ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 2, Summer 1980

ON ISOMETRIES OF THE BLOCH SPACE

BY

J. A. CIMA AND W. R. WOGEN

Introduction

Let Δ denote the open unit disc in the complex plane $\mathbf{C}$ and let Γ be the unit circle. The (normalized) set $\mathscr{B}$ of Bloch functions is defined as follows:

$$\mathscr{B} = \{f: f \text{ is holomorphic on } \Delta, f(0) = 0,$$
$$\text{and } \sup_{|z|<1} |f'(z)|(1 - |z|^2) \equiv M(f) < \infty\}.$$

With pointwise operations and $M(f) = \|f\|$, $\mathscr{B}$ becomes a nonseparable Banach space. Let $\mathscr{B}_0$ denote the closed subspace of $\mathscr{B}$ spanned by the polynomials. For many general properties of $\mathscr{B}$, see [1]. In [2], some function theoretic properties of the extreme points of the unit balls of $\mathscr{B}$ and $\mathscr{B}_0$ are investigated.

In this paper we characterize the (linear) isometries of $\mathscr{B}_0$ and the onto isometries of $\mathscr{B}$. Our description of these isometries closely parallels descriptions of isometries of many other spaces of analytic functions. See, for instance, [3], [4], [6], and [7]. Our work is patterned after a proof in [5, p. 141] of a theorem describing the isometries of function algebras. Thus as a first step we identify $\mathscr{B}$ with a subspace of $C_b(\Delta)$, the bounded continuous functions on Δ.

2. The Isometries of $\mathscr{B}_0$.

Let $C(\Delta)$ denote the continuous functions on Δ, and let

$$\mathscr{C} = \{f \in C(\Delta): \|f\|_{\mathscr{C}} \equiv \sup_{|z|<1} |f(z)|(1 - |z|^2) < \infty\}.$$

Let $\mathscr{D} = \{f \in \mathscr{C}: f \text{ is holomorphic on } \Delta\}$. Then $\mathscr{D}$ is a closed subspace of $\mathscr{C}$ and the derivative mapping $D: f \to f'$ is a linear isometry of $\mathscr{B}$ onto $\mathscr{D}$. Let $\mathscr{D}_0 = D(\mathscr{B}_0)$.

Now define a mapping Φ: $C_b(\Delta) \to \mathscr{C}$ by $(\Phi f)(z) = f(z)(1 - |z|^2)^{-1}$. Clearly Φ is an onto linear isometry. We denote Φ^{-1} by Ψ. Let $\Psi(\mathscr{D}) = \mathscr{A}$, $\Psi(\mathscr{D}_0) = \mathscr{A}_0$. We then have

$$\mathscr{A} = \{f'(z)(1 - |z|^2): f \in \mathscr{B}\}, \qquad \mathscr{A}_0 = \{f'(z)(1 - |z|^2): f \in \mathscr{B}_0\}$$

with $\mathscr{A}_0 \subset \mathscr{A} \subset C_b(\Delta)$ and $\mathscr{D}_0 \subset \mathscr{D} \subset C$. Indeed, $\mathscr{D}_0 \subset C_0(\Delta) \equiv$ all continuous functions on Δ which vanish on Γ.

Received May 5, 1978.

LEMMA 1. *Suppose that* $\tau\colon \Delta \to \Delta$ *is analytic. Then there is an analytic function* f *on* Δ *with*

$$|f(z)| = (1 - |\tau(z)|^2)(1 - |z|^2)^{-1} \quad \textit{for all } z \in \Delta$$

if and only if $\tau(z) = \lambda(z - \alpha)(1 - \bar{\alpha}z)^{-1}$ *for some* $\alpha \in \Delta$ *and* $\lambda \in \Gamma$.

