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A STRONG SPECTRAL RESIDUUM FOR EVERY CLOSED OPERATOR

BY

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1. Introduction

Decomposable operators (see, e.g., [2]) are linear operators, for which a weaker, geometric variant of the constructions, characteristic of spectral operators [3], is still possible. Residually decomposable operators, introduced by F.-H. Vasilescu [6], [7], and bounded S-decomposable operators, studied by I. Bacalu [1], are operators such that, loosely speaking, the property of decomposability holds only outside a certain part of the spectrum. F.-H. Vasilescu has proved [7] that for certain operators having the single-valued extension property there is a unique minimal closed subset of the spectrum, called the spectral residuum, outside which the operator has a good spectral behavior of this kind.

The main result of this paper is that, utilizing a similar concept of good spectral behavior, for an arbitrary closed operator there exists a unique minimal closed subset of the spectrum, called the strong spectral residuum, outside which the operator shows this behavior. It is proved that for a large class, close to that occurring in [7; Theorem 3.1], of operators strong and ordinary spectral residues coincide. If the strong spectral residuum is void, the operator is (bounded and) decomposable. Whether the converse is true, is equivalent to a well-known unsolved problem, raised by I. Colojoară and C. Foiaş [2; 6.5 (b)]. Though the proofs seem to remain valid after minor modifications in a Fréchet space, to make references more convenient, we have chosen the Banach space setting.

Let X be a complex Banach space and let $C(X)$ and $B(X)$ denote the class of closed and bounded linear operators on X, respectively. Let C and $\bar{C}$ denote the complex plane and its one-point compactification, respectively. Unless stated explicitly otherwise, all topological concepts for sets in $\bar{C}$ will be understood in the topology of $\bar{C}$. If $F \subset \bar{C}$, then F^c denotes $\bar{C} \backslash F$ and $\bar{F}$ denotes the closure of F. For $T \in C(X)$, $D(T)$ is its domain and $\sigma(T)$ denotes its extended spectrum, which coincides with the spectrum $s(T)$ if $T \in B(X)$, and is $s(T) \cup \{\infty\}$ otherwise. We set $\rho(T) = \sigma(T)^c$. If Y is a closed subspace of X and $T(Y \cap D(T)) \subset Y$, then we write $Y \in I(T)$ and $T \mid Y$ denotes the restriction of T to $Y \cap D(T)$.

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We recall some concepts and facts from [7]. For $x \in X$, $z \in \bar{C}$ we say that $z \in \delta_T(x)$ if in a neighborhood U of z there is a holomorphic $D(T)$-valued function f_x such that $(u - T)f_x(u) = x$ for $u \in U \cap C$. Such a function $f_x(u)$ is called T-associated with x. There is a unique maximal open set Ω_T in $\bar{C}$ with the following property: if $G \subset \Omega_T$ is an open set and f_0: $G \to D(T)$ is a holomorphic function such that $(u - T)f_0(u) = 0$ for $u \in G \cap C$ then $f_0(u) = 0$ on G. We put $S_T = \Omega_T^c$, and, for any x in X,

$$\gamma_T(x) = \delta_T(x)^c, \quad \sigma_T(x) = \gamma_T(x) \cup S_T \quad \text{and} \quad \rho_T(x) = \sigma_T(x)^c.$$

We say that T has the single-valued extension property if S_T is void. For any $T \in C(X)$, $H \subset \bar{C}$ we set $X_T(H) = \{x \in X; \sigma_T(x) \subset H\}$, then $X_T(H)$ is a linear manifold in X. A closed linear subspace Y in X belongs to the class I_T if $T \mid Y \in B(Y)$. If F is a closed set in $\bar{C}$, define

$$I_{T,F} = \{Y \in I_T; \sigma(T \mid Y) \subset F\}.$$

If $I_{T,F}$ has an upper bound (with respect to the relation $\subset$), which belongs to $I_{T,F}$, then it is denoted by $X_{T,F}$. Similarly, we define

$$I(T, F) = \{Y \in I(T); \sigma(T \mid Y) \subset F\}.$$

If $I(T, F)$ has an upper bound, belonging to $I(T, F)$, with respect to the relation $\subset$, then it is denoted by $X(T, F)$.

