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ON SATURATIONS OF EMBEDDED ANALYTIC RINGS

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Introduction

The theory of saturations of rings was started by Zariski in [15]. In this paper we present several results which hold for local rings of complex analytic hypersurfaces. The main results are in the following direction. It is known that if $f(x_1, \ldots, x_r)$ is a convergent power series, and its associated hypersurface V has an *isolated* singularity at the origin, then there is a number c such that for any convergent power series $g(x_1, \ldots, x_r)$ (with associated hypersurface V') satisfying $f \equiv g \bmod (x_1, \ldots, x_r)^{c+1}$), the local rings of V and V' at the origin are isomorphic. This is a special case of the main theorem of [4]. From this, it easily follows that an isolated singularity of a hypersurface is algebraic. (A more difficult theorem says that the assumption of codimension 1 can be dropped, see [1, Theorem 3.8].) These results are false for non-isolated singularities (look at $y^2 - x^2$ and $y^2 - x^2 + z^n$, see also [12], Section 14). Here we present some results which hold for non-isolated singularities. Our main result is Theorem (3.3). It says: given a hypersurface V of equation

$$f = y^n + A_1(x)y^{n-1} + \cdots + A_n(x), \quad x = (x_1, \ldots, x_r),$$

having branch locus $\Delta \subset \mathbf{C}^r$ (under the projection $(x, y) \to x$), there is a number c such that if V' is another hypersurface, of equation

$$g = y^n + B_1(x)y^{n-1} + \cdots + B_n(x),$$

satisfying (i) $A_i \equiv B_i \bmod (x)^{c+1}$, (ii) $\Delta' \subseteq \Delta$ (where Δ' is the branch locus of V' under $(x, y) \to x$), then the saturations of the respective local rings at the origin, with respect to the parameters $x_1, \ldots, x_r$, are isomorphic. By a Theorem of Zariski, they will be topologically equivalent (as embedded varieties). Using Artin's approximation lemma we present in Section 8 an application of this: If V is a hypersurface (defined in $\mathbf{C}^{r+1}$, near the origin) and for a suitable projection onto a hyperplane its branch locus is algebraic (at the origin), then there is an algebraic hypersurface V' (in $\mathbf{C}^{r+1}$, containing the origin), such that the saturations of the local rings of V and V' at the origin are isomorphic (see (8.4)). In particular, V and V' are topologically equivalent. Note that the assumptions are always satisfied for $r = 2$, i.e., an analytic embedded surface is topologically equivalent (as an embedded variety) to an algebraic surface. This answers

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affirmatively, in this case, a conjecture of R. Thom (see [11], page 228). For a different possible approach to solve this conjecture, see [10], Section 3.

The proof of Theorem (3.3) is presented in Sections 6 and 7. It is really very elementary, but there are many technical details involved, and we cannot avoid the use of a rather complicated notation. For these reasons, to help the reader, in this introduction we are going to discuss, informally, the ideas behind the proof.

Let V and V' be as before, $\pi\colon V \to \mathbf{C}^r$ (resp. $\pi'\colon V' \to \mathbf{C}^r$) the maps induced by the projection $(x, y) \to x$. The following is well known. Suppose we find a holomorphic function $\alpha\colon \pi^{-1}(U - \Delta) \to \mathbf{C}$ (U a suitable neighborhood of the origin in $\mathbf{C}^r$), such that:

(a) for $(x, y) \in \pi^{-1}(U - \Delta)$, $(x, \alpha(x, y)) \in V'$;

(b) the morphism $\phi\colon \pi^{-1}(U - \Delta) \to \pi'^{-1}(U - \Delta)$ given by $\phi(x, y) = (x, \alpha(x, y))$ is an isomorphism;

(c) if $x \in U - \Delta$, and $\pi^{-1}(x) = \{(x, y_1), \ldots, (x, y_n)\}$, then for $i \neq j$, we have $m < |y_i - y_j/\alpha(x, y_i) - \alpha(x, y_j)| < M$ where M, m are real constants (independent of x). Then, the saturations of V and V' at the origin will be isomorphic (cf. (3.6) and (3.8)). So, we try to find such a function α.