Proof. If τ is a conformal automorphism of Δ, then by the Schwarz-Pick lemma, we can take $f(z) = \tau'(z)$. Conversely, if such an f exists, then $\log |f(z)| = \log (1 - |\tau(z)|^2) - \log (1 - |z|^2)$ is harmonic on Δ. Now an elementary computation shows that the equation

$$\Delta \log (1 - |\tau(z)|^2) = \Delta \log (1 - |z|^2)$$

(here Δ denotes the Laplacian) reduces to

$$\frac{|\tau'(z)|^2}{(1 - |\tau(z)|^2)^2} = \frac{1}{(1 - |z|^2)^2}.$$

Thus $|\tau'(z)| = (1 - |\tau(z)|^2)(1 - |z|^2)^{-1}$ and by the Schwarz-Pick lemma τ must be of the desired form.

LEMMA 2. *Every* $\alpha \in \Delta$ *is a peak point of* $\mathscr{A}_0$.

Proof. Let $f_\alpha(z) = (1 - |z|^2)(1 - \bar{\alpha}z)^{-2}$. Then $f_\alpha \in \mathscr{A}_0$ and f_α peaks at α.

THEOREM 1. *Let* $S\colon \mathscr{D}_0 \to \mathscr{D}_0$ *be an isometry. Then there is a conformal automorphism* ϕ *of* Δ *and a* $\lambda \in \Gamma$ *so that* $Sf = (\lambda\phi')(f \circ \phi)$ *for every* $f \in \mathscr{D}_0$.

Proof. If X is a Banach space, $E(X)$ will denote the extreme points of the unit ball of X. The mapping $T = \Psi S \Phi|_{\mathscr{A}_0}$ is an isometry of $\mathscr{A}_0$. If $T\mathscr{A}_0 = \mathscr{R}_0$, we view the adjoint T^* as a mapping from $\mathscr{R}_0^*$ to $\mathscr{A}_0^*$. Thus T^* is an onto isometry. Then T^* maps $E(\mathscr{R}_0^*)$ injectively onto $E(\mathscr{A}_0^*)$. We now use the idea in [5] to show that each point γ of $E(\mathscr{R}_0^*)$ or $E(\mathscr{A}_0^*)$ extends to an evaluation functional on $C_0(\Delta)$. Let U = all norm preserving extensions of γ to $C_0(\Delta)$. U is weak* compact, and we let Λ be an extreme point of U. Then Λ is extreme in $E(C_0(\Delta))$ and thus $\Lambda = \mu e_z$ (e_z is evaluation at z) for some $z \in \Delta$, $\mu \in \Gamma$. By Lemma 2, z is unique. Of course $e_z|_{\mathscr{A}_0} \in E(\mathscr{A}_0^*)$.

Let $\Sigma(\mathscr{R}_0) = \{z \in \Delta\colon e_z|_{\mathscr{R}_0} \in E(\mathscr{R}_0^*)\}$. Hence there are functions

$$\tau\colon \Sigma(\mathscr{R}_0) \to \Delta \quad \text{and} \quad \alpha\colon \Sigma(\mathscr{R}_0) \to \Gamma$$

so that

$$T^*(e_z|_{\mathscr{R}_0}) = \alpha(z)(e_{\tau(z)}|_{\mathscr{A}_0}) \quad \text{for all } z \in \Sigma(\mathscr{R}_0).$$

Thus $Tf(z) = \alpha(z)f(\tau(z))$ for all $z \in \Sigma(\mathscr{R}_0)$, $f \in \mathscr{A}_0$. In particular, for $k = 0, 1, 2, \ldots,$

$$\begin{aligned} T(z^k(1 - |z|^2)) &= \alpha(z)(\tau(z))^k(1 - |\tau(z)|^2) \\ &\equiv G_k(z)(1 - |z|^2) \quad \text{for all } z \in \Sigma(\mathscr{R}_0), \end{aligned}$$

where G_k is a holomorphic function on Δ ($G_k \in \mathscr{D}_0$).

