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MULTIPLIERS OF TENSOR PRODUCTS OF CMA'S AND RADON-NIKODYM DERIVATIVES

BY

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I. Introduction

Throughout this paper, the expression (A, Γ) will denote a commutative semisimple convolution measure algebra (CMA) A with structure semigroup Γ. In particular, this means that A is a Banach algebra, Γ is a compact abelian topological semigroup, and there is an isometric algebra isomorphism of A into $M(\Gamma)$, the Banach algebra (under convolution product and total variation norm $\|\cdot\|$) of all complex-valued finite regular Borel measures on Γ. A multiplier of a CMA A is a linear operator $T: A \to A$ satisfying $T(a * b) = T(a) * b$, for all a, b in A. Each multiplier of A is a bounded operator on A, and, as a result the set $\mathscr{M}(A)$ of all multipliers of A is a Banach algebra with composition as product. It will be assumed that A has a weak approximate identity of norm one [9, 10]. Hence, there is an isometric algebra isomorphism of $\mathscr{M}(A)$ into $M(\Gamma)$ which extends the aforementioned imbedding of A in $M(\Gamma)$. Thus, we can, and will, regard A and $\mathscr{M}(A)$ as norm-closed subalgebras of the measure algebra $M(\Gamma)$. So regarded, A, being a CMA, is a complex L-subspace (defined below) of $M(\Gamma)$. Although not true in general, it will be assumed that $\mathscr{M}(A)$ is also an L-subspace of $M(\Gamma)$; this is equivalent to assuming that $\mathscr{M}(A)$ is a CMA. These and other results about multipliers of CMA's can be found in [10], [17], while the comprehensive monograph [13] deals with CMA's in general.

If X is a compact Hausdorff space, then an L-subspace $\mathscr{L}$ of the measure space $M(X)$ is a norm-closed complex linear subspace of $M(X)$ such that, if $\nu \in \mathscr{L}$, $\mu \in M(X)$, and μ is absolutely continuous with respect to ν, written $\mu \ll \nu$, then $\mu \in \mathscr{L}$. Thus, a measure $\mu \in \mathscr{L}$ if and only if its total variation measure $|\mu| \in \mathscr{L}$. In all that follows, X_1 and X_2 denote compact Hausdorff spaces, and $\mathscr{L}_1$ and $\mathscr{L}_2$ denote L-subspaces of $M(X_1)$ and $M(X_2)$, respectively. Then the projective tensor product $\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$ of $\mathscr{L}_1$ and $\mathscr{L}_2$ may be regarded as the completion in $M(X_1 \times X_2)$ of the usual algebraic tensor product $\mathscr{L}_1 \otimes \mathscr{L}_2$ [13, Prop. 2.5.2]. Thus, since $M(X_1)$, $M(X_2)$ are L-spaces, the inclusions

$$\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2 \subseteq M(X_1) \hat{\otimes} M(X_2) \subseteq M(X_1 \times X_2)$$

always obtain.

Our interest is in the multipliers of the projective tensor product $A_1 \hat{\otimes} A_2$ of two semisimple commutative CMA's A_1, A_2. It has been shown [16] that, if

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(A_1, Γ_1) and (A_2, Γ_2) are two such CMA's, then $A_1 \hat{\otimes} A_2$ is a semisimple commutative CMA whose structure semigroup is $\Gamma_1 \times \Gamma_2$. Our standing assumptions on A_2, A_2, $\mathscr{M}(A_1)$, $\mathscr{M}(A_2)$ imply that $\mathscr{M}(A_1) \hat{\otimes} \mathscr{M}(A_2)$ imbeds isometrically in $\mathscr{M}(A_1 \hat{\otimes} A_2) \subseteq M(\Gamma_1 \times \Gamma_2)$. The main purpose of this paper is to characterize $\mathscr{M}(A_1) \hat{\otimes} \mathscr{M}(A_2)$ as a multiplier-subspace of $\mathscr{M}(A_1 \hat{\otimes} A_2)$ and $M(\Gamma_1 \times \Gamma_2)$.