To fix ideas, suppose first that the branch locus Δ is the hyperplance $x_1 = 0$. In this special case, we may try this naive approach: consider a line l: $x_1 = t$, $x_i = \lambda_i t$, $i = 2, \ldots, r$. Above the line l, we have the curves

$$\pi^{-1}(l)\colon f(t, \lambda_1 t, \ldots, \lambda_r t, y) = 0 \quad \text{on} \quad V$$

and

$$\pi'^{-1}(l)\colon g(t, \lambda_1 t, \ldots, \lambda_r t, y) = 0 \quad \text{on} \quad V'.$$

If (using notations as before) $A_i \equiv B_i (\operatorname{mod} (x)^{c+1})$, c large enough, there is an isomorphism α_λ between these curves (near the origin). In fact, we need an isomorphism of $\mathbf{C}\{t\}$-algebras $\mathbf{C}\{t\}[y]/g(t, y) \approx \mathbf{C}\{t\}[y]/f(t, y) = A$, by Nakayama's lemma we need a root β of $g(t, y)$ in A, satisfying $\beta \equiv y_0 (\operatorname{mod} t . A)$, where y_0 is residue of y in A. But it is easy to see that the congruence above, with c large enough, implies $g(y_0) \in (g'(y_0))^2 (t . A)$ see, e.g. (6.12). Now we use Hensel's lemma to find β. The idea is to vary the line l. In this case (Δ a hyperplane) there is no problem, using $\Delta' \subset \Delta$, to extend the isomorphisms α to get a function $\alpha\colon \pi^{-1}(U - \Delta) \to \pi'^{-1}(U - \Delta)$ (U small enough) which satisfies (a) and (b). With more work (c) can also be proved. This simple approach does not seem to work when Δ is not smooth. In fact, if Δ is singular at the origin, it may happen that the variable line l intersects Δ at points arbitrarily close to the origin. It is not clear how to uniformly extend the isomorphisms α, to obtain a function α as before (consider, e.g., the case $r = 2$, Δ: $x_2^2 - x_1^3 = 0$). However, we may attempt a similar, but more complicated approach. Rather than using, as before, a system of lines, we try to "fill" a neighborhood of the origin in $\mathbf{C}^r$ (minus Δ) with several "families" of parametrized curves, say $\mathscr{L}_1, \ldots, \mathscr{L}_e$. The precise definitions are in (5.1–(5.5). Lying above each family $\mathscr{L}_b$ there will be, in V, several families of (parametrized) curves $\mathscr{N}_{b1}, \ldots, \mathscr{N}_{b, r(b)}$, here $r(b)$ is the number of irreducible components of $\pi^{-1}(C)$, where C is a typical curve of the family $\mathscr{L}_b$. To prove the existence of those families (cf. (5.7)–(5.14)), we apply a monoidal transform to a neighborhood U of the origin in $\mathbf{C}^r$ to get a morphism $q\colon F \to U$ (F a manifold), such that $q^{-1}(\Delta)$ has normal crossings. By using suitable coordinate neighborhoods on F, it is possible to construct families on F which, when "pushed down" to U, yield the desired families $\mathscr{L}_b$. The families $\mathscr{N}_{b\alpha}$ on V are "lifted" from $\mathscr{L}_b$, essentially by using analytic continuation. The details are in Section 5. Given V' as before (with c large enough), calling (as in Section 5) $N_{b\alpha}$ the union of the supports of the curves in $\mathscr{N}_{b\alpha}$ (minus the origin), it is not difficult to construct (using the parametrizations of the curves and Hensel's lemma) functions $\alpha_{b\alpha}\colon N_{b\alpha} \to C$, satisfying: $(x, a_{b\alpha}(x, y)) \in V'$ for $(x, y) \in N_{b\alpha}$. The problem is that $N_{b\alpha} \cap N_{b'\alpha'}$ might be non-empty, i.e. is not obvious that the functions $\alpha_{b\alpha}$ patch together to give a function α as we want. However, for a suitable choice of the families $\mathscr{L}_b$, it can be proved that the $\alpha_{b\alpha}$'s agree on intersections. This is done by carefully studying the convergence of the solution constructed in Hensel's lemma (by Newton's method, (cf. Section 4)), and a topological argument. In fact, we see that for $(x_0, y_0) \in V$ in a curve C in a family $\mathscr{N}_{b\alpha}$, and close enough to the origin, $\alpha_{b\alpha}(x_0, y_0)$ equals $\lim w_n$, where

