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FIBRATIONS OVER DOUBLE MAPPING CYLINDERS

BY

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1. Introduction

In the diagram of topological spaces

$$(1.1)\qquad \begin{tikzcd} & & E_1 \\ E_B \arrow[rru, "j"] \arrow[r, "j_B"] \arrow[d, "p_B"'] & E \arrow[ru, "p_1"'] \arrow[d, "p"] & \\ B \arrow[r, "\beta"'] & M & \end{tikzcd}$$

let $p\colon E \to M$ be a Hurewicz fibration, E_B obtained from p by pullback along $\beta\colon B \to M$, p_1 an arbitrary map and $j = p_1 j_B$. Let $\mu\colon \mathcal{M}(p_B, j) \to \mathcal{M}(p, p_1)$ denote the map which is induced on double mapping cylinders.

In this paper we study the map μ when the base space M is a homotopy pushout:

$$(1.2)\qquad \begin{tikzcd} B \arrow[r, "\beta"] & M \\ C \arrow[u, "g"] \arrow[r, "f"'] \arrow[ru, phantom, "F"] & A \arrow[u, "\alpha"'] \end{tikzcd}$$

Let $p_A\colon E_A \to A$ be the pullback of p over A and $f * p_A$ the fiberwise join of f and p_A. We prove:

THEOREM 1.3. *There is a map $W\colon E(f * p_A) \to \mathcal{M}(p_B, j)$ such that the following homotopy commutative square is a homotopy pushout.*

$$\begin{tikzcd} E(f * p_A) \arrow[r, "W"] \arrow[d] & \mathcal{M}(p_B, j) \arrow[d, "\mu"] \\ A \arrow[r] & \mathcal{M}(p, p_1) \end{tikzcd}$$

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The special case of this result having $E_1 = *$ and $A = *$ has been studied by Held-Sjerve [4]. For this case note that $E(f * p_A) = C * F$ where $F = p^{-1}(*)$ is the fiber of p and hence the result says that $\mu\colon C_{p_B} \to C_p$ is coclassified by a map $C * F \to C_{p_B}$. When also $B = *$ the result is classical (cf. [5]) and the corresponding coclassifying map $C * F \to SF$ may be taken to be the Hopf construction on the clutching function $\gamma\colon C \times F \to F$ of p. This follows readily from our explicit construction of the map W above. However Held-Sjerve only assert that the coclassifying map $C * F \to SF$ and the Hopf construction on γ are homotopic after suspension! Another advantage of our treatment as opposed to that of [4] is that we require no restrictions on the spaces involved. This is made possible by utilizing the track calculus to bypass standard difficulties with identification mappings.

Theorem 1.3 is rich in applications. In Section 6 we use it to recast the classical James-Whitehead homotopy decomposition of the total space of a fibration. In Section 7 we use it to prove a theorem on the homotopy structure of the product of two double mapping cylinders. Interestingly enough this latter result yields two different descriptions for the product of two mapping cones. One of these is the description recently found by Baues [3] while the other is closely related to the well-known formula of Atiyah [2] for the smash product of two Thom complexes.

Other applications of Theorem 1.3 may be found in [7] and [8].

2. Notation

We assume familiarity on the part of the reader with homotopy pushouts and pullbacks (see [6], [10], [9]). Double mapping cylinders are denoted $\mathscr{M}(f, g)$ in the unbased category and $\tilde{\mathscr{M}}(f, g)$ in the based category. This notational convention is used systematically (e.g., $C_f \equiv \mathscr{M}(0, f)$ and $\tilde{C}_f \equiv (\tilde{\mathscr{M}}(0, f)$ are respectively the unreduced and the reduced mapping cones of f).

Suppose $f\colon C \to A$ and $g\colon C \to B$ are based maps. Let

$$q\colon \mathscr{M}(f, g) \to \tilde{\mathscr{M}}(f, g)$$

be the canonical quotient map. Note that q becomes a based map by taking $[*_C, t] \in \mathscr{M}(f, g)$, for any t, as base point. When it is unimportant to fix the parameter t we say that $\mathscr{M}(f, g)$ has the *variable base point*. Recall that a based space is *well-pointed* if the inclusion of the base point is a closed cofibration (in the unbased category). Standard arguments may be used to prove the following.