Although the assumptions placed upon A_1, A_2 and $\mathscr{M}(A_1)$, $\mathscr{M}(A_2)$ may seem unduly restrictive, they are satisfied for the following classical example. Let G_1, G_2 be arbitrary locally compact abelian groups, and let $L^1(G_1)$, $L^1(G_2)$ be the usual group algebras of G_1, G_2, respectively. Then $L^1(G_1)$, $L^1(G_2)$ are semisimple commutative CMA's with approximate identities of norm one, and the multiplier algebras $\mathscr{M}(L^1(G_1)) = M(G_1)$, $\mathscr{M}(L^1(G_2)) = M(G_2)$ [15] are also CMA's. The structure semigroups of $L^1(G_1)$ and $L^1(G_2)$ are the Bohr compactifications $\bar{G}_1$ and $\bar{G}_2$ of G_1 and G_2, respectively [12, Example 4(i), p. 163], while $L^1(G_1) \hat{\otimes} L^1(G_2) = L^1(G_1 \times G_2)$ [5, Corollary 4, p. 61]. Thus,

$$\mathscr{M}(L^1(G_1) \hat{\otimes} L^1(G_2)) = \mathscr{M}(L^1(G_1 \times G_2)) = M(G_1 \times G_2),$$

and we seek to describe those measures in $M(G_1 \times G_2)$, or more generally in $M(\bar{G}_1 \times \bar{G}_2)$, which are contained in $M(G_1) \hat{\otimes} M(G_2)$. In the context of this example, it is shown that

$$M(G_1) \hat{\otimes} M(G_2) = M(G_1 \times G_2) \cap (M(\bar{G}_1) \hat{\otimes} M(\bar{G}_2)).$$

In particular, if $G_1 = G_2 = \mathbf{R}$, then the space $M(\mathbf{R}) \hat{\otimes} M(\mathbf{R})$ is properly contained in $M(\mathbf{R} \times \mathbf{R})$ by [4, pp. 784–785]; however, the characterization $M(\mathbf{R}) \hat{\otimes} M(\mathbf{R}) = M(\mathbf{R} \times \mathbf{R}) \cap (M(\bar{\mathbf{R}}) \hat{\otimes} M(\bar{\mathbf{R}}))$ appears to be new.

The major results of the paper appear in Section III. We open that section by giving characterizations (in Theorem 3.1, Corollary 3.2, and Corollary 3.3) of $\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$ in the general setting of arbitrary L-subspaces $\mathscr{L}_1$, $\mathscr{L}_2$ of measure spaces $M(X_1)$, $M(X_2)$, respectively. These results are presented in terms of the well-known fact that $M(X_1 \times X_2)$ is isometrically isomorphic to the Banach space $M(X_1, M(X_2))$ of all countably-additive vector-valued measures of finite variation from the Borel field $\mathfrak{B}(X_1)$ of X_1 into $M(X_2)$. Since each vector-valued measure m_1 in $M(X_1, M(X_2))$ has an associated positive variation measure μ^{m_1} in $M(X_1)$, the idea of m_1 possessing a (strong) Radon-Nikodym (RN) derivative with respect to μ^{m_1} is quite natural. These general L-space results are then applied to multiplier questions. For example, a multiplier m in $\mathscr{M}(A_1 \hat{\otimes} A_2) \subseteq M(\Gamma_1 \times \Gamma_2)$ is shown to belong to $\mathscr{M}(A_1) \hat{\otimes} \mathscr{M}(A_2)$ if and only if, as a vector-valued measure m_1, it has a strong RN derivative with respect to its variation measure μ^{m_1} (Theorem 3.4).

