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ON CONVOLUTION SQUARES OF SINGULAR MEASURES

BY

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A classical result of Wiener and Wintner [6] asserts that there exists a singular probability measure μ on the circle group T such that $\hat{\mu}(n) = O(|n|^{-1/2+\varepsilon})$ as $n \to \infty$ for every $\varepsilon > 0$. Such a measure μ has the property that $\mu^2 = \mu * \mu$ is absolutely continuous and its Radon-Nikodym derivative with respect to Lebesgue measure belongs to $L^p(T)$ for all positive real numbers p (cf. [2] and [5]). In the present paper, we shall construct a singular probability measure μ, with support having zero Lebesgue measure, such that μ^2 has uniformly convergent Fourier-Stieltjes series.

Let λ be the normalized Lebesgue measure on T and let Z be the additive group of integers. We denote by $C_0(Z)$ the space of all functions on Z (i.e., two-sided sequences) that vanish at infinity. A mapping of $C_0(Z)$ into itself is called continuous if it is continuous with respect to the supremum norm of $C_0(Z)$. Our result can be stated as follows.

THEOREM. *Let K be a measurable subset of T having positive Lebesgue measure, and let ϕ be a continuous mapping of $C_0(Z)$ into itself. Then there exists a singular probability measure μ on T satisfying these conditions:*

(a) $\operatorname{supp} \mu \subset K$ *and* $\lambda(\operatorname{supp} \mu) = 0$;
(b) $\sum_{n=-\infty}^{\infty} |\hat{\mu}(n)^2 \cdot \phi(\hat{\mu})(n)| < \infty$;
(c) *The Fourier-Stieltjes series of μ^2 converges uniformly.*

In order to prove this theorem, we need some notation and lemmas. For $f \in C(T)$, we define

$$\|f\|_A = \sum_{n=-\infty}^{\infty} |\hat{f}(n)| \quad \text{and} \quad \|f\|_U = \sup_N \left\| \sum_{n=-N}^{N} \hat{f}(n) e^{int} \right\|_\infty .$$

Notice that the set of all $f \in C(T)$ with $\|f\|_A < \infty$ (or $\|f\|_U < \infty$) forms a Banach space (cf. [3]). Given $f \in L^1(T)$, let $f^{(2)} = f * f$ and let $\operatorname{supp} f$ denote the closed support of f. Throughout the following lemmas, we fix an arbitrary continuous mapping ϕ of $C_0(Z)$ into itself and write $\Psi(P) = P^2 \cdot \phi(P)$ for $P \in C_0(Z)$. We begin with improving Lemma 3.2 of [5] by applying Körner's idea in [4].

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LEMMA 1. *Given* $g \in L^1_+(T)$ *and* $\eta > 0$, *there exists a simple function* $h \in L^1_+(T)$ *such that:*

(i) $\|h\|_1 = \|g\|_1$ *and* $\|\hat{g} - \hat{h}\|_\infty < \eta$;
(ii) $\operatorname{supp} h \subset \{g \neq 0\}$ *and* $\lambda(\operatorname{supp} h) \leq 2^{-1}\lambda(\{g \neq 0\})$;
(iii) $h \leq (2 + \eta)g$ *on* T.

Proof. We can write $g = g_1 + g_2 + \cdots + g_m$, where $g_j \in L^1_+(T)$, $g_j g_k = 0$ if $j \neq k$, and $\|g_j\|_1 < \eta/4$ for all $j = 1, 2, \ldots, m$. By induction, we choose simple functions $h_1, h_2, \ldots, h_m \in L^1_+(T)$ as follows.