PROPOSITION 2.1. *If each of A, B and C is a well-pointed space then $q\colon \mathscr{M}(f, g) \to \tilde{\mathscr{M}}(f, g)$ is a based homotopy equivalence (with respect to the variable base point of $\mathscr{M}(f, g)$).*

(2.2) Given a diagram

$$\begin{tikzcd} A \arrow[d, "\alpha"'] \arrow[dr, phantom, "\Downarrow F"] & C \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\gamma"] \arrow[dr, phantom, "\Downarrow G"] & B \arrow[d, "\beta"] \\ A' & C' \arrow[l, "f'"] \arrow[r, "g'"'] & B' \end{tikzcd}$$

having homotopies F: $\alpha f \simeq f'\gamma$ and G: $\beta g \simeq g'\gamma$ we obtain a homotopy class $\mu(\alpha, \gamma, \beta; F, G)$: $\mathscr{M}(f, g) \to \mathscr{M}(f', g')$ which depends only on the track classes of the homotopies F and G. An explicit representative of $\mu(\alpha, \gamma, \beta; F, G)$ is the map μ: $\mathscr{M}(f, g) \to \mathscr{M}(f', g')$ specified by

$$\mu[c, t] = \begin{cases} F(c, 4t), & 0 \le t \le \frac{1}{4} \\ [\gamma(c), \frac{1}{2}(4t - 1)], & \frac{1}{4} \le t \le \frac{3}{4} \\ G(c, 4 - 4t), & \frac{3}{4} \le t \le 1 \end{cases}$$

for $c \in C$. Whenever one or both of F and G is the static homotopy then other representatives of $\mu(\alpha, \gamma, \beta; F, G)$ may be more convenient (cf. [9, Section 2]).

DEFINITION 2.3. The fiberwise join $\alpha * \beta$: $E(\alpha * \beta) \to X$ of maps α: $A \to X$ and β: $B \to X$ is defined as usual [9, Section 2]. If now α: $A \to X$ and β: $B \to Y$ are maps (with X not necessarily equal to Y), the *exterior join*

$$\alpha \star \beta\colon E(\alpha \star \beta) \to X \times Y$$

is defined to be the join of the maps $\alpha \times 1$: $A \times Y \to X \times Y$ and $1 \times \beta$: $X \times B \to X \times Y$. In case $X = Y$ observe that we have a commutative square:

$$\begin{tikzcd} E(\alpha * \beta) \arrow[r] \arrow[d] & E(\alpha \star \beta) \arrow[d] \\ X \arrow[r, "\Delta"] & X \times X \end{tikzcd}$$

3. The total space as a homotopy pushout

As in Section 1, we suppose p: $E \to M$ is a Hurewicz fibration with M the homotopy pushout (1.2). We fix a lifting function λ: $\Lambda(p) \to E^I$ for p. Here $\Lambda(p) = \{(e, \omega) \in E \times M^I\colon \omega(0) = p(e)\}$ and by definition λ satisfies $\lambda_0(e, \omega) = e$ and $p\lambda_t(e, \omega) = \omega(t)$. For $(e, \omega) \in \Lambda(p)$ and $t \in I$, we let $\lambda_t(e, \omega) \in E$ denote the evaluation of the path $\lambda(e, \omega)$ at parameter t.

Now for each $c \in C$ let $F^{\#}(c)$ be the path in M defined by $F^{\#}(c)(t) = F(c, t)$. A map γ: $C \times_A E_A \to E_B$, called a *clutching function* for p, is defined by $\gamma(c, e) = (g(c), \lambda_1(e, F^{\#}(c)))$ for $(c, e) \in C \times_A E_A$. (Here $C \times_A E_A$ is the fibered product of C and E_A over A.)

PROPOSITION 3.1. *The square*

$$\begin{tikzcd} C \times_A E_A \arrow[r, "\gamma"] \arrow[d, "\pi_2"'] & E_B \arrow[d, "j_B"] \arrow[dl, phantom, "\overset{G}{\Leftarrow}"] \\ E_A \arrow[r, "j_A"'] & E \end{tikzcd}$$

with $\pi_2(c, e) = e$ *and* $G_t(c, e) = \lambda_t(e, F^{\#}(c))$ *for* $(c, e) \in C \times_A E_A$, *is an unbased homotopy pushout.*

Proof. This follows by applying [10, Theorem 25] to the cube which has the square containing G as top and the square (1.2) as bottom.

Let ϕ: $\mathcal{M}(\pi_2, \gamma) \to E$, $\phi[c, e, t] = \lambda_t(e, F^{\#}(c))$ for $(c, e) \in C \times_A E_A$, be the homotopy equivalence given by (3.1). Clearly the diagram

$$\begin{tikzcd} \mathcal{M}(\pi_2, \gamma) \arrow[rr, "\phi"] \arrow[dr] & & E \arrow[dl, "p"] \\ & M & \end{tikzcd}$$

is commutative where the unnamed map is the composite $\mathcal{M}(\pi_2, \gamma) \to \mathcal{M}(f, g) \to M$, the first factor induced by the triple of maps (p_A, π_1, p_B) and the second factor the homotopy equivalence induced by the homotopy F. We leave open the question of whether ϕ is a fiber homotopy equivalence.

We remark that in defining the clutching function for p we clearly could have chosen to fix B rather than A. Our choice of A is determined by the given parameter in the homotopy pushout (1.2); i.e., F_0 has image in A.

