ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 2, Summer 1980

RELATION WITH THE HOPF INVARIANT REVISITED

BY

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1. Introduction

The title of this note refers to Section 8 of Adams' paper *On the groups* $J(X)$, IV [4]. There Adams used his results from [2] to establish a formula which determines the mod p Hopf invariant in terms of the complex e-invariant (Proposition 8.2 [4] or Theorem 2 below). As an outgrowth of his 1970 lectures at Chicago [5], Adams reformulated the results of [2] in an article *Chern characters revisited* [1]. When related methods from the Chicago lectures are applied to a suitable version of the e-invariant, they yield a new proof of Proposition 8.2 which seems conceptually simpler. The object of this note, then, is to reformulate the e-invariant in a more general context and, with this and the Chicago "technology", revisit Proposition 8.2 in a spirit similar to the one with which Adams revisited his earlier Chern characters paper [2].

2. Definitions and statement of results

We begin by defining a homotopy invariant in a manner reminiscent of the definition of the invariant e_C which uses the Chern character [4; p. 41]. Let E be a ring spectrum with unit $i\colon S^0 \to E$ and let η_L and η_R respectively denote the homomorphisms

$$(E \wedge i)_*\colon \pi_*(E) = \pi_*(E \wedge S^0) \to \pi_*(E \wedge E)$$

$$\text{and} \quad (i \wedge E)_*\colon \pi_*(E) = \pi_*(S^0 \wedge E) \to \pi_*(E \wedge E).$$

Let $f \in \pi_n(S^0)$ be given and let

$$\begin{array}{ccc} S^n & \xrightarrow{\ f\ } & S^0 \\ {\scriptstyle Q(f)}\nwarrow & & \swarrow{\scriptstyle P(f)} \\ & C(f) & \end{array}$$

denote the associated cofiber triangle.

Now suppose that $f_*\colon E_*(S^n) \to E_*(S^0)$ is zero. (Let $\iota_k \in \pi_k(E \wedge S^k)$ denote the $E_*(S^0)$ generator $i \wedge S^k\colon S^0 \wedge S^k \to E \wedge S^k$.) As $f^*(\iota_0) = 0$, there is an extension

Received April 7, 1978.

$\tilde{\imath}_0 \in E^0C(f) = [C(f), E]_0$ so that $P(f)^*(\tilde{\imath}_0) = \imath_0$. Similarly, as $f_*(\imath_n) = 0$, there is a coextension $\bar{\imath}_n \in E_{n+1}(C(f))$ so that $Q(f)_*(\bar{\imath}_n) = \imath_n$. Then the element $\lambda(f) = (E \wedge \tilde{\imath}_0)_*(\bar{\imath}_n)$ lies in $\pi_{n+1}(E \wedge E)$. Since varying $\tilde{\imath}_0$ by an element in ker $P(f)^*$ varies $\lambda(f)$ by an element in im η_R and varying $\bar{\imath}_n$ by an element in ker $Q(f)_*$ varies $\lambda(f)$ by an element in im η_L, $\lambda(f)$ determines a unique element in $\pi_{n+1}(E \wedge E)/\text{ind}_E$ which we denote by $\lambda_E(f)$. (Here $\text{ind}_E = \text{im } \eta_L + \text{im } \eta_R$).

Before altering the scope of the definition of λ_E, a few remarks are in order. First of all, we emphasize that for maps of the n-stem which are homologically trivial, $\lambda_E(f)$ is defined without any special assumptions on the ring spectrum E such as the flatness of $E_*(E)$ over $E_*(S^0)$; consequently $\lambda_{bu}(f)$ is defined for any $f \in \pi_{2n-1}(S^0)$, $n > 0$, where bu is the connective BU-spectrum. Secondly, λ_E is clearly natural with respect to coefficient maps $c: E \to F$ of ring spectra. Thirdly, $\pi_{n+1}(E \wedge E)/\text{ind}_E$ can be identified with $\pi_{n+1}(E \wedge C(i))/\text{im } \eta_R$, where $C(i)$ is the mapping cone of $i: S^0 \to E$ and in terms of this identification $d'_{n+1}(\lambda_E(f)) = 0$, where

$$d'_{n+1}: \pi_{n+1}(E \wedge C(i)) \to \pi_{n+1}(E \wedge (C(i))^2)$$

is the Adams differential. Consequently $\lambda_E(f)$ lies in $E_2^{1,n+1}(S^0)$, the E_2 term of the Adams spectral sequence associated to E and S^0.

Now whenever we choose to consider a ring spectrum E which is commutative and for which $E_*(E)$ is flat over $E_*(S^0)$, we may broaden the class of maps we may consider to those $f: X \to Y$ which induce 0 on E_* homology and whose domains satisfy the isomorphism condition of [6; p. 609]. In this case there are elements $\tilde{\imath}_Y \in [C(f), E \wedge Y]_0$ and $\bar{\imath}_X \in [X, E \wedge C(f)]_1$ so that

$$P(f)^*(\tilde{\imath}_Y) = \imath_Y = i \wedge Y \in [Y, E \wedge Y]_0$$

and

$$Q(f)_*(\bar{\imath}_X) = \imath_X = i \wedge X \in [X, E \wedge X]_0.$$

Thus the element $(E \wedge \tilde{\imath}_Y)_*(\bar{\imath}_X) \in [X, E \wedge E \wedge Y]_1$ determines an element $\lambda_E(f)$ in $\text{Ext}^{1,1}_{E_*(E)}(E_*(X), E_*(Y))$.