Let $h_0 = 0$, and suppose that $h_0, \ldots, h_{j-1}$ have been chosen for some $j \in \{1, 2, \ldots, m\}$. By Lemma 3.2 of [5] and its proof, there is a simple function $h_j \in L^1_+(T)$ satisfying these conditions: $\|h_j\|_1 = \|g_j\|_1$, $\operatorname{supp} h_j \subset \{g_j \neq 0\}$,

$$\lambda(\operatorname{supp} h_j) \leq 2^{-1}\lambda(\{g_j \neq 0\}), \quad h_j \leq (2 + \eta)g_j$$

and

$$|\hat{g}_j - \hat{h}_j| < \eta/(2m) \quad \text{on} \bigcup_{i=1}^{j-1} \{|\hat{g}_i - \hat{h}_i| \geq \eta/(2m)\}. \tag{1}$$

This completes the induction.

Setting $h = h_1 + h_2 + \cdots + h_m$, we claim that h has all the required properties. Evidently we need only confirm that $\|\hat{g} - \hat{h}\|_\infty < \eta$. To this end, take an arbitrary integer n. If $|\hat{g}_j(n) - \hat{h}_j(n)| < \eta/(2m)$ for all j, then we have $|\hat{g}(n) - \hat{h}(n)| < \eta/2$. If $|\hat{g}_j(n) - \hat{h}_j(n)| \geq \eta/(2m)$ for some index j, then (1) implies that $|\hat{g}_i(n) - \hat{h}_i(n)| < \eta/(2m)$ for all $i \neq j$. It follows that

$$|\hat{g}(n) - \hat{h}(n)| < (m - 1)\eta/(2m) + \|g_j - h_j\|_1 < \eta/2 + 2\|g_j\|_1 < \eta,$$

which completes the proof.

LEMMA 2. *Let* $g_1, g_2, \ldots, g_p \in L^2_+(T)$ *and* $\varepsilon > 0$ *be given. Then there exists a simple function* h *in* $L^1_+(T)$ *satisfying the following five conditions:*

(i) $\|h\|_1 = \|g_1\|_1$ *and* $\|\hat{g}_1 - \hat{h}\|_\infty < \varepsilon$;
(ii) $\operatorname{supp} h \subset \{g_1 \neq 0\}$ *and* $\lambda(\operatorname{supp} h) \leq 2^{-1}\lambda(\{g_1 \neq 0\})$;
(iii) $h \leq (2 + \varepsilon)g_1$ *on* T;
(iv) $\|g_k * (g_1 - h)\|_A < \varepsilon$ *for all* $k = 1, 2, \ldots, p$;
(v) $\sum_{n=-\infty}^{\infty} |\Psi(\sum_{k=1}^{p} \hat{g}_k)(n) - \Psi(\hat{h} + \sum_{k=2}^{p} \hat{g}_k)(n)| < \varepsilon$.

Proof. Write $C = (1 + \|g_1\|_2 + \cdots + \|g_p\|_2)^2$. Since $\phi(\sum_{k=1}^{p} \hat{g}_k)$ is in $C_0(Z)$, we can find a finite subset Y of Z such that

$$\left|\phi\left(\sum_{k=1}^{p} \hat{g}_k\right)\right| < (30C)^{-1}\varepsilon \quad \text{on } Z \backslash Y. \tag{1}$$

By the continuity of the mapping ϕ, there exists a positive real number $\eta < \min(\varepsilon, 1)$ such that, for $P \in C_0(Z)$,

(2) $$\left\|\sum_{k=1}^{p} \hat{g}_k - P\right\|_\infty < \eta \Rightarrow \left\|\phi\left(\sum_{k=1}^{p} \hat{g}_k\right) - \phi(P)\right\|_\infty < (30C)^{-1}\varepsilon.$$

Applying Lemma 1 with $g = g_1$, we obtain a simple function $h \in L^1_+(T)$ which satisfies conditions (i), (ii) and (iii) with η in place of ε. By Lemma 3.1 of [5], we may assume that h also satisfies (iv). Notice that (2) and the inequality in (i) with η in place of ε imply

(3) $$\left\|\phi\left(\sum_{k=1}^{p} \hat{g}_k\right) - \phi\left(\hat{h} + \sum_{k=2}^{p} \hat{g}_k\right)\right\|_\infty < (30C)^{-1}\varepsilon.$$

Since $\phi(\sum_{k=1}^{p} \hat{g}_k)$ is a bounded function and since η may be chosen arbitrarily small, we may also assume that

(4) $$\left\|\left[\left(\sum_{k=1}^{p} \hat{g}_k\right)^2 - \left(\hat{h} + \sum_{k=2}^{p} \hat{g}_k\right)^2\right] \cdot \phi\left(\sum_{k=1}^{p} \hat{g}_k\right)\right\|_\infty < (3|Y| + 1)^{-1}\varepsilon,$$

where $|Y|$ is the number of the elements of Y.