We have so far in this section worked in the unbased category. To enunciate (3.1) in the based category we suppose that (1.2) is a based homotopy pushout and that the based map p: $E \to M$ is a Hurewicz fibration in the based category. This latter means that p is a Hurewicz fibration for which the given base point $e \in E$ is a *regular point* in the sense that p admits a lifting function λ satisfying $\lambda(e_0, pe_0^*) = e_0^*$ (where the notation x^* indicates the path with constant value x). For such a λ, the above definitions of γ and G respect base points and the identical argument (but made in the based category) shows that the square in (3.1) is a based homotopy pushout.

Remark 3.2. We may apply Proposition 3.1 to the trivial fibration

$$\mathcal{M}(f, g) \times Y \to \mathcal{M}(f, g).$$

This has clutching function $g \times 1$: $C \times Y \to B \times Y$. Thus by (1.3) the continuous bijection

$$\phi: \mathcal{M}(f \times 1_Y, g \times 1_Y) \to \mathcal{M}(f, g) \times Y$$

is a homotopy equivalence. In the based case, the map (no longer a bijection)

$$\phi: \tilde{\mathcal{M}}(f \times 1_Y, g \times 1_Y) \to \tilde{\mathcal{M}}(f, g) \times Y$$

is a based homotopy equivalence. This shows that on both the unbased and the based categories of topological spaces, the functor — $\times$ Y preserves homotopy pushouts.

DEFINITION 3.3. If (X, X_0) and (Y, Y_0) are pairs of topological spaces we define $(X, X_0) \dot{\times} (Y, Y_0)$ (abbreviated to $X \dot{\times} Y$ if the subspaces X_0 and Y_0 are (clear) to be the subspace $X \times Y_0 \cup X_0 \times Y$ of the product $X \times Y$. In the evident manner $\dot{\times}$ is a functor in two variables. If (X, X_0) and (Y, Y_0) are based pairs of spaces then $(X, X_0) \dot{\times} (Y, Y_0)$ is a based space with the obvious base point. Observe that $(X, *) \dot{\times} (Y, *) = X \vee Y$ and that $(X, X_0) \dot{\times} (Y, Y_0) = X \times Y$ if either $X_0 = X$ or $Y_0 = Y$.

Suppose given a pair (Y, Y_0) with inclusion i: $Y_0 \to Y$. Then with the maps f and g as in (1.2) we have a homotopy

$$C \times Y_0 \times I \to (\mathcal{M}(f, g), B) \dot{\times} (Y, Y_0)$$

given by $(c, y_0, t) \mapsto ([c, t], y_0)$ for $c \in C$, $y_0 \in Y$, $t \in I$.

PROPOSITION 3.4. *The above homotopy induces a map*

$$\phi: \mathcal{M}(A \times Y_0 \xleftarrow{f \times 1} C \times Y_0 \xrightarrow{g \times i} B \times Y) \longrightarrow (\mathcal{M}(f, g), B) \dot{\times} (Y, Y_0)$$

which is a homotopy equivalence. This proposition holds in the reduced case also.

Proof. We consider the diagram

$$\begin{tikzcd}
C \times Y_0 \arrow[r, "g \times 1"] \arrow[d, "f \times 1"'] \arrow[dr, phantom, "\overset{F \times 1}{\Rightarrow}"] & B \times Y_0 \arrow[r, "1 \times i"] \arrow[d, "i_1 \times 1"] & B \times Y \arrow[d, "i_1 \times 1"] \\
A \times Y_0 \arrow[r, "i_0 \times 1"'] & \mathcal{M}(f, g) \times Y_0 \arrow[r] & \mathcal{M}(f, g) \dot{\times} Y
\end{tikzcd}$$

in which F is the defining homotopy for $\mathcal{M}(f, g)$. By (3.2) the square on the left is a homotopy pushout. The square on the right is a topological pushout and hence a homotopy pushout since $i_1 \times 1$: $B \times Y_0 \to \mathcal{M}(f, g) \times Y_0$ is a closed cofibration. Therefore the outer rectangle is a homotopy pushout with a homotopy which is clearly track equivalent to the above homotopy. Hence ϕ is a homotopy equivalence.

4. Commuting parameters in double mapping cylinders

We suppose given diagram (4.1) provided with homotopies as shown.

(4.1)
$$\begin{tikzcd}
A_1 & C_1 \arrow[l, "a_1"'] \arrow[r, "b_1"] & B_1 \\
A \arrow[u, "f_1"] \arrow[d, "f_0"'] & C \arrow[l, "a"'] \arrow[r, "b"] \arrow[u, "g_1"'] \arrow[d, "g_0"] & B \arrow[u, "h_1"'] \arrow[d, "h_0"] \\
A_0 & C_0 \arrow[l, "a_0"] \arrow[r, "b_0"'] & B_0
\end{tikzcd}$$

Taking the double mapping cylinders of the horizontal pairs (as in (2.2)) we get maps