Definition 1. A closed subspace Y in $I(T)$ is a spectral maximal space of $T \in C(X)$ if for any $Z \in I(T)$ the relation $\sigma(T \mid Z) \subset \sigma(T \mid Y)$ implies $Z \subset Y$.

It is easily seen that if F is closed in $\bar{C}$ and $X(T, F)$ exists, then $X(T, F)$ is a spectral maximal space of T. Conversely, if Y is a spectral maximal space of T and $F = \sigma(T \mid Y)$, then $Y = X(T, F)$.

The following result is taken from [4] and will be utilized later.

Lemma 1. *If $T \in C(X)$, the closed set $F \subset \bar{C}$ contains S_T and $X_T(F)$ is closed in X, then $X_T(F) = X(T, F)$.*

Let S be closed in $\bar{C}$. A finite family of open sets $(G_1, \ldots, G_n; G_s)$ is an S-covering of the closed set $H \subset \bar{C}$ if $\bigcup_{i=1}^n G_i \cup G_s \supset H \cup S$ and $\bar{G}_i \cap S = \varnothing$ for $i = 1, \ldots, n$.

The next definition is an extension from the case of a bounded operator [1].

Definition 2. Suppose $T \in C(X)$ and the closed set S is contained in $\sigma(T)$. Call T strongly S-decomposable if for any open S-covering $(G_1, \ldots, G_n; G_s)$ of $\sigma(T)$ there are spectral maximal spaces of T, $X_i \subset D(T)$ $(i = 1, \ldots, n)$, $X_s \subset X$ such that:

(1) $\sigma(T \mid X_i) \subset \bar{G}_i$ $(i = 1, \ldots, n)$ and $\sigma(T \mid X_s) \subset \bar{G}_s$;
(2) for any spectral maximal space Y of T, $Y = Y \cap X_s + \sum_{i=1}^n (Y \cap X_i)$.

T is called S-decomposable if we postulate (2) only for $Y = X$.

The following results will be utilized later. For their proofs we refer to [4] (cf. also [1]).

LEMMA 2. *If $T \in C(X)$ is S-decomposable then $S_T \subset S$.*

LEMMA 3. *If $T \in C(X)$ is S-decomposable and F is a closed set containing S then $X_T(F) = X(T, F)$.*

2. The strong spectral residuum

DEFINITION 3. Let $T \in C(X)$ and $R = R(T)$ be the family of all closed sets S such that $S_T \subset S \subset \sigma(T)$ and T is strongly S-decomposable. If there is $S^* \in R$ such that S^* is contained in each $S \in R$, then S^* is called the strong spectral residuum of T.

Now we state the main result of this paper.

THEOREM 1. *The strong spectral residuum exists for each operator $T \in C(X)$.*

Proof. It will be divided into several steps.

(1) R is nonvoid, for $\sigma(T)$ clearly belongs to R. If $\{S_a;\ a \in A\}$ is a totally ordered subfamily of R with intersection $S_0 = \cap \{S_a;\ a \in A\}$ and $H \subset \bar{C}$ is a closed set disjoint from S_0 then, since $\bar{C}$ is compact, there is $a_0 \in A$ such that $H \cap S_{a_0}$ is void. Hence an S_0-covering of $\sigma(T)$ is an S_a-covering of $\sigma(T)$ for some $a \in A$. Since T is strongly S_a-decomposable, it is also strongly S_0-decomposable. By Zorn's lemma, there exists a minimal element in R.

(2) If T is S_1- and S_2-decomposable, $S = S_1 \cap S_2$, the set H is closed in $\bar{C}$ and is disjoint from S, then the subspace $X_{T,H}$ exists.