$$w_0 = y_0 \quad \text{and} \quad w_{j+1} = w_j - g(x_0, w_j)/g'(x_0, w_j)$$

(the "Newton sequence"). Unfortunately, in order that the union of the supports of the curves (and the critical locus) form a neighborhood of the origin in $\mathbf{C}^r$, we cannot take these curves "arbitrarily small", consequently it does not seem possible to avoid that curves of different families meet at points where the functions $a_{b\alpha}$ are not defined by the Newton sequence (but rather by analytic continuation). To guarantee that the functions $a_{b\alpha}$ agree at these points, a special analysis is needed. In fact, using an argument involving covering spaces (and Lemma (6.8), which makes clear, in a crucial situation, what "close enough" means), we are reduced to points close to the origin, where $\alpha_{b\alpha}, \alpha_{b'\alpha'}$ are both given by the same formula (namely, the limit above). Unfortunately, in the topological argument we need our functions $\alpha_{b\alpha}$ to be defined in a set larger than $N_{b\alpha}$. To "control" the situation, we are forced to work with three systems of curves $\mathscr{L}_b^1 \supset \mathscr{L}_b^2 \supset \mathscr{L}_b^3$ (cf. (5.3)), rather than just $\mathscr{L}_b$. Moreover, several times we must "shrink" our families, i.e., rather than a single $\mathscr{L}_b^i$, we must consider collections $\mathscr{L}_b^i(\delta)$, where δ is the radius of the disk parametrizing each curve, and accordingly with the $\mathscr{N}_b$'s. In other words, we shall work with systems $\mathscr{L}_b = \mathscr{L}_b^3(\delta)$, $\mathscr{N}_{b\alpha} = \mathscr{N}_{b\alpha}^3(\rho)$, and eventually we'll fix δ, ρ "small enough". Once we proved that α is well defined, (a) and (b) easily follow. The details are in Section 6. Finally we prove that (after replacing, perhaps, c by a larger number, and shrinking U) (c) also holds. The main point of the proof of this is Lemma (7.3), which essentially says that, given any family $\mathscr{L}_b$, and a general curve C of this family, then

$$\lim_{x \to P} |(\alpha(x, y_p) - \alpha(x, y_q)/y_p - y_q) - 1| = 0,$$

where $x \in C$, P is the origin, and (x, y_p), (x, y_q) are in V. This is proved using parametrizations. With this, and the monoidal transform $q: F \to U$, which essentially reduces our situation to the case when Δ has normal crossings (hence we can use parametrizations of the variety, etc.), it is easy to prove that (c) holds. The details are in Section 7.

In Section 1 we review some basic, well known results on saturations, in Section 2 we present some other results, which allow us to give a more intrinsic formulation to our main result, using absolute saturations (cf. Corollary (3.5)).

Most of these results appear in the author's Thesis, presented at M.I.T. in 1970, written under the direction of Michael Artin. We wish to thank him for his important help during that period.

1. Review of results on saturations

(1.1) In this section we review some known results about saturation of local rings. We shall restrict ourselves, in general, to the case of hypersurfaces. These results are taken from [15], [7] and [8]. The review is done for the reader's convenience, especially as [7] has not been published yet.

(1.2) Throughout this paper we shall use the following terminology and notations.

(a) $\mathbf{C}$ denotes the complex numbers. If $a \in \mathbf{C}$, ε is a positive real number, write $B(a, \varepsilon) = \{z \in \mathbf{C} / |z - a| < \varepsilon\}$. If $a = 0$, we just write $B(\varepsilon)$, i.e., $B(\varepsilon) = \{z \in \mathbf{C} / |z| < \varepsilon\}$.