II. Preliminaries

A semicharacter χ on the structure semigroup Γ of the CMA A is a nonzero continuous function from Γ into the complex unit disc such that $\chi(st) = \chi(s)\chi(t)$ for all s, t in Γ. The set $\hat{\Gamma}$ of all semicharacters on Γ can be identified with the maximal ideal space of A. If $A_\Gamma \subset M(\Gamma)$ denotes the canonical image of A in $M(\Gamma)$, then each $\chi \in \hat{\Gamma}$ determines a complex homomorphism on A by the map

$$\mu \mapsto \langle \chi, \mu \rangle = \int_\Gamma \chi(s)\, d\mu(s),$$

for μ in A_Γ, and each complex homomorphism of A may be so expressed. Now, if $C(\Gamma)$ denotes the Banach space of all continuous complex-valued functions on Γ endowed with the sup norm $\|\cdot\|_\infty$, then it turns out that $C(\Gamma)$ is the closed linear span of $\hat{\Gamma}$ in $C(\Gamma)$. A measure μ in $M(\Gamma)$ will be regarded both as a countably additive set function on the Borel field $\mathfrak{B}(\Gamma)$ of Γ and as a linear functional on $C(\Gamma)$. Thus, we will use the symbols $\mu(E)$, for E in $\mathfrak{B}(\Gamma)$, and $\langle f, \mu \rangle$, for f in $C(\Gamma)$, where, as is customary,

$$\langle f, \mu \rangle = \int_\Gamma f(s)\, d\mu(s).$$

In general, the notation $\langle \cdot, \cdot \rangle$ will denote the action of a dual Banach space on its pre-dual Banach space. Of course, $M(\Gamma)$ becomes a CMA by defining a convolution product

$$(\mu * \nu)(E) = \int_\Gamma \int_\Gamma K_E(st)\, d\mu(s)\, d\nu(t),$$

where $\mu, \nu \in M(\Gamma)$, $E \in \mathfrak{B}(\Gamma)$, and K_E is the characteristic function of E.

A major portion of our analysis relies upon the interplay between vector-valued measures and tensor products developed in [3], [4]. We begin with a result of Grothendieck. Let $\nu \in M(X_1)$ be a positive measure, and $\mathscr{L}$ an L-subspace of $M(X_2)$. If $L^1(\nu, \mathscr{L})$ denotes the Banach space of all strongly ν-measurable, Bochner-integrable functions from X_1 to $\mathscr{L}$, then $L^1(\nu, \mathscr{L}) = L^1(\nu) \hat{\otimes} \mathscr{L}$, the projective tensor product of $L^1(\nu)$ and $\mathscr{L}$, where $L^1(\nu)$ is the Banach space of all ν-integrable scalar-valued functions on X_1 [5, Theorem 2, p. 59]. Thus, $L^1(\nu, \mathscr{L})$ can be regarded as the Banach space of all $\mathscr{L}$-valued measures in $M(X_1, M(X_2)) = M(X_1 \times X_2)$ [4, p. 784] that have strong RN derivatives with respect to ν. For standard definitions and facts about the various kinds of measurability and integrability for vector-valued measures, consult [7, Sec. III, 1].

The isometric isomorphisms

$$M(X_1 \times X_2) = M(X_1, M(X_2)) \quad \text{and} \quad M(X_1 \times X_2) = M(X_2, M(X_1))$$

are implemented via the mappings $m \mapsto m_1$ and $m \mapsto m_2$, $m \in M(X_1 \times X_2)$, where $m_1\colon \mathfrak{B}(X_1) \to M(X_2)$ and $m_2\colon \mathfrak{B}(X_2) \to M(X_1)$ are defined by

$$m_1(E)(F) = m(E \times F) = m_2(F)(E), \qquad E \in \mathfrak{B}(X_1),\ F \in \mathfrak{B}(X_2);$$

in terms of linear functionals,

$$\langle g, m_1(E)\rangle = \int_{E\times X_2} g(x_2)\, dm(x_1, x_2), \quad g \in C(X_2),\ E \in \mathfrak{B}(X_1),$$