Indeed, if $S \subset F \subset \bar{C}$ then $F = \bigcap_{i=1}^{2} (F \cup S_i)$, hence

$$X_T(F) = \bigcap_{i=1}^{2} X_T(F \cup S_i).$$

If, in addition, F is closed, then $X_T(F \cup S_i)$ is closed in X, by Lemma 3, for T is S_i-decomposable ($i = 1, 2$). Thus $X_T(F)$ is closed in X and, by Lemma 1, $X_T(F) = X(T, F)$. Putting $F = H \cup S$, $Z = X_T(H \cup S)$, we obtain that $Z = X(T, H \cup S)$ is a Banach space. Thus the operator $V = T|Z$ is in $C(Z)$ and $\sigma(V) \subset H \cup S$. The sets $\sigma_H = \sigma(V) \cap H$ and $\sigma_S = \sigma(V) \cap S$ are disjoint spectral sets [5; p. 299] of V. If P_H, P_S denote the associated projections and Z_H, Z_S denote their ranges, then $Z = Z_H + Z_S$. [5; Theorems 5.7–A–B] yield that $Z_H \in I(T, H)$. Moreover, if ∞ belonged to σ_H, then we should have $S \subset C$, hence $S_i \subset C$ for $i = 1$ or $i = 2$. Since T is S_i-decomposable, this is easily seen to imply $T \in B(X)$. But then $V \in B(Z)$ would yield $\infty \notin \sigma(V)$, a contradiction. Thus σ_H is bounded, which implies $Z_H \in I_{T,H}$.

Further, if $Y \in I_{T,H}$ then $\sigma(T|Y) \subset H \cup S$ implies $Y \subset Z$. Hence $T|Y = V|Y$ and $\sigma(V|Y) \subset H$. If D is a Cauchy domain (bounded or not, cf. [5;

pp. 288–293]) such that $H \subset D$, $\bar{D} \subset S^c$, with positively oriented boundary $B(D)$, then for every $y \in Y$ we have

$$\begin{aligned} P_H y &= (2\pi i)^{-1} \int_{B(D)} (z - V)^{-1} y \, dz + cy \\ &= (2\pi i)^{-1} \int_{B(D)} (z - V \mid Y)^{-1} y \, dz + cy \\ &= y, \end{aligned}$$

where $c = 1$ if D is unbounded and $c = 0$ otherwise. Thus $Y \subset Z_H$, hence the subspace $X_{T,H} = Z_H$ exists.

(3) If the closed set $E \subset \bar{C}$ contains S_T and $X_T(E)$ is closed in X, then $\sigma(T \mid X_T(E)) \supset S_T$.

Denote by $\sigma_p^0(T)$ the set of all $z \in C$ such that there is a connected open neighborhood V of z and a $D(T)$-valued holomorphic function $f(v)$, not identically 0 and satisfying $(v - T)f(v) = 0$ on V. As in the case $T \in B(X)$, $\sigma_p^0(T)$ is open and its closure in $\bar{C}$ is S_T. If there is a point $z \in \bar{C}$ such that $z \in S_T \cap \rho(T \mid X_T(E))$, then there exists an open disk $G \subset C$ such that $G \subset \sigma_p^0(T) \cap \rho(T \mid X_T(E))$. Further, there is a holomorphic function $f(z)$, not identically 0 and satisfying $(z - T)f(z) \equiv 0$ on G. By [6; Proposition 2.2], $\sigma_T(f(z)) = \sigma_T(0) = S_T$. Thus there is $z_0 \in G$ such that $f(z_0) \neq 0$ and $f(z_0) \in X_T(E)$, which contradicts $z_0 \in \rho(T \mid X_T(E))$.

(4) If T is S-decomposable, $S \subset G \subset \bar{C}$ and G is open, then $\sigma(T \mid X_T(\bar{G})) \supset S$.