$\mathbf{C}\{x_1, \ldots, x_n\}$ denotes the ring of convergent power series in the variables x_1, $\ldots$, x_n. Sometimes we use vector notation for the variables, i.e., we write $(x) = (x_1, \ldots, x_n)$. An *analytic ring* is a ring of the form $\mathbf{C}\{x_1, \ldots, x_n\}/I$, I and ideal. We shall say that an analytic ring A is *embedded* if A has no non-trivial nilpotents, is equidimensional and

$$\dim (A) + 1 = \dim_{\mathbf{C}}(M/M^2)$$

where M is the maximal ideal of A.

(b) We use the definition of an analytic space given in [5]. If X is an analytic space, $\mathcal{O}_X$ denotes its structure sheaf. The stalks of $\mathcal{O}_X$ could have nilpotents (but most of the time we shall work with reduced spaces, i.e., with no nilpotents). If $x \in X$, $\mathcal{O}_{X,x}$ denotes the local ring of X at x. An analytic variety is a reduced analytic space. If X is an analytic space, a subvariety of X is a closed reduced subspace. A subvariety of an open $U \subset \mathbf{C}^n$ of pure codimension 1 at each point is called a hypersurface. Equivalently, a hypersurface is locally defined by the vanishing of a single holomorphic function (whose germs have no multiple factors). Embedded analytic rings are precisely local rings of hypersurfaces. Finally, if X is an analytic space,

$$\operatorname{Sing}(X) = \{x \in X / \mathcal{O}_{X,x} \text{ is not regular}\}.$$

(c) Let $f: X \to Y$ be a morphism of analytic spaces, W a subspace of Y corresponding to the sheaf of ideals $\mathcal{I}$. Then, $f^*(W)$ is the subspace of X

corresponding to $\mathscr{I}\mathcal{O}_X$; $f^{-1}(W) = f^*(W)_{\text{red}}$, the reduced subspace associated for $f^*(W)$.

(d) If π: $X \to Y$ is a finite (i.e., finite-to-one and proper) surjective morphism of analytic spaces, the set

$$\Delta = \{y \in Y // \text{at some } x \in \pi^{-1}(y),\ \pi \text{ is not a local isomorphism}\}$$

is closed. This set, with the reduced structure of analytic subspace of Y, is called the branch locus of π.

(e) If X, Y are analytic spaces, then

$$X \underset{f}{\overset{\sim}{\to}} Y,$$

when used without further explanations, will mean that f is an isomorphism from X onto Y.

(f) Let A be an analytic ring, $x_1, \ldots, x_n$ elements of M, the maximal ideal of A. They are a system of parameters if $n = \dim A$ and they generate an ideal which is M-primary. If A is embedded, we shall say (following [14], Section 2) that parameters $x_1, \ldots, x_n$ are a system of local parameters of A if there is some $y \in M$, such that $\{x_1, \ldots, x_n, y\}$ generate M.

(g) A morphism f: $X \to Y$ of analytic spaces is called strongly surjective if it is surjective and the induced linear mapping of tangent spaces df_x: $T_{X,x} \to T_{Y,y}$ is surjective, for all $x \in X$. For instance, if X is a hypersurface in $\mathbf{C}^{r+1}$ and Y is an r-dimensional manifold, this means that, in suitable coordinates, locally f looks like the morphism induced by the projection $(x_1, \ldots, x_r, x_{r+1}) \to (x_1, \ldots x_r)$.

(1.3) Let V be a hypersurface defined near the origin 0 of $\mathbf{C}^{n+1}$, $\mathfrak{O} = \mathcal{O}_{V,P}$, $x_1, \ldots, x_n$ a system of local parameters of $\mathfrak{O}$. We may assume that V has an equation

$$y^m + A_1(x)y^{m-1} + \cdots + A_m(x) = 0.$$

In [15], Zariski introduced a relation, called dominance, among the elements of $\bar{\mathfrak{O}}$. Let f, $g \in \mathfrak{O}$. They are germs of functions (which we still denote by f and g) defined on $U - \operatorname{Sing}(V)$ (where U is some neighborhood of 0 in V), holomorphic and bounded. Let $H = \{(x_1 \cdots x_n, y) \mid y = 0\}$, π: $U \to H$ be the natural projection and $\Delta \subset H$ its branch locus. We say that f dominates g (written $f \succ g$) if (after shrinking U, if necessary) there is a real number $M > 0$ such that for any pair of points $P_1, P_2 \in U$, satisfying $\pi(P_1) = \pi(P_2) \notin \Delta$, we have