and similarly for $m_2(F)$, $F \in \mathfrak{B}(X_2)$. Associated with $m \in M(X_1 \times X_2)$ are its variation measures $\mu^{m_1} \in M(X_1)$ and $\mu^{m_2} \in M(X_2)$ defined by

$$\mu^{m_1}(E) = |m|(E \times X_2) = |m|_2(X_2)(E), \quad E \in \mathfrak{B}(X_1),$$
$$\mu^{m_2}(F) = |m|(X_1 \times F) = |m|_1(X_1)(F), \quad F \in \mathfrak{B}(X_2).$$

Clearly, $\|m_1(E)\| \leq \mu^{m_1}(E)$ for all $E \in \mathfrak{B}(X_1)$, and it turns out [2, p. 532] that μ^{m_1} is the least positive measure in $M(X_1)$ for which this is true. Now, the norm of $m_1 \in M(X_1, M(X_2))(m_2 \in M(X_2, M(X_1)))$ is $\|m_1\| = \|\mu^{m_1}\| = \|m\|$ ($\|m_2\| = \|\mu^{m_2}\| = \|m\|$), establishing the isometry

$$M(X_1 \times X_2) = M(X_1, M(X_2)) \quad (= M(X_2, M(X_1))).$$

Also if $f \in L^1(X_1, \mu^{m_1})$, then

$$\int_{X_1} f(x_1)\, d\mu^{m_1}(x_1) = \int_{X_1\times X_2} f(x_1)\, d|m|(x_1, x_2),$$

and similarly for $g \in L^1(X_2, \mu^{m_2})$.

Each $m \in M(X_1 \times X_2)$ determines a bounded linear transformation

$$T_{m_1}\colon L^1(X_1, \mu^{m_1}) \to M(X_2)$$

defined by

$$T_{m_1}(f)(F) = \int_{X_1\times F} f(x_1)\, dm(x_1, x_2), \quad f \in L^1(X_1, \mu^{m_1}),\ F \in \mathfrak{B}(X_2).$$

This integral is well defined since $|m|_2(F) \leq |m|_2(X_2) = \mu^{m_1}$, for all $F \in \mathfrak{B}(X_2)$. As a linear functional, T_{m_1} satisfies.

$$\langle g, T_{m_1}(f)\rangle = \int_{X_1\times X_2} f(x_1) g(x_2)\, dm(x_1, x_2), \quad f \in L^1(X_1, \mu^{m_1}),\ g \in C(X_2).$$

Of course, there exists a map $T_{m_2}\colon L^1(X_2, \mu^{m_2}) \to M(X_1)$, defined analogously. For all $f \in C(X_1)$, $g \in C(X_2)$, and $m \in M(X_1 \times X_2)$, T_{m_1} and T_{m_2} are related by $\langle g, T_{m_1}(f)\rangle = \langle f \otimes g, m\rangle = \langle f, T_{m_2}(g)\rangle$.

We conclude the preliminaries with certain key results about L-subspaces and RN derivatives. Since $\mathscr{L}_1$ and $\mathscr{L}_2$ are L-subspaces of $M(X_1)$ and $M(X_2)$, respectively, $\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$ imbeds in $M(X_1 \times X_2)$. A straightforward generalization of [4, Theorem 6.3] yields the following characterization of $\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$. Let $m \in M(X \times X_2)$. If $m \in \mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$, then $\mu^{m_1} \in \mathscr{L}_1$, $m_1 \in M(X_1, \mathscr{L}_2)$ and m_1 has a strong RN derivative with respect to μ^{m_1}. Conversely, if $\mu^{m_1} \in \mathscr{L}_1$ and m_1 has a strong $\mathscr{L}_2$-valued RN derivative with respect to μ^{m_1}, then $m \in \mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$.