This invariant is related to Adams' e-invariant in the following result.

Proposition 1. *With the data of the preceding paragraph,*

$$\lambda_E(f) = e(f) \quad \textit{in } \text{Ext}^{1,1}_{E_*(E)}(E_*(X), E_*(Y)).$$

The proof resembles Proposition 1 of [6] and is omitted.

The next data will be useful in the statement and proof of Adams' Proposition 8.2, given below as Theorem 2. As usual, let H denote the mod p Hopf invariant, $e_{\mathbf{C}}$ the complex e-invariant, $H\mathbf{Z}_p$ the $\mathbf{Z}_p$ Eilenberg-MacLane spectrum, K the BU spectrum and bu the connective BU-spectrum. Further let

$f \in \pi_{2n-1}(S^0)$ with $n = k(p-1) > 0$ so that $\lambda_{H\mathbf{Z}_p}(f) = H(f)$ and $\lambda_K(f) = e_{\mathbf{C}}(f)$ are defined. Finally, let $\mathbf{Z}_{(p)}$ denote the integers localized at p and $\rho'\colon \mathbf{Z}_{(p)} \to \mathbf{Z}/p = \mathbf{Z}_p$ be reduction.

THEOREM 2 (Adams). *With the above data*
(i) $p^k e_{\mathbf{C}}(f) \in \mathbf{Z}_{(p)}$ *and*
(ii) $H(f) = -\rho'(p^k e_{\mathbf{C}}(f))$.

3. Proof of Theorem 2

We give the details for $p = 2$ and open the proof of (i) with several preliminary remarks. First, $\mathrm{Ext}^{1,2n}_{K_*(K)}(K_*(S^0), K_*(S^0))$ is the cyclic group of order $m(n)$ generated by $(v^n - u^n)/m(n)$ where $m(n)$ is as in [3, p. 139]. Secondly, let $a\colon bu \to K$ be the coefficient map. As

$$(a \wedge a)_*\colon \pi_*(bu \wedge bu) \to \pi_*(K \wedge K)$$

maps ind_{bu} isomorphically onto ind_K in positive dimensions and

$$(a \wedge a)_\#(\lambda_{bu}(f)) = \lambda_K(f) = e_{\mathbf{C}}(f),$$

the image of any representative $x \in \pi_*(K \wedge K)$ of $\lambda_K(f)$ in $\mathbf{Q}[u, u^{-1}, v, v^{-1}]$ may be assumed to lie in the image of $\pi_*(bu \wedge bu)$. Finally we may assume bu and K are localized at 2.

Consequently, the representative

$$g(u, v) = \frac{M(f)}{m(n)}(v^n - u^n)$$

of $e_{\mathbf{C}}(f)$ may be assumed to lie in $\mathbf{Z}_{(2)}[u/2, v/2]$ according to condition (2′) of 17.5 [5; p. 288]. But

$$g(u, v) = \frac{2^n M(f)}{m(n)}((v/2)^n - (u/2)^n) \in \mathbf{Z}_{(2)}[u/2, v/2]$$

implies that $2^n M(f)/m(n) \in \mathbf{Z}_{(2)}$ and (i) is established.

Now let $b\colon bu \to H\mathbf{Z}_2$ be the coefficient map equal to the composite

$$bu \xrightarrow{j} H\mathbf{Z} \xrightarrow{f^0} H\mathbf{Z}_{(2)}$$

of [5; p. 262]. Recalling that $(b \wedge H\mathbf{Z}_2)_*$ identifies $\pi_*(bu \wedge H\mathbf{Z}_2)$ with the subalgebra of $\pi_*(H\mathbf{Z}_2 \wedge H\mathbf{Z}_2) = \mathscr{A}_*$ generated by $\xi_1^2, \xi_2^2, \xi_3, \xi_4, \ldots$, we see that $(bu \wedge b)_*\colon \pi(bu \wedge bu) \to \pi_*(bu \wedge H\mathbf{Z}_2)$ maps $u/2 \to \xi_1^2$ and $v/2 \to 0$. Thus, if $x \in \pi_*(bu \wedge bu)$ represents $\lambda_{bu}(f)$ (and projects to $g(u, v)$ in $\mathbf{Z}_{(2)}[u/2, v/2]$),

$$(*) \qquad (b \wedge b)_*(x) = H(f)\xi_1^{2n},$$

according to the coefficient naturality of the invariant λ on the one hand, and

$$(**) \qquad (b \wedge b)_*(x) = \rho'\left(\frac{-2^n M(f)}{m(n)}\right)\xi_1^{2n} + (b \wedge H\mathbf{Z}_2)_*(\beta_2(k))$$

according to the methods of 16.5 [5; p. 270] on the other hand. Here β_2 is the mod 2 Bockstein $\beta_2\colon \pi_{2n+1}(bu \wedge H\mathbf{Z}_2) \to \pi_{2n}(bu \wedge H\mathbf{Z}_2)$. Equating $(*)$ and $(**)$, we obtain the formula (ii).

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