For $z \in \Sigma(\mathscr{R}_0)$,

$$G_0(z) = \alpha(z)(1 - |\tau(z)|^2)(1 - |z|^2)^{-1} \quad \text{and} \quad \tau(z) = G_1(z)/G_0(z).$$

Thus τ has a meromorphic extension from $\Sigma(\mathscr{R}_0)$ to all of Δ. This extension is unique since $\Sigma(\mathscr{R}_0)$ is uncountable. If $p_n(z) = z^n$, then for all $n \geq 0$, Sp_n is holomorphic on Δ. Further it is easy to see that

$$(Sp_n)(z) = G_0(z)\left(\frac{G_1(z)}{G_0(z)}\right)^n \quad \text{for } z \in \Sigma(\mathscr{R}_0),$$

and since extension from $\Sigma(\mathscr{R}_0)$ is unique this equation holds for $z \in \Delta$.

It follows that G_1/G_0 can have no poles, so that the meromorphic extension of τ is actually holomorphic.

Also

$$\|p_n\|_{\mathscr{D}} = \left(\frac{n}{n+2}\right)^n\left(\frac{2}{n+2}\right) < 1,$$

so $\|Sp_n\|_{\mathscr{D}} < 1$ for $n \geq 0$. Thus

$$|G_0(z)|\ |G_1(z)/G_0(z)|^n(1 - |z|^2) \leq \|Sp_n\|_{\mathscr{D}} < 1 \quad \text{for all } z \in \Delta,\ n \geq 0.$$

Hence the range of G_1/G_0 is contained in Δ. We apply Lemma 1 to G_0 to conclude that G_1/G_0 is a conformal automorphism ϕ of Δ and that $G_0 = \lambda\phi'$ for some $\lambda \in \Gamma$. Then for each polynomial q, we have $Sq = (\lambda\phi')(q \circ \phi)$, and this establishes the theorem.

COROLLARY 1. *If $S\colon \mathscr{B}_0 \to \mathscr{B}_0$ is an isometry, then there is a conformal automorphism ϕ of Δ and a $\lambda \in \Gamma$ so that $(Sf) = \lambda(f \circ \phi - f(\phi(0)))$ for all $f \in \mathscr{B}_0$.*

COROLLARY 2. *Every isometry of $\mathscr{B}_0$ is onto.*

There are other equivalent (but less natural) norms on $\mathscr{B}$. For example, one could use $|||f||| = \sup_{|z|<1} |f'(z)|(1 - |z|)$. Then Lemma 1 can be modified as follows.

LEMMA 1′. *Suppose that $\tau\colon \Delta \to \Delta$ is analytic. Then there is an analytic function f on Δ with $|f(z)| = (1 - |\tau(z)|)(1 - |z|)^{-1}$ for all $z \in \Delta$ if and only if $\tau(z) = \lambda z$ for some $\lambda \in \Gamma$.*

Minor modifications of the statements and proofs of Theorem 1 and Corollary 1 lead to the following.

PROPOSITION. *If S is an isometry of $(\mathscr{B}_0, |||\ \ |||)$, then there are $\lambda, \mu \in \Gamma$ such that $(Sf)(z) = \lambda f(\mu z)$ for all $f \in \mathscr{B}_0$.*

3. The onto isometries of $\mathscr{B}$

The onto isometries of $\mathscr{B}$ turn out to be the natural ones appearing in Corollary 1. The proof of this is similar to that given in Section 2, but it differs in an essential way. We let $\beta\Delta$ denote the Stone-Cech compactification of Δ.