In particular, if $\mathscr{L}_2$ has the RN property (see [1]), then $M(X_1, \mathscr{L}_2) = M(X_1) \hat{\otimes} \mathscr{L}_2$. Moreover, it is a consequence of [5, Theorem 11, p. 141] that

$$\mu^{m_2} = \vee \{ | T_{m_1}(f) | : f \in C(X_1), \|f\|_\infty \leq 1 \}.$$

Further, $m_1 \in M(X_1, \mathscr{L}_2)$ if and only if $\mu^{m_2} \in \mathscr{L}_2$ if and only if $|m|_1 \in M(X_1, \mathscr{L}_2)$. (Of course, analogous statements hold when the indices 1 and 2 are interchanged.)

III. Multipliers and Radon-Nikodym derivatives

We begin the section with some additional characterizations of $\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$. The measure space $\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$ is

$$\{m \in M(X_1 \times X_2) : m \ll \nu \otimes \tau, \nu \in \mathscr{L}_1, \tau \in \mathscr{L}_2\} \quad [12, \text{pp. } 155\text{–}6].$$

The next theorem offers natural choices for ν and τ.

THEOREM 3.1. *The subspace $\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$ of $M(X_1 \times X_2)$ consists precisely of those measures m in $M(X_1 \times X_2)$ which satisfy the following three conditions:*

(1) *$m_1 \in M(X_1, \mathscr{L}_2)$ or, equivalently, $\mu^{m_2} \in \mathscr{L}_2$.*
(2) *$m_2 \in M(X_2, \mathscr{L}_1)$ or, equivalently, $\mu^{m_1} \in \mathscr{L}_1$.*
(3) *$m \in L^1(\mu^{m_1}) \hat{\otimes} L^1(\mu^{m_2})$ or, equivalently, $m \ll \mu^{m_1} \otimes \mu^{m_2}$.*

Proof. First, for both conditions (1) and (2), results from Section II imply that the stated equivalences obtain, while the equivalence of the two statements in (3) follows from the fact that $L^1(\mu^{m_1}) \hat{\otimes} L^1(\mu^{m_2}) = L^1(\mu^{m_1} \otimes \mu^{m_2})$ [5, Corollary 4, p. 61].

Now, suppose that $m \in \mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$. From the preliminaries, (1) and (2) are both clearly satisfied; in fact, since $m \in \mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$, the vector-valued measure m_1 has a strong RN derivative with respect to μ^{m_1}. Thus, $m_1 \in L^1(\mu^{m_1}, \mathscr{L}_2)$, or equivalently,

$$m \in L^1(\mu^{m_1}) \hat{\otimes} \mathscr{L}_2.$$

Now by similar reasoning, the vector-valued measure m_2 is in $M(X_2, L^1(\mu^{m_1}))$, and has a strong RN derivative with respect to μ^{m_2}. Thus,

$$m_2 \in L^1(\mu^{m_2}, L^1(\mu^{m_1})),$$

or, equivalently, $m \in L^1(\mu^{m_1}) \hat{\otimes} L^1(\mu^{m_2})$; that is, (3) is satisfied.

Conversely, if m is a measure in $M(X_1 \times X_2)$ for which (1), (2), and (3) are satisfied, then m is clearly contained in

$$\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2 = \{m \in M(X_1 \times X_2) : m \ll \nu \otimes \tau, \nu \in \mathscr{L}_1, \tau \in \mathscr{L}_2\}. \quad \blacksquare$$

Theorem 3.1 may be restated simply as the identity

$$\mathscr{L}_1 \hat{\otimes} \mathscr{L}_2 = \{m \in M(X_1 \times X_2) : m \ll \mu^{m_1} \otimes \mu^{m_2}, \mu^{m_1} \in \mathscr{L}_1, \mu^{m_2} \in \mathscr{L}_2\}.$$

The next result shows that if $m \in \mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$, then $\mu^{m_1} \otimes \mu^{m_2}$ is the "least" product measure with respect to which m is absolutely continuous.