In order to confirm (v), first notice that

(5) $$\sum_{n=-\infty}^{\infty} \left|\hat{h}(n) + \sum_{k=2}^{p} \hat{g}_k(n)\right|^2 = \left\|h + \sum_{k=2}^{p} g_k\right\|_2^2 \le \left(\|h\|_2 + \sum_{k=2}^{p} \|g_k\|_2\right)^2 < 9C$$

by the Parseval formula and (iii). Now we write

$$\begin{aligned}\sum_n &\left|\Psi\left(\sum_{k=1}^{p} \hat{g}_k\right)(n) - \Psi\left(\hat{h} + \sum_{k=2}^{p} \hat{g}_k\right)(n)\right| \\ &\le \sum_n \left|\left(\sum_{k=1}^{p} \hat{g}_k\right)^2(n) - \left(\hat{h} + \sum_{k=2}^{p} \hat{g}_k\right)^2(n)\right| \cdot \left|\phi\left(\sum_{k=1}^{p} \hat{g}_k\right)(n)\right| \\ &\quad + \sum_n \left|\hat{h}(n) + \sum_{k=2}^{p} \hat{g}_k(n)\right|^2 \cdot \left|\phi\left(\sum_{k=1}^{p} \hat{g}_k\right)(n) - \phi\left(\hat{h} + \sum_{k=2}^{p} \hat{g}_k\right)(n)\right| \\ &= A + B,\end{aligned}$$

say. By (4), (1) and (5), we have

$$\begin{aligned}A &= \sum_Y + \sum_{Z\setminus Y} \\ &< |Y|(3|Y| + 1)^{-1}\varepsilon + (30C)^{-1}\varepsilon \sum_n \left|\left(\sum_{k=1}^{p} \hat{g}_k\right)^2(n) - \left(\hat{h} + \sum_{k=2}^{p} \hat{g}_k\right)^2(n)\right| \\ &< \varepsilon/3 + (30C)^{-1}\varepsilon \cdot 10C \\ &= 2\varepsilon/3.\end{aligned}$$

Similarly we have

$$B \le (30C)^{-1}\varepsilon \sum_n \left| \left(\hat{h} + \sum_{k=2}^{p} \hat{g}_k \right)(n) \right|^2 < \varepsilon/3$$

by (3) and (5). This establishes (v) and the proof is complete.

LEMMA 3. *Let $f \in L^2_+(T)$, $\|f\|_1 = 1$, and $\varepsilon > 0$ be given. Then there exists a simple function g in $L^1_+(T)$ such that:*

(a) $\|g\|_1 = 1$ *and* $\|\hat{f} - \hat{g}\|_\infty < \varepsilon$;
(b) $\operatorname{supp} g \subset \{f \neq 0\}$ *and* $\lambda(\operatorname{supp} g) \le 2^{-1}\lambda(\{f \neq 0\})$;
(c) $\sum_{n=-\infty}^{\infty} |\Psi(\hat{f})(n) - \Psi(\hat{g})(n)| < \varepsilon$;
(d) $\|f^{(2)} - g^{(2)}\|_U < \varepsilon$.