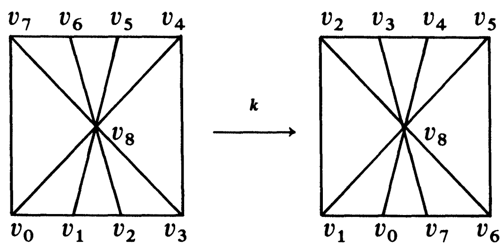
$$\mathcal{M}(a_0, b_0) \xleftarrow{\theta'} \mathcal{M}(a, b) \xrightarrow{\theta''} \mathcal{M}(a_1, b_1).$$

Likewise taking the double mapping cylinders of the vertical pairs we get maps

$$\mathcal{M}(f_0, f_1) \xleftarrow{\delta'} \mathcal{M}(g_0, g_1) \xrightarrow{\delta''} \mathcal{M}(h_0, h_1).$$

LEMMA 4.2. *There is a homeomorphism* $\mathcal{M}(\theta', \theta'') \cong \mathcal{M}(\delta', \delta'')$.

Proof. In the proof we adopt (see (2.2)) the $0 - \frac{1}{4} - \frac{3}{4} - 1$ division of the unit interval I to define the maps θ', θ'', δ' and δ''. Let $k\colon I^2 \to I^2$ be the self-homeomorphism of the unit square I^2 which is defined barycentrically on each triangle of I^2 as indicated below:



Let U be the disjoint union space

$$C \times I^2 \amalg A \times I \amalg B \times I \amalg C_0 \times I \amalg C_1 \times I \amalg A_0 \amalg B_0 \amalg A_1 \amalg B_1.$$

Now each of $\mathcal{M}(\theta', \theta'')$ and $\mathcal{M}(\delta', \delta'')$ is a quotient space of U. Define $k'\colon U \to U$ to be $1_C \times k$ on $C \times I$ and the identity elsewhere. A careful check of the corresponding identifications then shows that k' induces a homeomorphism

$$\mathcal{M}(\theta', \quad \theta'') \to \mathcal{M}(\delta', \quad \delta'').$$

Remarks 4.3. (i) Lemma 4.2 is also valid (with the same proof) in the based category.

(ii) In the proof of (4.2) the maps θ', θ'', δ' and δ'' were defined very precisely using explicit homotopies and a fixed division of the parameter interval. Actually we may allow ourselves to vary the homotopies of (4.1) up to track equivalence and to change the parameter division if we replace the homeomorphism of (4.2) by a homotopy equivalence.

More exactly, note that (4.2) provides a (canonical) homotopy making the square

$$\begin{tikzcd}
\mathscr{M}(a, b) \arrow[r, "\theta''"] \arrow[d, "\theta'"'] & \mathscr{M}(a_1, b_1) \arrow[d, "\mathrm{inc}"] \\
\mathscr{M}(a_0, b_0) \arrow[r, "\mathrm{inc}"'] & \mathscr{M}(\delta', \delta'')
\end{tikzcd}$$

a homotopy pushout. Now this square (but with a different homotopy of course) remains a homotopy pushout if the maps θ', θ'', δ' and δ'' are replaced within their homotopy classes. Also, and in the usual way, any of the four spaces occurring in the square may be replaced within its homotopy type.

(iii) In the presence of well-pointed spaces, Proposition 2.1 makes it possible to formulate "mixed" versions of the homotopy pushout in (4.3.ii).

From the point of view of applications in this paper, the following consequence of (4.3.ii) is all we shall need.

THEOREM 4.4. *Suppose we are given a homotopy commutative diagram:*

$$\begin{tikzcd}
 & B & \\
C' \arrow[ur, "g'"] \arrow[d, "f'"'] & C \arrow[l, "c'"] \arrow[r, "c''"'] \arrow[d, "f"] & C'' \arrow[ul, "g''"'] \arrow[d, "f''"] \\
A' & A \arrow[l, "a'"] \arrow[r, "a''"'] & A''
\end{tikzcd}$$

Then the double mapping cylinders on the induced pairs

$$A' \leftarrow \mathscr{M}(c', f) \rightarrow \mathscr{M}(g'', f''),$$
$$\mathscr{M}(f', g') \leftarrow \mathscr{M}(f, c'') \rightarrow A''$$

are homotopy equivalent.

Proof. Consider the following diagram (4.1) situation with corresponding vertical and horizontal double mapping cylinders as shown:

$$\begin{tikzcd}
B & C'' \arrow[l, "g''"'] \arrow[r, "f''"] & A'' & \mathscr{M}(g'', f'') \\
C' \arrow[u, "g'"] \arrow[d, "f'"'] & C \arrow[l, "c'"'] \arrow[r, "f"] \arrow[u, "c''"'] \arrow[d, "f"] & A \arrow[u, "a''"'] \arrow[d, "1"] & \mathscr{M}(c', f) \arrow[u] \arrow[d] \\
A' & A \arrow[l, "a'"'] \arrow[r, "1"] & A & \mathscr{M}(a', 1)
\end{tikzcd}$$

$$\mathscr{M}(f', g') \leftarrow \mathscr{M}(f, c'') \rightarrow \mathscr{M}(1, a'')$$

Since $\mathscr{M}(1, a'') \simeq A''$ and $\mathscr{M}(a', 1) \simeq A'$ the result follows from (4.3.ii).