COROLLARY 3.2. *If $m \in M(X_1 \times X_2)$ and $m \ll \nu \otimes \tau$, where $\nu \in M(X_1)$ and $\tau \in M(X_2)$, then $\mu^{m_1} \ll \nu$, $\mu^{m_2} \ll \tau$, $\mu^{m_1} \otimes \mu^{m_2} \ll \nu \otimes \tau$, and $m \ll \mu^{m_1} \otimes \mu^{m_2}$.*

Proof. Let $\mathscr{L}_1 = L^1(\nu)$ and $\mathscr{L}_2 = L^1(\tau)$. Then, by hypothesis, $m \in L^1(\nu \otimes \tau) = \mathscr{L}_1 \hat{\otimes} \mathscr{L}_2$; hence, all of the conclusions follow immediately from Theorem 3.1. ∎

Corollary 3.2 yields an alternate (equivalent) formulation of condition (3) in Theorem 3.1, which is often easier to apply.

COROLLARY 3.3. *If $m \in M(X_1 \times X_2)$, then $m \ll \mu^{m_1} \otimes \mu^{m_2}$ if and only if $m \in M(X_1) \hat{\otimes} M(X_2)$.*

We now apply the accumulated results on L-subspaces to multipliers. A consequence of our standing assumptions about the CMA-semigroup pairs (A_1, Γ_1) and (A_2, Γ_2) is that the inclusions

$$\mathscr{M}(A_1) \hat{\otimes} \mathscr{M}(A_2) \subseteq \mathscr{M}(A_1 \hat{\otimes} A_2) \subseteq M(\Gamma_1 \times \Gamma_2)$$

obtain, and that the results on L-spaces are applicable.

Our first multiplier theorem implies that a multiplier measure m in

$$\mathscr{M}(A_1 \hat{\otimes} A_2) \subseteq M(\Gamma_1 \times \Gamma_2)$$

belongs to $\mathscr{M}(A_1) \hat{\otimes} \mathscr{M}(A_2)$ if and only if it possesses a strong RN derivative with respect to μ^{m_1}.

THEOREM 3.4. *If a measure m in $M(\Gamma_1 \times \Gamma_2)$ is a multiplier of $A_1 \hat{\otimes} A_2$, then:*

(1) *$m_1 \in M(\Gamma_1, \mathscr{M}(A_2))$ or, equivalently, $\mu^{m_2} \in \mathscr{M}(A_2)$.*
(2) *$m_2 \in M(\Gamma_2, \mathscr{M}(A_1))$ or, equivalently, $\mu^{m_1} \in \mathscr{M}(A_1)$.*

Proof. By symmetry, it clearly suffices to show that (1) holds. The equivalence of the two assertions in (1) follows from Section II. Now, for each ϕ in $\hat{\Gamma}_1$, define θ_ϕ: $M(\Gamma_1 \times \Gamma_2) \to M(\Gamma_2)$ by setting

$$\langle \psi, \theta_\phi(\mu) \rangle = \langle \phi \otimes \psi, \mu \rangle, \quad \psi \in \hat{\Gamma}_2,$$

for μ in $M(\Gamma_1 \times \Gamma_2)$. Then, since $m \in \mathscr{M}(A_1 \hat{\otimes} A_2)$ by hypothesis,

$$m * (a_1 \otimes a_2) \in A_1 \hat{\otimes} A_2 \quad \text{for all } a_i \in A_i,\ i = 1, 2.$$

Hence, by [14, Lemma 2], $\theta_\phi(m * (a_1 \otimes a_2)) \in A_2$, for every ϕ in $\hat{\Gamma}_1$. Now, if $\phi \in \hat{\Gamma}_1$, $a_1 \in A_1$, and $a_2 \in A_2$, then

$$(\dagger) \qquad \theta_\phi(m * (a_1 \otimes a_2)) = \langle \phi, a_1 \rangle (T_{m_1}(\phi) * a_2).$$