Proof. Choose and fix a sufficiently large natural number p such that

$$\int_{2\pi(j-1)/p}^{2\pi j/p} \{f(t)\}^2\, dt < \varepsilon \quad (j = 1, 2, \ldots, p). \tag{1}$$

For each $j = 1, 2, \ldots, p$, let g_j be the restriction of f to the interval $[2\pi(j-1)/p, 2\pi j/p)$. Take a natural number N_0 so large that

$$\sum_{|n| \ge N_0} |\hat{g}_j(n)|^2 < \varepsilon/p \quad (j = 1, 2, \ldots, p). \tag{2}$$

An inductive application of Lemma 2 will yield simple functions $h_1, h_2, \ldots, h_p$ and natural numbers $N_1, N_2, \ldots, N_p$ satisfying the following conditions for $j = 1, 2, \ldots, p$:

(3) $\|h_j\|_1 = \|g_j\|_1$ and $\|\hat{g}_j - \hat{h}_j\|_\infty < \varepsilon/(2pN_{j-1})$;

(4) $\operatorname{supp} h_j \subset \{g_j \neq 0\}$ and $\lambda(\operatorname{supp} h_j) \le 2^{-1}\lambda(\{g_j \neq 0\})$;

(5) $h_j \le 3g_j$ on T;

(6) $\sum_{k=1}^{j-1} \|(g_j - h_j) * h_k\|_A + \sum_{k=j}^{p} \|(g_j - h_j) * g_k\|_A < \varepsilon/p$;

(7) $\sum_{n=-\infty}^{\infty} |\Psi(\hat{f}_{j-1})(n) - \Psi(\hat{f}_j)(n)| < \varepsilon/p$;

(8) $\sum_{|n| \ge N_j} |\hat{h}_j(n)|^2 < \varepsilon/p$.

Here and elsewhere $f_j = (h_1 + \cdots + h_j) + (g_{j+1} + \cdots + g_p)$ for $j = 0, 1, \ldots, p$. We may assume that $N_0 < N_1 < \cdots < N_p$.

Now we define $g = f_p = h_1 + \cdots + h_p$. It is easy to check that g satisfies conditions (a), (b) and (c). So we need only prove that $\|f^{(2)} - g^{(2)}\|_U < C\varepsilon$ for some absolute constant C. To this end, we write

$$f_{j-1}^{(2)} - f_j^{(2)} = g_j^{(2)} - h_j^{(2)} + 2(g_j - h_j) * \left(\sum_{k=1}^{j-1} h_k + \sum_{k=j+1}^{p} g_k \right)$$

for $j = 1, 2, \ldots, p$. Since $\|h\|_U \leq \|h\|_A$ for $h \in A(T)$, it follows from (6) that

$$\begin{aligned} \|f^{(2)} - g^{(2)}\|_U &= \left\| \sum_{j=1}^{p} \{f_{j-1}^{(2)} - f_j^{(2)}\} \right\|_U \\ &< \left\| \sum_{j=1}^{p} \{g_j^{(2)} - h_j^{(2)}\} \right\|_U + 2\varepsilon. \end{aligned}$$

Therefore it will suffice to show that

$$\left\| \sum_{j=1}^{p} S_N\{g_j^{(2)} - h_j^{(2)}\} \right\|_\infty < 36\varepsilon \quad (N = 0, 1, 2, \ldots), \tag{9}$$

where $S_N(h)$ denotes the Nth partial sum of the Fourier series of $h \in L^1(T)$.

Now we claim that

$$\left\| \sum_{j=1}^{k} \{g_j^{(2)} - h_j^{(2)}\} \right\|_\infty < 20\varepsilon \quad (k = 1, 2, \ldots, p). \tag{10}$$

Indeed our definition of g_j and (4) imply that

$$\{g_j^{(2)} - h_j^{(2)} \neq 0\} \subset (4\pi(j-1)/p, 4\pi j/p) \mod 2\pi$$

for all indices j. Therefore we infer from (5) and (1) that

$$\begin{aligned} \left\| \sum_{j=1}^{k} \{g_j^{(2)} - h_j^{(2)}\} \right\|_\infty &\leq 2 \sup \{\|g_j^{(2)} - h_j^{(2)}\|_\infty : 1 \leq j \leq k\} \\ &\leq 2 \sup \{\|g_j\|_2^2 + \|h_j\|_2^2 : 1 \leq j \leq p\} \\ &\leq 20 \sup \{\|g_j\|_2^2 : 1 \leq j \leq p\} \\ &< 20\varepsilon. \end{aligned}$$

This establishes (10).