Remark 4.5. Theorem 4.4 is a generalization of Lemma 3.3 in [9]. This is seen by applying (4.4) to the following diagram:

$$\begin{tikzcd}
 & Y & \\
C \arrow[ur, "u"] \arrow[d, "f"'] & C \arrow[l, "1"'] \arrow[r, "g"] \arrow[d, "f"] \arrow[dr, phantom, "F"] & B \arrow[ul, "v"'] \arrow[d, "\beta"] \\
A & A \arrow[l, "1"] \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

5. Proof of Theorem 1.3

With notation as in Section 3 we consider the following diagram where $G' = p_1 G$:

$$\begin{tikzcd}
 & B & \\
C \arrow[ur, "g"] \arrow[d, "f"'] & C \times_A E_A \arrow[l, "\pi_1"'] \arrow[r, "\gamma"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\Nearrow G'"] & E_B \arrow[ul, "p_B"'] \arrow[d, "j"] \\
A & E_A \arrow[l, "p_A"] \arrow[r] & E_1
\end{tikzcd} \tag{5.1}$$

We define a map $W\colon E(f * p_A) \to \mathscr{M}(p_B, j)$ by the formulas

$$W[c, e, t] = \begin{cases} [\gamma(c, e), 2t], & 0 \leq t \leq \frac{1}{2} \\ p_1 \lambda_{2-2t}(e, F^{\#}(c)), & \frac{1}{2} \leq t \leq 1 \end{cases}$$

for $(c, e) \in C \times_A E_A$, and

$$W(c) = g(c),\ c \in C, \qquad W(y) = p_1 j_A(y),\ y \in E_A.$$

Applying Theorem 4.4 to diagram (5.1) we obtain a homotopy pushout

$$\begin{tikzcd} E(f * p_A) \arrow[r, "W"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \mathcal{M}(p_B, j) \arrow[d] \\ A \arrow[r] & \cdot \end{tikzcd} \tag{5.2}$$

in which the lower right corner of the square is the double mapping cylinder on the pair of maps

$$\mathcal{M}(f, g) \xleftarrow{\delta'} \mathcal{M}(\pi_2, \gamma) \xrightarrow{\delta''} E_1$$

with δ' induced functorially by the triple (p_A, π_1, p_B) and δ'' induced by the homotopy G'. Observe that the diagram

$$\begin{tikzcd} \mathcal{M}(f, g) \arrow[d] & \mathcal{M}(\pi_2, \gamma) \arrow[l, "\delta'"'] \arrow[r, "\delta''"] \arrow[d, "\phi"] & E_1 \arrow[d, "1"] \\ M & E \arrow[l, "p"] \arrow[r, "p_1"'] & E_1 \end{tikzcd}$$

is commutative with the vertical maps homotopy equivalences. Hence square (5.2) is a homotopy pushout with the lower right corner being $\mathcal{M}(p, p_1)$. It is routine to check that the map $\mathcal{M}(p_B, j) \to \mathcal{M}(p, p_1)$ is μ. This completes the proof of (1.3).

Remark 5.3. The classical case of Theorem 1.3 arises when $A = B = E_1 = *$, yielding a homotopy pushout

$$\begin{tikzcd} C * F \arrow[r, "W"] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & SF \arrow[d, "\mu"] \\ * \arrow[r] & C_p \end{tikzcd}$$

where F is the fiber of $p\colon E \to SC$. Since the homotopy G' in this case is clearly the static homotopy, W is homotopic to the Hopf construction $h(\gamma)$ on the clutching function $\gamma\colon C \times F \to F$.

6. The James-Whitehead decomposition

In this section we use Theorem 1.3 to derive the classical James-Whitehead decomposition of the total space of a fibration [5].

We assume that $p: E \to X$ is a Hurewicz fibration and that the pair (X, X_0) is coclassified by a map $C \to X_0$. Further we assume that the fiber F of p over the "vertex" of X admits the finite homotopy decomposition

$$
(6.1)\qquad
\begin{tikzcd}
 & Y_{n+1} = F \\
B_n \arrow[r, "\beta_n"] & Y_n \arrow[u, "l_n"'] \\
B_{n-1} \arrow[r, "\beta_{n-1}"] & Y_{n-1} \arrow[u, "l_{n-1}"'] \\
 & \vdots \arrow[u] \\
B_0 \arrow[r, "\beta_0"] & Y_0 \arrow[u, "l_0"']
\end{tikzcd}
$$