Indeed, by Lemma 3, $X_T(\bar{G})$ is closed in X, thus $S \supset S_T$ and (3) imply $\sigma(T \mid X_T(\bar{G})) \supset S_T$. Hence, if the statement of (4) is false, there is $z \in (S \backslash S_T) \cap \rho(T \mid X_T(\bar{G}))$. Thus there exists a neighborhood U of z such that $U \subset \Omega_T \cap \rho(T \mid X_T(\bar{G}))$, and for $u \in U$, $y \in X_T(\bar{G})$ we have

$$(u - T)(u - T \mid X_T(\bar{G}))^{-1} y = y.$$

Therefore $z \notin \sigma_T(y)$ for every $y \in X_T(\bar{G})$. Further, let (G_1, G) be an open S-covering of $\sigma(T)$. Since T is S-decomposable, for every $x \in X$ we have $x = x_1 + y$ where $x_1 \in X_{T,\bar{G}_1}$ and $y \in X_T(\bar{G})$. Hence $\gamma_T(x_1) \subset \bar{G}_1$ and $\sigma_T(x_1) \subset \bar{G}_1 \cup S_T$. Since $\sigma_T(x) \subset \sigma_T(x_1) \cup \sigma_T(y)$, we have $z \notin \sigma_T(x)$ for each $x \in X$, and $z \in S \subset \sigma(T)$. On the other hand, for any $T \in C(X)$ we have $\sigma(T) = \cup \{\sigma_T(x); x \in X\}$ (see [6; p. 513]), a contradiction, which proves (4).

(5) If T is S-decomposable, $S \subset G \subset \bar{C}$, G is open and Y is a spectral maximal space of T, then $W = Y \cap X_T(\bar{G})$ is a spectral maximal space of T.

Indeed, by Lemma 3, $X_T(\bar{G}) = X(T, \bar{G})$. Further, put $H = \sigma(T \mid X_T(\bar{G}))$, then (4) implies $S \subset H \subset \bar{G}$, and we have $X_T(\bar{G}) = X(T, H)$. If $F = \sigma(T \mid Y)$, then $Y = X(T, F)$. We shall show that $W = X(T, H \cap F)$.

It is clear that $W \in I(T)$. Suppose now that $z \in (H^c \cup F^c) \cap C$. If $(z - T \mid W)w = 0$ and $z \in H^c$, then $w = 0$, for $z - T$ is injective on all of $X(T, H)$. Similarly for $z \in F^c$, thus we have shown that $z - T \mid W$ is injective.

Choose an arbitrary $w \in W$ and assume that $z \in (H^c \cap F) \cap C$. Then there is $h \in X(T, H)$ such that $(z - T)h = w$, for $z - T$ is surjective on $X(T, H)$. Further, we can prove similarly as in [6; Proposition 3.1] that a spectral maximal space of T is a T-absorbing subspace of X, hence $z \in \sigma(T \mid Y)$ implies $h \in Y$, thus $h \in W$. In a similar way we obtain that $z - T \mid W$ is surjective also for $z \in (H \cap F^c) \cap C$. Finally, if $z \in H^c \cap F^c \cap C$, then there exist $h \in X(T, H)$ and $f \in X(T, F)$ such that $(z - T)h = w = (z - T)f$, hence $(z - T)(h - f) = 0$. Since $H \supset S$, the subspace $X_T(H \cup F) = X(T, H \cup F)$, by Lemma 3. The operator $z - T$ is injective on this subspace, and clearly $h - f \in X(T, H \cup F)$. Hence $h = f \in W$, thus we have shown that $z - T \mid W$ is surjective for $z \in (H^c \cup F^c) \cap C$.

Suppose now that $\infty \in H^c \cup F^c$, then one of the closed sets, say F, is bounded. Then $\sigma(T \mid Y) = F$ implies that $T \mid Y \in B(Y)$, hence $T \mid W \in B(W)$ and $\infty \in \rho(T \mid W)$. Thus we have proved that in any case $W \in I(T, H \cap F)$.