Now let N be an arbitrary nonnegative integer, and let M_N denote the left-hand side of (9). If $N \leq N_0$, then we have

$$\begin{aligned} M_N &\leq \sum_{j=1}^{p} \sum_{n=-N}^{N} |(\hat{g}_j(n))^2 - (\hat{h}_j(n))^2| \\ &\leq 2 \sum_{j=1}^{p} \sum_{n=-N}^{N} |\hat{g}_j(n) - \hat{h}_j(n)| \\ &< 2 \sum_{j=1}^{p} (2N+1)\varepsilon/(2pN_{j-1}) \\ &\leq 4\varepsilon \end{aligned} \tag{11}$$

by (3). If $N_{k-1} < N \leq N_k$ for some $k = 1, 2, \ldots, p$, then

$$(12) \qquad M_N \leq \left\| \sum_{j=1}^{k-1} S_N\{g_j^{(2)} - h_j^{(2)}\} \right\|_\infty + \|S_N\{g_k^{(2)} - h_k^{(2)}\}\|_A$$
$$+ \left\| \sum_{j=k+1}^{p} S_N\{g_j^{(2)} - h_j^{(2)}\} \right\|_A$$
$$= P + Q + R,$$

say. By (5) and (1), we have

$$(13) \qquad Q \leq \|g_k\|_2^2 + \|h_k\|_2^2 \leq 10\|g_k\|_2^2 < 10\varepsilon.$$

A similar estimate as in (11) yields

$$(14) \qquad R < 2 \sum_{j=k+1}^{p} (2N+1)\varepsilon/(2pN_{j-1}) < 4\varepsilon.$$

Furthermore, we have

$$(15) \qquad P \leq \left\| \sum_{j=1}^{k-1} \{g_j^{(2)} - h_j^{(2)}\} \right\|_\infty + \sum_{j=1}^{k-1} \sum_{|n|>N} \{|\hat{g}_j(n)|^2 + |\hat{h}_j(n)|^2\}$$
$$\leq \left\| \sum_{j=1}^{k-1} \{g_j^{(2)} - h_j^{(2)}\} \right\|_\infty + (k-1) \cdot 2\varepsilon/p$$
$$< 20\varepsilon + 2\varepsilon$$
$$= 22\varepsilon,$$

by (2), (8), and (10). It follows from (12)–(15) that $M_N < 36\varepsilon$ whenever $N_{k-1} < N \leq N_k$ for some index k.

Finally, if $N > N_p$, then (15) with $k = p + 1$ shows that $M_N < 22\varepsilon$. This establishes (9) and the proof is complete.

Proof of the theorem. Let $K \subset T$ and $\phi\colon C_0(Z) \to C_0(Z)$ be as in the hypotheses of the present theorem. Choose and fix an arbitrary simple function $f_0 \in L_+^1(T)$ such that $\|f_0\|_1 = 1$ and $\operatorname{supp} f_0 \subset K$. We inductively apply Lemma 3 to obtain a sequence (f_k) of simple functions in $L_+^1(T)$, subject to these four conditions ($k \geq 1$):

(1) $\|f_k\|_1 = 1$ and $\|\hat{f}_{k-1} - \hat{f}_k\|_\infty < 2^{-k}$;

(2) $\operatorname{supp} f_k \subset \{f_{k-1} \neq 0\}$ and $\lambda(\operatorname{supp} f_k) \leq 2^{-1}\lambda(\{f_{k-1} \neq 0\})$;

(3) $\displaystyle\sum_{n=-\infty}^{\infty} |\Psi(\hat{f}_{k-1})(n) - \Psi(\hat{f}_k)(n)| < 2^{-k}$;

(4) $\|f_{k-1}^{(2)} - f_k^{(2)}\|_U < 2^{-k}$.