To see this, observe that, for $\psi \in \hat{\Gamma}_2$,

$$\begin{aligned}\langle \psi, \theta_\phi(m * (a_1 \otimes a_2))\rangle &= \langle \phi \otimes \psi, m * (a_1 \otimes a_2)\rangle \\ &= \langle \phi \otimes \psi, m\rangle\langle \phi, a_1\rangle\langle \psi, a_2\rangle \\ &= \langle \phi, T_{m_1}(\phi)\rangle\langle \phi, a_1\rangle\langle \psi, a_2\rangle \\ &= \langle \phi, a_1\rangle\langle \psi, T_{m_1}(\phi) * a_2\rangle.\end{aligned}$$

From (†) it follows that, for each ϕ in $\hat{\Gamma}_1$, $T_{m_1}(\phi) * a_2 \in A_2$, for all a_2 in A_2. Thus, T_{m_1} maps $\hat{\Gamma}_1$ into $\mathcal{M}(A_2)$, and, as a result, T_{m_1} maps $L^1(X_1, \mu^{m_1})$ into $\mathcal{M}(A_2)$. Therefore, $m_1(E) = T_{m_1}(K_E) \in \mathcal{M}(A_2)$, for $E \in \mathfrak{B}(\Gamma_1)$. ∎

COROLLARY 3.5. *For a multiplier m in* $\mathcal{M}(A_1 \hat{\otimes} A_2) \subset M(\Gamma_1 \times \Gamma_2)$, *the following statements are equivalent:*

(1) $m \in \mathcal{M}(A_1) \hat{\otimes} \mathcal{M}(A_2)$.
(2) m_1 *has a strong RN derivative with respect to* μ^{m_1}.
(3) $m \in M(\Gamma_1) \hat{\otimes} M(\Gamma_2)$.
(4) $m \ll \mu^{m_1} \otimes \mu^{m_2}$.

Proof. Theorem 3.4 and Corollary 3.3 together yield the equivalence of (1) and (4), while the equivalence of (3) and (4) follows from Corollary 3.3. Finally the equivalence of (2) and (3) is proved in [4, Theorem 6.3]. ∎

The equivalence of statements (1) and (4) of Corollary 3.5 implies that the multiplier-subspace $\mathcal{M}(A_1) \hat{\otimes} \mathcal{M}(A_2)$ has the characterization

$$\mathcal{M}(A_1) \hat{\otimes} \mathcal{M}(A_2) = \{m \in \mathcal{M}(A_1 \hat{\otimes} A_2): m \ll \mu^{m_1} \otimes \mu^{m_2}\},$$

while the equivalence of statements (1) and (3) yields the identity

$$\mathcal{M}(A_1) \hat{\otimes} \mathcal{M}(A_2) = \mathcal{M}(A_1 \hat{\otimes} A_2) \cap (M(\Gamma_1) \hat{\otimes} M(\Gamma_2)).$$

It is natural to ask under what circumstances

$$\mathcal{M}(A_1) \hat{\otimes} \mathcal{M}(A_2) = \mathcal{M}(A_1 \hat{\otimes} A_2).$$

This is the case, for example, if A_1 and A_2 are both L^1-algebras under order convolution [8]. Below, another sufficient condition to effect this decomposition is given in terms of the notion of an l_1-algebra. By an l_1-algebra, we mean a commutative semisimple CMA which, as a Banach space, is isometrically isomorphic to $l_1(W)$, for some set W.