THEOREM 2. *Let $S: \mathscr{D} \to \mathscr{D}$ be an onto isometry. Then there is a conformal automorphism ϕ of Δ and a $\lambda \in \Gamma$ so that $Sf = (\lambda\phi')(f \circ \phi)$ for all $f \in \mathscr{D}$.*

Proof. Now $\mathscr{A} \subset C_b(\Delta) \cong C(\beta\Delta)$. Let $T = \Psi S \Phi$, so that T is an isometry of $\mathscr{A}$ onto $\mathscr{A}$. Thus T^* maps $E(\mathscr{A}^*)$ injectively onto $E(\mathscr{A}^*)$. As before, each $\gamma \in E(\mathscr{A}^*)$ has an extension to some e_x, $x \in \beta\Delta$. Lemma 2 shows that for each $z \in \Delta$, $e_z|_{\mathscr{A}}$ is in $E(\mathscr{A}^*)$. (i.e., $z \in \Sigma(\mathscr{A})$.) Thus there are mappings $\tau: \Delta \to \beta\Delta$ and $\alpha: \Delta \to \Gamma$ such that $T^*e_z|_{\mathscr{A}} = \alpha(z)e_{\tau(z)}|_{\mathscr{A}}$ for every $z \in \Sigma(\mathscr{A})$. Then

$$(Tf)(z) = \alpha(z)f(\tau(z)) \quad \text{for all } z \in \Delta, f \in \mathscr{A}.$$

If $f \in \mathscr{A}_0$, then f extends to be 0 on $\beta\Delta - \Delta$. Thus for $h_0(z) = - |z|^2$, we have

$$(Th_0)(z) = \begin{cases} \alpha(z)h_0(\tau(z)) = \alpha(z)(1 - |\tau(z)|^2, & \tau(z) \in \Delta, \\ 0, & \tau(z) \in \beta\Delta - \Delta. \end{cases}$$

But $(Th_0)(z)(1 - |z|^2)^{-1}$ is in $\mathscr{D}_0$ and is not the zero function. Thus except possibly for a sequence $\{z_n\}$ with $|z_n| \to 1$ we have $\tau(z) \in \Delta$. Let $h_1(z) = z(1 - |z|^2)$. If $z \neq z_n$, then

$$(Th_1)(z) = \alpha(z)\tau(z)(1 - |\tau(z)|^2) = \tau(z)(Th_0)(z).$$

Hence τ is analytic on $\Delta - \{z_n\}$ and τ is bounded. Thus τ has a unique analytic extension to Δ. As in the proof of Theorem 1, apply Lemma 1 to show that τ is a conformal automorphism and that S has the desired form.

COROLLARY 3. *If $S: \mathscr{B} \to \mathscr{B}$ is an onto isometry, then there is a conformal automorphism ϕ of Δ and a $\lambda \in \Gamma$ so that $Sf = \lambda(f \circ \phi - f(\phi(0)))$ for all $f \in \mathscr{B}_0$.*

If the norm on $\mathscr{B}$ is modified as in Section 2, then S above has the form $(Sf)(z) = \lambda f(\mu z)$, where λ, $\mu \in \Gamma$.

REFERENCES

1. J. M. ANDERSON, J. CLUNIE and CH. POMMERENKE, *On Bloch functions and normal functions*, J. Reine Angew. Math., vol. 270 (1974), pp. 12–37.
2. J. A. CIMA and W. R. WOGEN, *Extreme points of the unit ball of the Bloch space B_0*, Mich. Math. J., vol. 25 (1978), pp. 213–222.
3. K. DELEEUW, W. RUDIN and J. WERMER, *The isometries of some function spaces*, Proc. Amer. Math. Soc., vol. 11 (1960), pp. 694–698.
4. F. FORELLI, *The isometries of H^p*, Canad. J. Math., vol. 16 (1964), pp. 721–728.
5. K. H. HOFFMAN, *Banach Spaces of Analytic Functions*, Prentice-Hall, Englewood Cliffs, N.J.
6. J. N. MCDONALD, *Isometries of the disc algebra*, Pacific J. Math., vol. 58 (1975), pp. 143–154.
7. W. RUDIN, *L^p-isometries and equimeasurability*, Indiana Univ. Math. J., vol. 25 (1976), pp. 215–228.

THE UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL, NORTH CAROLINA