$$|f(P_1) - f(P_2)/g(P_1) - g(P_2)| < M.$$

(1.4) Let V be as in (1.3). We keep the notations used there. The set $\{f \in \bar{\mathfrak{O}}/f \succ y\}$ is a local subring of $\bar{\mathfrak{O}}$, independent of the choice of y. This ring is called the saturation of $\mathfrak{O}$ with respect to the local parameters $(x) = (x_1, \ldots, x_n)$ and is denoted $\tilde{\mathfrak{O}}_x$ (or sometimes $\mathfrak{O}_x^{\sim}$). See [15] for details.

(1.5) There is another approach to the question of saturations, followed in [7] and [8] by Pham and Teissier. It is the so called theory of Lipschitz saturation. Let V be an analytic variety, $P \in V$, $\mathfrak{O} = \mathcal{O}_{V,P}$, $\bar{\mathfrak{O}}$ be integral closure of $\mathfrak{O}$. Then,

$$\{g \in \bar{\mathfrak{O}} \mid |g(P') - g(P'')| < K|P' - P''| \text{ for some real } K > 0, \text{ for all } P', P'' \text{ non-singular, near } P\}$$

(where we are identifying the germ g with a bounded holomorphic function defined at nonsingular points closed to P) is an analytic ring, called the (absolute) Lipschitz saturation of $\bar{\mathfrak{O}}$. We denote it by $\tilde{\mathfrak{O}}$ or $\mathfrak{O}^{\sim}$.

There is also a relative version: if $\mathfrak{O}$, P, V are as before, x is a system of parameters of $\mathfrak{O}$, H is the germ of linear space associated with the parameters (x), and $\pi\colon V \to H$ the germ of morphism induced by $\mathbf{C}\{x\} \subset \mathfrak{O}$, we define the Lipschitz saturation, relative to (x), to be the set of germs $g \in \bar{\mathfrak{O}}$ satisfying an inequality

$$|g(P') - g(P'')| < K|P' - P''|,$$

for P', P'' non-singular, near P, and such that $\pi(P') = \pi(P'')$, for some positive real number K (independent of the fibers of π). This is again an analytic ring, denoted $\tilde{\mathfrak{O}}_x$ or $(\mathfrak{O}_x)^{\sim}$. In [7] and [8] there are other equivalent definitions of this notion. For an algebraic treatment of the theory see [6].

(1.6) If $\mathfrak{O}$ is an embedded analytic ring and $(x) = (x_1, \ldots, x_r)$ is a system of local parameters (cf. (1.2)), then the Zariski and Lipschitz saturations of $\mathfrak{O}$, relative to (x), coincide (see [7], page 27). Thus, when we are in this situation, we can talk simply about (x)-saturation. This justifies the use of the same symbol to denote both saturations in the embedded case.

(1.7) A nice feature of Lipschitzian saturation is that it easily globalizes. Namely, if V is a variety, there is a coherent sheaf of algebras $\mathscr{A}$ such that for all P in V, $\mathscr{A}_P = (\mathcal{O}_{V,P})^{\sim}$. If $\pi\colon V \to H$ is a finite surjective morphism, with H nonsingular, then there is a coherent sheaf of algebras $\mathscr{A}(\pi)$ on V such that for every $P \in V$, if $Q = \pi(P)$ and $u_1, \ldots, u_n$ are regular parameters of $\mathcal{O}_{H,Q}$, then $\mathscr{A}(\pi)_0 = (\mathcal{O}_{V,P})_v^{\sim}$, where $v_i = \pi^*(u_i)$ (the pull-back of u_i), $i = 1, \ldots, r = \dim V$.

Hence, using the functor Specan (see [5], Exp. 19) we get in the absolute case a space $\tilde{V}$ and a finite morphism

$$p\colon \tilde{V} \to V, \tag{1.7.1}$$

in the relative case a space $V(\pi)$ and a definite morphism

$$p\colon \tilde{V}(\pi) \to V. \tag{1.7.2}$$

The space $\tilde{V}$ (resp. $\tilde{V}(\pi)$) is called the saturation (resp. saturation relative to π) of V. It can be shown (see [7], page 10) that these morphisms induce homeomorphisms of the underlying topological spaces.

