the induction hypothesis we can write

$$f(z) - z_{k+1} \cdot h_{k+1}(z) = z_{k+2} \cdot h_{k+2}(z) + \cdots + z_n \cdot h_n(z) \quad \text{for all} \quad z \in D$$

with $h_{k+1}, h_{k+2}, \ldots, h_n \in A^\infty(D)$ as we wanted to show.

## 2. The general situation

Through all of this section $D$ will be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with $C^\infty$ boundary; that is $D = \{z \in D'/\rho(z) < 0\}$ where $\rho$ is a $C^\infty$ real valued strictly plurisubharmonic function on a neighborhood $D'$ of $\bar{D}$ and grad $\rho \neq 0$ on $\partial D$. We let $M'$ be a $k$-dimensional complex manifold in $D'$ that intersects $\partial D$ transversally; this means at every point $z \in \partial D \cap M'$ the intersection of the complex tangent planes to $\partial D$ and $M'$ has complex dimension $k-1$. We set $M = M' \cap D$.

2.1. *Statement of the results of this section.* Our main result is:

THEOREM 3. *If $f \in A^\infty(M)$ then there exists $F \in A^\infty(D)$ so that $F/M = f$.*

Theorem 3 is proved as follows: In 2.2 we give (without a proof) a slight modification of Lemma 1.1 in Henkin [7] which expresses the fact that the local version of Theorem 3 is the case when $D$ is strictly convex domain and $M$ is a plane section. In 2.3 we use this localization lemma and the results of Section 1 to identify certain sheaves we use in 2.4 together with an analogue of Cartan's "Theorem B" to finish the proof of Theorem 3.

Finally in 2.5 we use an embedding result of Fornaess [4] Theorem 9 and Theorem 3 above to obtain the following approximation result.

THEOREM 4. *Let $D$ be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$ with $C^\infty$ boundary. Then there exist a neighborhood $\Omega$ of $\bar{D}$ and functions $\psi_1, \ldots, \psi_m \in H(\Omega)$ so that given $f \in A^\infty(D)$, $\varepsilon > 0$ and $0 \leq l < \infty$ there exists a polynomial $h$ in the functions $\psi_1, \ldots, \psi_m$ so that $\|f - h\|_l^{\bar{D}} < \varepsilon$ where*

$$\|f - h\|_l^{\bar{D}} = \sum_{|\alpha| \leq l} \sup_{z \in \bar{D}} |D_\alpha f(z) - D_\alpha h(z)|.$$

We note that the fact that functions in $A^\infty(D)$ can be approximated in the above sense by functions in $H(\Omega)$ for a suitable neighborhood $\Omega$ of $\bar{D}$ can be obtained as a consequence of Lemma 2′ of Cirka [3] which is proved by different methods.

2.2. *Localization.* For $z^* \in \partial M$ let

$$F(z, z^*) = \sum_{i=1} \frac{\partial \rho}{\partial z_i}(z^*)(z_i - z_i^*) + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z_i \, \partial z_j}(z^*) \cdot (z_i - z_i^*)(z_j - z_j^*).$$

If $U$ is an open set in $D'$ we denote by $\theta_U$ the space of holomorphic functions on $U$ and we set $\mathcal{F}_U(M') = \{f \in \theta_U \mid f|_{M'} = 0\}$.

We can state now:

LEMMA 2.1. *Let $z^* \in \partial M$. Then for any $\varepsilon > 0$ there exist constants $0 < \delta < \sigma < \varepsilon$, functions*

$$F_1, \ldots, F_{n-k} \in \mathcal{T}_{B(z^*,\sigma)}(M')$$

*and numbers $n_1, \ldots, n_{k-1} \in \{1, \ldots, n\}$ so that the map*

$$z \to (z_{n_1} - z^*_{n_1}, \ldots, z_{n_{k-1}} - z^*_{n_{k-1}}, F(z, z^*), F_1(z), \ldots, F_{n-k}(z))$$

*is a biholomorphic change of variables from the ball $B(z^*, \sigma)$ onto a neighborhood $U_{z^*}$ of $0$ in the space of complex variables $\omega = (\omega_1, \ldots, \omega_n)$. Moreover; the preimage $G_z^*$ of some strictly convex subdomain, $V_z^*$ of $U_z^*$ with $C^\infty$ boundary and with $\bar{V}_{z^*} \subset U_z^*$, satisfies*

$$D \cap B(z^*, \delta) \subset G_{z^*} \subset D.$$

For the proof we refer to Lemma 1.11 in Henkin [7].

2.3. *Identification of sheaves.* We denote by $\theta$ the sheaf of germs of holomorphic functions on $D'$ and by $W$ the sheaf of germs of $C^\infty$ functions on $\bar{D}$ which are holomorphic in $D$. We set

$$\mathcal{T}(M') = \{f \in \theta / f / M' \equiv 0\} \quad \text{and} \quad \mathcal{T}^W(M) = \{\omega \in W \mid \omega \mid M \equiv 0\}.$$

We define ${}_MW = W/_{\mathcal{T}^W(M)}$; it follows from Lemma 2.1 and Corollary 1 that ${}_MW$ can be identified with the sheaf of germs of $C^\infty$ functions on $\bar{M}$ that are holomorphic when restricted to $M$.