It is easy to check that the measures $f_k\lambda$ converge weak* to a singular probability measure μ with the required properties. This establishes the theorem.

Remarks. (I) In order to give an example of a continuous mapping of $C_0(Z)$ into itself, let $P \in C_0(Z)$, let F be any continuous function on the complex plane with $F(0) = 0$, and let α be any mapping of Z into itself such that $\alpha(n) \to \infty$ as $n \to \infty$. Define $\phi(Q) = P + F \circ Q + |Q \circ \alpha|$ for $Q \in C_0(Z)$. Then ϕ is a continuous mapping on $C_0(Z)$. It is obvious that for an appropriate choice of ϕ, condition (b) of the present theorem implies that $\hat{\mu}$ belongs to l^p for all $p > 2$ (cf. Hewitt and Ritter [1]). However, our method does not seem to yield a singular probability measure μ such that $|\hat{\mu}(n)| \leq \eta(n)$, where η is a preassigned function on Z subject to suitable conditions (cf. Remark (IV) stated below).

(II) A weak version of our theorem holds for every nondiscrete locally compact abelian group. Let G be such a group with Haar measure λ_G and dual Γ, and let ϕ be a continuous mapping of $C_0(\Gamma)$ into itself. Suppose $f \in L^1_+ \cap L^2(G)$, $\|f\|_1 = 1$, and $\varepsilon > 0$. Then there exists a probability measure μ in $M(G)$ such that:

(a) $\operatorname{supp} \mu \subset \{f \neq 0\}$ and $\lambda_G(\operatorname{supp} \mu) = 0$;
(b) $\int_\Gamma |\Psi(\hat{\mu}) - \Psi(\hat{f})| \, d\gamma < \varepsilon$, where $\Psi(P) = P^2\phi(P)$ for $P \in C_0(\Gamma)$.

If, in addition, every neighborhood of $0 \in G$ contains an element of order larger than 2, then such a measure μ can be chosen to satisfy

(c) $\mu^2 = g\lambda_G$ and $\|g - f * f\|_\infty < \varepsilon$ for some $g \in C_c(G)$.

These facts can be proved along the same lines as the present theorem. However, in the case that G contains an open subgroup of bounded order, then the proof of the second result stated above requires some *ad hoc* (structural) technique. We omit the details.

(III) The additional assumption in the second result in Remark (II) is necessary. To see this, first notice that the conditions $f \in L^1 \cap L^\infty(G)$ and $\hat{f} \geq 0$ imply $\hat{f} \in L^1(\Gamma)$. Now suppose that $G = \{0, 1\}^\alpha$ for some infinite cardinal α and that μ is a real measure in $M(G)$. Then $\hat{\mu}$ is a real-valued function on Γ and hence $\hat{\mu}^2 \geq 0$. It follows from the above remark that $\mu^2 \in L^\infty(G)$ implies $\hat{\mu}^2 \in L^1(\Gamma)$, and, in particular, $\mu \in L^1(G)$. Using the last fact, one can easily show that if G contains an open subgroup of the form $\{0, 1\}^\alpha$ for some infinite cardinal α, then there exists *no* singular probability measure μ in $M(G)$ such that $\mu^2 \in L^\infty(G)$.

(IV) One might ask if the measure μ in our theorem can be chosen so that $\mu^2 = g\lambda$ for some $g \in C(T)$ satisfying a Hölder condition. However, the answer is negative. The following observation is due to the referee. If g satisfies a Hölder condition of order $\alpha > 0$, then $\hat{g}(n) = O(|n|^{-\alpha})$. Thus $\hat{\mu}(n) = O(|n|^{-\alpha/2})$ and μ is supposed to be carried by an arbitrary set of positive Lebesgue measure. This is contrary to Zygmund's classical work on $U(\varepsilon)$-sets (see p. 351 of [7]). However, if we drop the condition on the support of μ, then we can get any Hölder condition of order less than 1/2, by using random processes.

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