PROPOSITION 3.6. *If either* $\mathcal{M}(A_1)$ *or* $\mathcal{M}(A_2)$ *is an* l_1-*algebra, then*

$$\mathcal{M}(A_1 \hat{\otimes} A_2) = \mathcal{M}(A_1) \hat{\otimes} \mathcal{M}(A_2).$$

Proof. Suppose that $\mathscr{M}(A_2)$ is an l_1-algebra; then $\mathscr{M}(A_2)$ has the RN property [1, p. 31]. Therefore, from Section II, $M(\Gamma_1, \mathscr{M}(A_2))$ is isometrically isomorphic to

$$M(\Gamma_1) \hat{\otimes} \mathscr{M}(A_2) \subseteq M(\Gamma_1) \hat{\otimes} M(\Gamma_2).$$

Hence, if $m \in \mathscr{M}(A_1 \hat{\otimes} A_2)$, then by Theorem 3.4, $m_1 \in M(\Gamma_1, \mathscr{M}(A_2))$, so $m \in M(\Gamma_1) \hat{\otimes} M(\Gamma_2)$, and the conclusion now follows from Corollary 3.5. Clearly, an analogous argument can be given when $\mathscr{M}(A_1)$ is an l_1-algebra. ∎

If S is a commutative separative semigroup, then the Banach algebra $l_1(S)$ will be called a semigroup-algebra. Such algebras were among the first examples of commutative CMA's to be studied [6]. When $l_1(S)$ has a weak approximate identity of norm one, the multiplier algebra $\mathscr{M}(l_1(S))$ is also a semigroup-algebra; more precisely, $\mathscr{M}(l_1(S)) = l_1(\Omega(S))$, where $\Omega(S)$ is the translational hull (or multiplier semigroup) of S [10, Theorem 4.9]. Thus, $\mathscr{M}(A_1 \hat{\otimes} l_1(S)) = \mathscr{M}(A_1) \hat{\otimes} \mathscr{M}(l_1(S))$ by Proposition 3.6.

It might seem possible for $\mathscr{M}(A)$ to be an l_1-algebra when A is not a semigroup-algebra. Our final proposition shows that this cannot occur.

PROPOSITION 3.7. *Let (A, Γ) be a commutative semisimple CMA, realized as an L-subalgebra of $M(\Gamma)$.*

(1) *Then A is an l_1-algebra if and only if A is the semigroup-algebra $l_1(S)$ for some separative subsemigroup S of Γ.*

(2) *If, in addition, A has a weak approximate identity of norm one, then A is an l_1-algebra (equivalently, a semigroup-algebra) if and only if $\mathscr{M}(A)$ is an l_1-algebra. Moreover, if $A = l_1(S)$, where S is a subsemigroup of Γ, then $\mathscr{M}(A) = l_1(\Omega(S))$, and $\Omega(S)$, the translational hull of S, is also a subsemigroup of Γ.*

Proof. (1) Suppose that A is an l_1-algebra. Then every measure in A is discrete; indeed, $A = l_1(S)$, where $S = \{x \in \Gamma\colon x \in \text{support of } \mu, \mu \in A\}$. Since A is an algebra, $\delta_{xy} = \delta_x * \delta_y \in A$, for all x, y in S, and so S is a subsemigroup of the commutative semigroup Γ. Further, the subsemigroup S is separative, because A is semisimple. Thus, A is a semigroup-algebra. The converse is obvious.

(2) If A is an l_1-algebra, then, from (1), $A = l_1(S)$, for some subsemigroup S of Γ. Thus, $\mathscr{M}(l_1(S)) = l_1(\Omega(S))$, where $\Omega(S)$ is a subsemigroup of Γ (containing S as an ideal) [10], and so, in particular, $\mathscr{M}(A)$ is an l_1-algebra.

Conversely, suppose that $\mathscr{M}(A)$ is an l_1-algebra. Then $\mathscr{M}(A)$ has the RN property, hence, the closed subspace A of $\mathscr{M}(A)$ also has the RN property [1, p. 30]. Arguing as in the proof of (1), it follows readily that A is the semigroup-algebra $l_1(S)$, where

$$S = \{x \in \Gamma\colon x \in \operatorname{supp} \mu, \mu \in A\}.$$

Finally, from the preceding paragraph, $\mathscr{M}(A) = l_1(\Omega(S))$. ∎

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