(1.8) Given a morphism of analytic spaces $\pi\colon X \to H$ (with H nonsingular) and a nonsingular subvariety $W \subset X$, an analytic retraction of π onto W is a pair (r, q), where $r: X \to W$ and $q: H \to W$ are morphisms, $q\pi = r$ and $ri = id_W$ ($i: W \to X$ is the canonical injection).

(1.9) Let $\pi: X \to H$, W be as in (1.8), but we also assume that π is finite and surjective, (r, q) a retraction of π. We say that X is equisaturated along W, with respect to π and the given retraction, if there is a point $0 \in W$ and a commutative diagram

$$\begin{array}{ccc} \tilde{V}(\pi) & \xrightarrow{\sim} & \tilde{V}(\pi_0) \times W \\ {\scriptstyle p'}\big\downarrow & & \big\downarrow{\scriptstyle q' = p_0' \times id_W} \\ H & \xrightarrow{\sim} & H_0 \times W \end{array}$$

where we use the notations of (1.2) and (1.7), π_0 is the morphism from $V_0 = r^{-1}(0)$ to $H_0 = q^{-1}(0)$ induced by π, and p' (resp. p_0') is the composition of the canonical morphism of (1.7.2) with π (resp. π_0).

(1.10) We maintain the notations and assumptions of (1.9), but we also assume that X is a hypersurface. Let Δ be the branch locus of π. We say that π has trivial branch locus, with respect to the retraction (r, q), if there is a point $0 \in W$ and a commutative diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\sim} & (\Delta \cap H_0) \times W \\ \big\downarrow & & \big\downarrow{\scriptstyle j_0 \times id_W} \\ H & \underset{\alpha}{\xrightarrow{\sim}} & H_0 \times W \end{array}$$

where $j: \Delta \to H, j_0: \Delta \cap H_0 \to H_0$ are the inclusions; moreover we require that $\Delta \cap H_0$ be the branch locus of π_0 and ($p_2: H_0 \times W \to W$ being the second projection) $\pi r = P_2 \alpha\pi$.

(1.11) In [7], Theorem 5, it is proved the following result, that we shall use later. Let X be a hypersurface, H a manifold, $W \subset X$ a submanifold, $\pi: X \to H$ a finite, strongly surjective morphism, and (r, q) a retraction of π onto W, assume π has trivial branch locus (relative to (r, q)). Then, given $0 \in W$, there are neighborhoods U of 0, U' of $\pi(0)$ such that $\pi(U) = U'$, (r, q) induces a retraction (r', q') of $\pi': U \to U'$ (π' induced by π) onto $W \cap U$, and U is equisaturated along $W \cap U$ (with respect to π' and (r', q')).

2. Other basic theorems on saturations

(2.1) LEMMA. *Let V be an analytic hypersurface in $\mathbf{C}^{r+1}$, H a manifold $\pi: V \to H$ a finite, strongly surjective, morphism, $\Delta \subset H$ its branch locus, nonsingular at $\pi(P)$, $P \in V$. Then:* (a) *there is a unique component W of* Sing (V) *passing through P;* (b) *there are neighborhoods U of P (in V), U' of $\pi(P)$ (in H)*

such that $\pi(U) = U'$, and a retraction (r, q) of the induced morphism $\pi'\colon U \to U'$ onto $W \cap U$ such that π' has trivial branch locus relative to (r', q')).