This means that for each $i = 0, 1, \ldots, n$, there is a homotopy F_i such that the square

$$
\begin{tikzcd}
B_i \arrow[r, "\beta_i"] \arrow[d] & Y_i \arrow[d, "l_i"] \\
* \arrow[r] \arrow[ur, phantom, "F_i"] & Y_{i+1}
\end{tikzcd}
$$

is a homotopy pushout. For what follows we fix a choice of such homotopies F_i as well as a lifting function λ for p. Let

$$\gamma: C \times F \to E_{X_0} = p^{-1}(X_0), \qquad \gamma(c, y) = \lambda_1(y, F^{\#}(c)),$$

be the associated clutching function (where F is a fixed homotopy giving X as a mapping cone). For each $i = 0, 1, \ldots, n$, we set

$$\gamma_i = \gamma \circ (1 \times l_i): C \times Y_i \to E_{X_0}$$

and define maps

$$\bar{\beta}_i: B_i * C \longrightarrow M_i, \quad M_i \equiv \mathcal{M}(Y_i \xleftarrow{\text{proj}} C \times Y_i \xrightarrow{\gamma_i} E_{X_0})$$

by

$$\bar{\beta}_i[b, c, t] = \begin{cases} [(c, \beta_i b), 2t], & 0 \le t \le \frac{1}{2} \\ \gamma_{i+1}(c, F_i(b, 2 - 2t)), & \frac{1}{2} \le t \le 1 \end{cases}$$

(where $\gamma_{n+1} = \gamma$).

THEOREM 6.2. *E admits the homotopy decomposition*

$$\begin{tikzcd}
 & E \\
B_n * C \arrow[r, "\bar{\beta}_n"] & M_n \arrow[u] \\
B_{n-1} * C \arrow[r, "\bar{\beta}_{n-1}"] & M_{n-1} \arrow[u] \\
 & \vdots \arrow[u] \\
B_0 * C \arrow[r, "\bar{\beta}_0"] & M_0 \arrow[u]
\end{tikzcd}$$

with the composite

$$E_{X_0} \xrightarrow{i_1} M_0 \to \cdots \to E$$

being the inclusion.

Proof. From the following diagram (1.1) situation

$$\begin{tikzcd}
 & & E_{X_0} \\
C \times Y_i \arrow[r, "1 \times l_i"'] \arrow[d, "\mathrm{proj}"'] \arrow[urr, "\gamma_i"] & C \times Y_{i+1} \arrow[d, "\mathrm{proj}"] \arrow[ur, "\gamma_{i+1}"'] & \\
Y_i \arrow[r, "l_i"] & Y_{i+1} &
\end{tikzcd}$$

we obtain a homotopy pushout:

$$\begin{tikzcd}
B_i * C \arrow[r, "\bar{\beta}_i"] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & M_i \arrow[d, "\mu(l_i,\, 1 \times l_i,\, 1_{E_{X_0}})"] \\
* \arrow[r] & M_{i+1}
\end{tikzcd}$$

Since (X, X_0) is coclassified by a map $C \to X_0$, there is a homotopy equivalence $M_{n+1} \to E$. Hence E has the desired homotopy decomposition and clearly the composite $E_{X_0} \to M_0 \to \cdots \to E$ is the inclusion.

As in [9, Remark 5.2], the above fibration $p\colon E \to X$ admits a characteristic function $e\colon C \to E_{X_0}$ given by $e(c) = \lambda_1(*, F^{\#}(c))$ where $*$ is some fixed base point in F. Observe that in giving the homotopy decomposition (6.1), we have necessarily fixed base points (the "vertices") in Y_i for each $i = 0, 1, \ldots, n+1$. If we suppose that the maps l_i preserve these given base points, that Y_0 is a one point space and that the characteristic function e is defined with respect to the given base point in F then $e = \gamma_0\colon C \to E_{X_0}$. Hence, in this case, we obtain

$$E \simeq E_{X_0} \bigcup\nolimits_e C(C) \bigcup\nolimits_{\bar{\beta}_0} C(B_0 * C) \bigcup\nolimits_{\bar{\beta}_1} \cdots \bigcup\nolimits_{\bar{\beta}_n} C(B_n * C) \tag{6.3}$$

We have proven Theorem 6.2 in the unbased category. A similar result is available in the based category. For this one supposes that p is a Hurewicz fibration in the based category and that all homotopy pushouts are in the based category. We leave the exact formulation and details to the reader. Finally observe that if, in the presence of the hypothesis for the based case, the spaces B_i and C are well-pointed then a mixed version is valid. This mixed version says that the decomposition of E given in (6.2), but with the M_i's replaced by based double mapping cylinders and with the $B_i * C$'s having variable base points, is a based homotopy decomposition.