The stalk over $z \in D'$ of the sheaf $\theta$ will be denoted by $\theta_z$ and similarly for the other sheaves considered. We can state now:

LEMMA 2.2. *If $z \in \bar{D}$ then the map $\tilde{f} \otimes_{\theta_z} \tilde{\omega} \to \tilde{f} \cdot \tilde{\omega}$ induces an isomorphism $\mathcal{T}_z(M') \otimes_{\theta_z} W_z \approx \mathcal{T}_z^W(M)$.*

*Proof.* If $z \in \bar{D} - \partial M$ there is nothing to show; so we assume $z = z^* \in \partial M$. Since $W_z$ is a flat $\theta_z$-module (see [10, Theorems 5.1 and 5.2]), tensoring the inclusion $0 \to \mathcal{T}_z(M') \leftarrow \theta_z$ with $W_z$, the above map is injective. The fact that it is onto follows from Theorem 2 via Lemma 2.1. The lemma is proved.

2.4. *Proof of Theorem* 3. The sequence $0 \to \mathcal{T}^W(M) \to W \to {}_MW \to 0$ is an exact sequence of sheaves. Since $\mathcal{T}(M')$ is a coherent sheaf in $D'$ [6, Theorem 2, page 138] using Theorem 5.4 in Nagel [10] we get

$H^q(\bar{D}, \mathcal{T}(M')\otimes_\theta W)=0$ for all $q\geq 1$. So, via Lemma 2.2, we obtain

$$H^q(\bar{D}, \mathcal{T}^W(M))=0 \quad \text{for all} \quad q\geq 1.$$

Now Theorem 3 can be obtained by passing to long exact cohomology sequences.

2.5. *Proof of Theorem* 4. We first need:

LEMMA 2.3. *Let $C$ be a bounded strictly convex domain in $\mathbf{C}^m$ with $C^\infty$ boundary and let $f\in A^\infty(C)$. Then given any $\varepsilon>0$ and $0\leq l<\infty$ there exists a polynomial $p$ over $\mathbf{C}^m$ so that $\|f-p\|_l^{\bar{C}}\leq\varepsilon$.*

*Proof.* Without loss of generality we assume $0\in C$. For $0<r<1$ we put

$$C_r=\{z\in\mathbf{C}^m/r\cdot z\in C\}=\frac{1}{r}\cdot C.$$

Then $C_r$ is strictly convex and we set $f_r(z)=f(r\cdot z)$. Now $f_r\in A^\infty(C_r)$ and since $f\in A^\infty(C)$ we can find $0<r_1<1$ so that $\|f-f_{r_1}\|_l^C<\varepsilon/2$. Pick $r_2$ so that $r_1<r_2<1$. Then $\bar{C}\subset C_{r_2}\subset\bar{C}_{r_2}\subset C_{r_1}$ and $f_{r_1}\in H(\bar{C}_{r_2})$. Via integral formulas we can find a constant $K$ so that

$$\|h\|_l^{\bar{C}}\leq K\cdot\|h\|_\infty^{\bar{C}_{r_2}} \quad \text{for all} \quad h\in H(\bar{C}_{r_2})$$

Finally since $C_{r_2}$ is convex and hence polynomially convex, there exists a polynomial $p$ over $\mathbf{C}^m$ so that

$$\|p-f_{r_1}\|_l^{\bar{C}_{r_2}}<\varepsilon/2K.$$

Then we have

$$\|f-p\|_l^{\bar{C}}\leq\|f-f_{r_1}\|_l^{\bar{C}}+\|f_{r_1}-p\|_l^{\bar{C}_{r_2}}\leq\varepsilon/2+K\cdot\varepsilon/2K=\varepsilon.$$

This ends the proof of the lemma.

In order to finish the proof of Theorem 4 we use Corollary 9 on page 276 of [6] to find a domain of holomorphy $\Omega$ so that $\bar{D}\subset\Omega$ and $\bar{D}$ is holomorphically convex in $\Omega$. The embedding theorem for complex manifolds guarantees the hypothesis we need in order to apply Fornaess's result [4, Theorem 9]. Therefore we can find an embedding $\Psi:\Omega\to\mathbf{C}^m$ and a strictly convex domain $C$ on $\mathbf{C}^m$ with $C^\infty$ boundary so that:

(i) $$\Psi:\Omega\xrightarrow{\text{onto}}M'$$

is biholomorphic; where $M'$ is a closed submanifold in $\mathbf{C}^m$,

(ii) $\Psi(D)\subset C$ and $\Psi(\Omega-\bar{D})\subset\mathbf{C}^m-\bar{C}$,
(iii) $M'$ intersects $\partial C$ transversally.

Let $\Psi^{-1}: M'\to\Omega$ be the inverse of $\Psi$ and put $g=f\circ\Psi^{-1}$. Set $M=C\cap M'$. Then $g\in A^\infty(M)$ and by Theorem 3 we can find $G\in A^\infty(C)$ so that $G/M=g$.

There exist constants $K_l$, $0 \leq l < \infty$, depending on the derivatives of the components of $\Psi$, so that $\|S \circ \Psi\|_l^D \leq K_l \cdot \|S\|_l^C$ for all $S \in A^\infty(C)$.

Finally use Lemma 2.3 to find a polynomial $p$ over $\mathbf{C}^m$ so that $\|p - G\|_l^C \subset \varepsilon / K_l$ and put $h = p \circ \Psi \in H(\Omega)$. Then we have $\|f - h\|_l^{\bar{D}} < \varepsilon$ as desired.

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