If a subspace U is in $I(T, H \cap F)$, then $\sigma(T \mid U) \subset H \cap F$, hence $U \subset X(T, H) \cap X(T, F) = W$. Thus $W = X(T, H \cap F)$ is a spectral maximal space of T.

(6) If $S_1, S_2 \in R$ and $S = S_1 \cap S_2$, then $S \in R$.

Indeed, suppose $(G_j\ (j = 1, \ldots, n), G_s)$ is an open S-covering of $\sigma(T)$. The sets $Z_k = S_k \backslash G_s$ $(k = 1, 2)$ are closed in $\bar{C}$ and they are disjoint, for $S \subset G_s$. Hence there are open sets H_k $(k = 1, 2)$ such that $H_k \supset Z_k$ and $\bar{H}_1 \cap \bar{H}_2 = \varnothing$. Put $G_{s_k} = G_s \cup H_k$, then $G_{s_k} \supset S_k \cup G_s$ $(k = 1, 2)$ and $\bar{G}_{s_1} \cap \bar{G}_{s_2} = \bar{G}_s$. There exist open sets B_k such that $S_k \subset B_k$, $\bar{B}_k \subset G_{s_k}$ $(k = 1, 2)$. For every G_j $(j = 1, \ldots, n)$ let $G_j^k = G_j \cap \bar{B}_k^c$; then $G_j^k \subset G_j$, $\overline{G_j^k} \cap S_k = \varnothing$ and $G_j^k \cup G_{s_k} \supset G_j$ $(k = 1, 2)$. Thus $(G_j^k\ (j = 1, \ldots, n), G_{s_k})$ is an open S_k-covering of $\sigma(T)$. Since T is strongly S_1-decomposable, for any spectral maximal subspace Y of T we have, by Lemma 3 and (2),

$$Y = Y \cap X_T(\bar{G}_{s_1}) + \sum_{j=1}^{n} (Y \cap X_{T,\overline{G_j^1}}).$$

According to (2), the spectral maximal spaces $X_{T,\bar{G}_j}$ exist for $j = 1, \ldots, n$, and

$$X_{T,G_j^1} \subset X_{T,\bar{G}_j}.$$

Hence

$$Y = Y \cap X_T(\bar{G}_{s_1}) + \sum_{j=1}^{n} (Y \cap X_{T,\bar{G}_j}).$$

By (5), $W = Y \cap X_T(\bar{G}_{s_1})$ is a spectral maximal space of T. Since T is strongly S_2-decomposable, we obtain

$$W = W \cap X_T(\bar{G}_{s_2}) + \sum_{j=1}^{n} (W \cap X_{T,G_j^2}) \subset Y \cap X_T(\bar{G}_s) + \sum_{j=1}^{n} (Y \cap X_{T,\bar{G}_j}),$$

for we have $\bigcap_{k=1}^{2} X_T(\bar{G}_{s_k}) = X_T(\bigcap_{k=1}^{2} \bar{G}_{s_k})$. Hence

$$Y = Y \cap X_T(\bar{G}_s) + \sum_{j=1}^{n} (Y \cap X_{T,\bar{G}_j});$$

thus T is strongly S-decomposable.

(7) According to (1), there exists a minimal element S_1 in R. If $S_2 \in R$, then (6) yields $S_1 \cap S_2 \in R$, hence $S_2 \supset S_1$. Thus S_1 is the strong spectral residuum of T, and the proof is complete.