7. The product of two double mapping cylinders

In this final section we consider two arbitrary pairs of maps

$$X \xleftarrow{\alpha} A \xrightarrow{p} P, \qquad Y \xleftarrow{\beta} B \xrightarrow{q} Q.$$

We may consider the following Theorem 4.4 situation.

$$\begin{tikzcd}
 & P \times Y & \\
A \times Y \arrow[ur, "p \times 1"] \arrow[d, "\alpha \times 1"'] & A \times B \arrow[l, "1 \times \beta"'] \arrow[r, "p \times 1"] \arrow[d, "\alpha \times 1"'] & P \times Q \arrow[ul, "1 \times \beta"'] \arrow[d, "i_1 \times q"] \\
X \times Y & X \times B \arrow[l, "1 \times \beta"] \arrow[r, "i_0 \times q"'] \arrow[ur, phantom, "\Rightarrow"] & \mathcal{M}(\alpha, p) \times Q
\end{tikzcd}$$

We obtain a homotopy pushout

$$\begin{tikzcd}
E(\alpha \bar{\star} \beta) \arrow[r] \arrow[d] & \mathcal{M}(P \times Y \arrow[d, phantom, "\Rightarrow"] & P \times B \arrow[l, "1 \times \beta"'] \arrow[r, "i_1 \times q"] \arrow[d] & \mathcal{M}(\alpha, p) \times Q) \\
X \times Y \arrow[r] & \mathcal{M}(\mathcal{M}(\alpha, p) \times Y & \mathcal{M}(\alpha, p) \times B \arrow[l, "1 \times \beta"'] \arrow[r, "1 \times q"] & \mathcal{M}(\alpha, p) \times Q)
\end{tikzcd}$$

But from Proposition 3.4 we have a commutative square

$$\begin{tikzcd}
\mathcal{M}(1_P \times \beta, i_1 \times q) \arrow[r, "\phi"] \arrow[d] & (\mathcal{M}(\alpha, p), P) \dot{\times} (\mathcal{M}(\beta, q), Q) \arrow[d] \\
\mathcal{M}(1_{\mathcal{M}(\alpha, p)} \times \beta, 1_{\mathcal{M}(\alpha, p)} \times q) \arrow[r, "\phi"] & \mathcal{M}(\alpha, p) \times \mathcal{M}(\beta, q)
\end{tikzcd}$$

with each of the maps ϕ being a homotopy equivalence. The composite

$$W\colon E(\alpha \star \beta) \to (\mathcal{M}(\alpha, p), P) \dot{\times} (\mathcal{M}(\beta, q), Q)$$

is given by

$$W[a, b, t] = \begin{cases} (pa, [b, 2t]), & 0 \le t \le \frac{1}{2}, \\ ([a, 2-2t], qb), & \frac{1}{2} \le t \le 1, \end{cases}$$

$$W[a, y] = (pa, [y]),$$

$$W[x, b] = ([x], qb).$$

Therefore we obtain:

THEOREM 7.1. *For some homotopy the square*

$$\begin{tikzcd}
E(\alpha \star \beta) \arrow[r, "W"] \arrow[d] & (\mathcal{M}(\alpha, p), P) \dot{\times} (\mathcal{M}(\beta, q), Q) \arrow[d, "\mathrm{inc}"] \\
X \times Y \arrow[r, "\mathrm{inc}"] \arrow[ur, phantom, "\Nearrow"] & \mathcal{M}(\alpha, p) \times \mathcal{M}(\beta, q)
\end{tikzcd}$$

is a homotopy pushout, with W given by the above formula.

The based version of (7.1) also holds and consequently so does the mixed version whenever the spaces A, B, X and Y are well-pointed.

As remarked in the introduction, Theorem 7.1 yields two different representations for the product of two mapping cones. Firstly, if $X = *$ and $Y = *$ then we recover the following theorem of Baues [3, Satz (3.2)].

THEOREM 7.2. *For some homotopy the square*

$$\begin{tikzcd}
A * B \arrow[r, "W"] \arrow[d] & (C_p, P) \dot{\times} (C_q, Q) \arrow[d] \\
* \arrow[r] \arrow[ur, phantom, "\Nearrow"] & C_p \times C_q
\end{tikzcd}$$

is a homotopy pushout, with W defined by

$$W[a, b, t] = \begin{cases} (pa, [b, 2t]), & 0 \le t \le \frac{1}{2} \\ ([a, 2-2t], qb), & \frac{1}{2} \le t \le 1 \end{cases}$$

It should be noted that our formula for W in (7.2) differs from Baues' formula for W in that we consider a mapping cone to have vertex at parameter $t = 0$ while Baues takes the vertex at parameter $t = 1$. Note also that we could have obtained (7.2) by working out a James-Whitehead decomposition for $C_p \times C_q$ relative to the trivial fibration $C_p \times C_q \to C_p$.

Secondly if $P = *$ and $Q = *$ and if in applying Theorem 7.1 we replace $\mathscr{M}(\alpha, p)$ and $\mathscr{M}(\beta, q)$ by C_α and C_β respectively then we obtain the following theorem.