Proof. Let $Q = \pi(P)$, $\mathfrak{O} = \mathcal{O}_{V,P}$, $\mathfrak{O}' = \mathcal{O}_{H,Q}$, $y_1, \ldots, y_r$ a regular system of parameters of $\mathfrak{O}'$. Then it is clear that $x_i = \pi^*(y_i)$, $i = 1, \ldots, r$ form an equisingular system of local parameters of $\mathfrak{O}$ (see [14], Definition 4.3). Then (a) follows from Theorem 4.5 of [14]. To show (b), note that the same theorem, near P, the morphism π induces an isomorphism ϕ of W with Δ. We may assume, after changing coordinates if necessary, that on some coordinate neighborhood U' of Q, Δ is given by $x_r = 0$. Now it is clear that, for U' small enough and an open U containing P suitably chosen, the morphisms

$$r' = \phi^{-1}p\pi\colon U \to U \cap W \quad \text{and} \quad q' = \phi^{-1}p\colon U' \to U \cap W,$$

where $p\colon U' \to U' \cap \Delta$ is given by $p(x_1, \ldots, x_r) = (x_1, \ldots, x_{r-1}, 0)$, have the required properties.

Let X be an analytic variety. A singular subvariety of X is a subvariety W of X such that X such that $W \subset \operatorname{Sing}(X)$. To verify that an irreducible subvariety $W \subset X$ is singular, it suffices to get a nonempty open set $U \subset W$, such that $U \subset \operatorname{Sing}(X)$.

(2.2) Proposition. *Let V be a hypersurface, H a manifold, $\pi\colon V \to H$ a finite, strongly surjective morphism, $p\colon \tilde{V}(\pi) \to V$ the saturation of V relative to π, W an irreducible subvariety of V. Then, W is a singular subvariety of V if and only if $p^{-1}(W)$ is a singular subvariety of $\tilde{V}(\pi)$.*

Proof. Let W be non-singular in V (i.e. $W \not\subset \operatorname{Sing}(V)$). Then for some open dense set U in W, every $P \in U$ is non-singular, a fortiori normal. Recall that p is a homeomorphism. Its inverse p^{-1} is holomorphic at normal points (since $\tilde{V}(\pi)$ is dominated by the normalization of V). Hence $p^{-1}(W)$ contains an open set of non-singular points, and is non-singular. Assume now W singular. To show the converse, it suffices to show that if W is an irreducible component of $\operatorname{Sing}(V)$, then $p^{-1}(W)$ is a singular subvariety of $\tilde{V}(\pi)$. Let W be an irreducible component of $\operatorname{Sing}(V)$. If $\dim W < r - 1$, then W contains a dense open set of normal points of V (because V is a hypersurface). Hence, p^{-1} is holomorphic there and $p^{-1}(W)$ is again singular. If the dimension of W is $r - 1$, consider $\pi(W) = W' \subset H$. It must be a component of the branch locus Δ of π. Take a non-singular point Q of W', let $P \in W$ such that $\pi(P) = Q$. We may apply Lemma (2.1), part (b) to get neighborhoods U, U' of P, Q respectively, and induced morphism $\pi'\colon U \to U'$ and a retraction (r, q) of π', such that π' has trivial branch locus. By (1.11), if $\pi_P\colon r^{-1}(P) \to q^{-1}(P)$ is the morphism induced by π, and U, U' are small enough, we have

$$\tilde{U}(\pi') \approx \tilde{C}(\pi_p) \times ((U \cap W) \quad \text{where} \quad C = r^{-1}(P)$$

and we use the notation of (1.7.2). But C is singular at P, hence by [15], Prop. 1.2, $\tilde{C}(\pi_P)$ is singular at $p^{-1}(P)$. It follows that $U \cap W$ is a singular subvariety of $\tilde{U}(\pi')$. Since there is a commutative diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "p"'] & & \tilde{U}(\pi') \arrow[dl, "p'"] \\ & U & \end{tikzcd}$$

(cf. (1.7), p' is the canonical morphism of (1.7.2)) it follows that $p^{-1}(W)$ is a singular subvariety of $\tilde{V}(\pi)$.

(2.3) In the embedded case, the important topological consequence of the notion of saturation is the following: let V (resp. V') be a hypersurface, defined in some neighborhood of P in $\mathbf{C}^{r+1}$ (resp. P' in $\mathbf{C}^{r+1}$), let $\mathfrak{O}$ (resp. $\mathfrak{O}'$) be its local ring at P (resp. P'), and $(x_1, \ldots, x_r) = x$ (resp. $(x'_1, \ldots, x'_r) = x'$) a system of local parameters of 0 (resp. 0'). Suppose there is an isomorphism ϕ: $\tilde{\mathfrak{O}}_x \to \tilde{\mathfrak{O}}'_{x'}$ such that $\phi(x_1) = x'_i$, $i = 1, \ldots, r$. Then, there is a neighborhood U of P (resp. U' of P') in $\mathbf{C}^{r+1}$ and a homeomorphism f: $U \to U'$ inducing a homeomorphism f': $U \cap V \to U' \cap V'$ (see [15], Theorem 6.1).