Now we recall some definitions and results from [7]. $T \in C(X)$ is called S-residually decomposable ($S \subset \sigma(T)$ is a closed set) with localized spectrum if for every closed $F \subset \bar{C}$ with $F \cap S = \varnothing$ the subspace $X_{T,F}$ exists, for every S-covering $(G_1, \ldots, G_n, G_s)$ of $\sigma(T)$ there exist $X_1, \ldots, X_n \in I_T$ such that $\sigma(T \mid X_i) \subset \bar{G}_i$ $(i = 1, \ldots, n)$ and any $x \in X$ has a decomposition $x = x_1 + \cdots + x_n + x_s$ where $x_i \in X_i$, $\gamma_T(x_i) \subset \gamma_T(x)$ $(i = 1, \ldots, n)$ and $\sigma_T(x_s) \subset \bar{G}_s$. In this case we shall write $S \in Q(T) = Q$. If there is $S_0 \in Q$ such that $S \in Q$ implies $S_0 \subset S$, then S_0 is called the spectral residuum of T.

F.-H. Vasilescu proved [7; Theorem 3.1] that if $T \in C(X)$ has the single-valued extension property, and for any closed $F_1, F_2 \subset \bar{C}$ the property that $X_T(F_1)$, $X_T(F_2)$ are in $D(T)$ and are closed implies that $X_T(F_1 \cup F_2)$ is in $D(T)$ and is closed, then the spectral residuum of T exists.

THEOREM 2. *Suppose $T \in C(X)$ has the single-valued extension property and for any closed $F \subset \bar{C}$ the set $X_T(F)$ is closed in X. For any closed set $S \subset \sigma(T)$ then $S \in Q(T)$ if and only if $S \in R(T)$. Hence the spectral residuum of T exists and coincides with the strong spectral residuum of T.*

Proof. Under the given conditions Lemma 1 implies that for any closed $F \subset \bar{C}$ the set $X_T(F) = X(T, F)$ is a spectral maximal space of T. Assume first that $S \in Q(T)$, $(G_1, \ldots, G_n, G_s)$ is an open S-covering of $\sigma(T)$ and Y is a spectral maximal space of T. Setting $F = \sigma(T \mid Y)$ then $Y = X_T(F)$ and, in view of [7; Proposition 3.1], we may assume that the sets $G_1, \ldots, G_n$ are bounded. For any $y \in Y$, $y = y_1 + \cdots + y_n + y_s$ where $y_i \in X_T(\bar{G}_i)$ $(i = 1, \ldots, n, s)$, further $S_T = \varnothing$ implies that $\sigma_T(y_i) \subset \sigma_T(y) \subset F$ $(i = 1, \ldots, n)$, since T has localized spectrum. Hence also $\sigma_T(y_s) \subset F$. The spectral maximal spaces $X_i = X_T(\bar{G}_i)$ $(i = 1, \ldots, n, s)$ exist, $X_i \subset D(T)$ for $i = 1, \ldots, n$, by [7; Proposition 2.5], and $Y = Y \cap X_s + \sum_{i=1}^{n} (Y \cap X_i)$; thus $S \in R(T)$.

Conversely, if $S \in R(T)$, and F is closed in $\bar{C}$ with $F \cap S = \varnothing$, then $X(T, F) = X_T(F)$ exists. If F is bounded, then [7; Proposition 2.5] yields $X_T(F) \subset D(T)$. If F is unbounded, then S is bounded, which implies $T \in B(X)$. In either case, $X_{T,F} = X_T(F)$ exists. For any $x \in X$ the closed set $H = \sigma_T(x)$ defines the spectral maximal space $X_T(H)$. By assumption, for every open S-covering $(G_1, \ldots, G_n, G_s)$ of $\sigma(T)$,

$$X_T(H) = X_T(H \cap \bar{G}_s) + \sum_{i=1}^{n} X_T(H \cap \bar{G}_i).$$

Hence $x = x_1 + \cdots + x_n + x_s$, where $x_i \in X_T(\bar{G}_i)$, and $S_T = \varnothing$ implies $\gamma_T(x_i) \subset H = \gamma_T(x)$. Thus $S \in Q(T)$, and the proof is complete.

Added in proof. After submitting the manuscript, the author learned that E. Albrecht (Manuscripta Math., vol. 25 (1978), pp. 1–15) had shown that there is a decomposable operator for which the strong spectral residuum is not void.

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