THEOREM 7.3. *For some homotopy the square*

$$\begin{tikzcd} E(\alpha \star \beta) \arrow[r, "w"] \arrow[d] & C_\alpha \vee C_\beta \arrow[d, "\mathrm{inc}"] \\ X \times Y \arrow[r, "i_1 \times i_1"'] \arrow[ur, phantom, "\Rightarrow"] & C_\alpha \times C_\beta \end{tikzcd}$$

is a homotopy pushout, with w defined by

$$w[a, b, t] = \begin{cases} (*, [b, 1 - 2t]), & 0 \le t \le \frac{1}{2}, \\ ([a, 2t - 1], *), & \frac{1}{2} \le t \le 1, \end{cases}$$

$$w[a, y] = (*, [y])$$

$$w[x, b] = ([x], *).$$

Here, of course, the vertices of the unreduced mapping cones C_α and C_β are used in constructing the wedge $C_\alpha \vee C_\beta$.

As a corollary we obtain Atiyah's formula for the Thom space of $\alpha \star \beta$ ([2], [4]).

COROLLARY 7.4. $C_{\alpha \star \beta} \simeq C_\alpha \wedge C_\beta$ *(as unbased spaces).*

Whenever each of the spaces X, Y, P and Q is a one point space then Theorems 7.2 and 7.3 coincide in the sense that there is a commutative diagram

$$\begin{tikzcd} A * B \arrow[r, "W"] \arrow[dr, "w"'] & SA \vee_1 SB \arrow[d, "r \vee r"] \\ & SA \vee_0 SB \end{tikzcd}$$

with W as in (7.2), w as in (7.3) and r denoting reversal of parameter in the suspensions. Here we have used $\vee_i$, $i = 0, 1$, to indicate that parameter $t = i$ is used in constructing the wedge $SA \vee_i SB$. (Of course this distinction is unnecessary in the based or mixed cases.) The map w: $A * B \to SA \vee SB$ is just the generalized Whitehead product map of Arkowitz [1].

Since $\Sigma(\Sigma A \times \Sigma B)$ homotopy retracts onto $\Sigma(\Sigma A \vee \Sigma B)$, (7.3) implies that $\Sigma w \simeq 0$ where $w\colon A * B \to \Sigma A \vee \Sigma B$ is the generalized Whitehead product map (mixed form). In fact Corollary 7.4 together with Hilton's formula

$$\Sigma(\tilde{C}_\alpha \times \tilde{C}_\beta) \simeq \Sigma\tilde{C}_\alpha \vee \Sigma\tilde{C}_\beta \vee \Sigma(\tilde{C}_\alpha \wedge \tilde{C}_\beta)$$

would suggest that we have $\Sigma w \simeq 0$ for a general $w\colon E(\alpha \star \beta) \to \tilde{C}_\alpha \vee \tilde{C}_\beta$ (with well-pointed spaces). To see that this is not true we may consider the following example. α is $1_X\colon X \to X$ and β is $* \to X$. Then $E(\alpha \star \beta)$ has the homotopy type of $X \times X$ and $\tilde{C}_\alpha \vee \tilde{C}_\beta$ has the homotopy type of X. With these identifications the map w corresponds to the projection $X \times X \to X$ onto the second factor.

PROPOSITION 7.5. *If A, B, X and Y are well-pointed spaces then the following diagram is based homotopy commutative (with $E(\alpha \star \beta)$ having the variable base point):*

$$\begin{tikzcd}
\Sigma E(\alpha \star \beta) \arrow[r, "\Sigma w"] \arrow[d] & \Sigma\tilde{C}_\alpha \vee \Sigma\tilde{C}_\beta \arrow[d, "\Sigma(i_1\,\mathrm{proj}_X) \vee \Sigma(i_1\,\mathrm{proj}_Y)"] \\
\Sigma(X \times Y) \arrow[r, "\omega"] & \Sigma(X \times Y) \vee \Sigma(X \times Y)
\end{tikzcd}$$

(Here ω is the comultiplication on $\Sigma(X \times Y)$.)

Proof. By Theorem 7.3 (mixed version) we have

$$\begin{aligned}\Sigma w &\simeq (\Sigma\,\mathrm{proj}_{\tilde{C}_\alpha} + \Sigma\,\mathrm{proj}_{\tilde{C}_\beta}) \circ \Sigma(i_1 \times i_1) \circ \Sigma(\alpha \star \beta) \\ &= (\Sigma(i_1\,\mathrm{proj}_X) \vee \Sigma(i_1\,\mathrm{proj}_Y)) \circ \omega \circ \Sigma(\alpha \star \beta)\end{aligned}$$

since $(\Sigma\,\mathrm{proj}_{\tilde{C}_\alpha} + \Sigma\,\mathrm{proj}_{\tilde{C}_\beta}) \circ \mathrm{inc} \simeq 1_{\Sigma\tilde{C}_\alpha \vee \Sigma\tilde{C}_\beta}$.

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