Actually, it is possible to prove this somewhat finer result:

(2.4) Theorem. *We keep the assumptions and notations of* (2.3). *Then, if* U, U' *are small enough, the homeomorphism* f: $U \cap V \to U' \cap V'$ *has the following properties:*

(a) $W \subset (U \cap V)$ *is an irreducible component of* Sing $(U \cap V)$ *if and only if* $f(W)$ *is an irreducible component of* Sing $(U' \cap V')$.

(b) *Let* S_1 (*resp.* S'_1) *be the union of the components of* Sing $(U \cap V)$ (*resp.* Sing $(U' \cap V')$) *of codimension* 1, *then* f *induces an isomorphism of analytic spaces:*

$$f_0\colon (U \cap V) - S_1 \to (U' \cap V') - S'_1.$$

Proof. It is clear that if U, U' are small enough we obtain a commutative diagram

$$(2.4.1)\qquad \begin{tikzcd} \tilde{V}(\pi) \arrow[r, "\phi"] \arrow[d, "p"] & \tilde{V}'(\pi) \arrow[d, "p'"] \\ V \arrow[r, "f'"] \arrow[d, "\pi"] & V' \arrow[d, "\pi'"] \\ H \arrow[r, "\alpha"] & H' \end{tikzcd}$$

where we used the notations of (1.7) and we wrote, to simplify, V rather than $V \cap U$, etc.; here all the arrows except f' are morphisms of complex spaces; H, H' are manifolds, π, π' are finite and strongly surjective morphisms, etc. and this geometric situation induces the algebraic one described in (2.3) in the obvious way. Moreover, f' is the mapping of our theorem (cf. [15], Theorem 6.1). Then (a) follows from the commutativity of (2.4.1) and Proposition 2.2. (b) is a consequence of the fact that p^{-1}, p'^{-1} are holomorphic at normal points and V is a hypersurface, hence non-singular in codimension 1.

(2.5) We recall the definition of transversal parameters, introduced (in the algebroid case) in [15], Definition 8.1. In analytic terms, this means the following. Let $\mathfrak{O}$ be a reduced, equidimensional analytic ring, $(x_1, \ldots, x_r) = (x)$ a system of parameters of $\mathfrak{O}$. The inclusion $\mathbf{C}\{x\} \subset \mathfrak{O}$ induces a surjective morphism $\pi\colon V \to H$ of germs of varieties (with H non-singular; V centered at $P \in V$, $\mathcal{O}_{V,P} = \mathfrak{O}$). Let W be an irreducible component of Sing (V) of codimension 1. Then $\pi(W) \subset H$ has again codimension 1, and is defined by the vanishing of some $\xi \in \mathbf{C}\{x\}$. Let $q\colon \bar{V} \to V$ be the normalization of V. Then $q^*(W)$ (cf. (1.2)c) is the union of several subspaces of $\bar{V}$, say $W_1, \ldots, W_l$, $l \geq 1$, of codimension 1. By the normality of $\bar{V}$, on an open dense of W_i $(i = 1, \ldots, l)$, $\bar{V}$ is non-singular, and W_i is defined, in local coordinates, by an equation $z_1^{h(i)} = 0$, where $h(i)$ is a well defined integer, depending only on W_i. The function $\xi' = \xi\pi q$ vanishes on $(W_i)_{\text{red}}$ with multiplicity $\geq h(i)$. If, for $i = 1, \ldots, t$, the order of ξ' along $(W_i)_{\text{red}}$ is exactly $h(i)$, we say that the parameters (x) are transversal with respect to W. We shall say that the parameters (x) are generic if they are transversal with respect to any component of codimension 1 of Sing V, passing through P.

(2.6) Given a variety X and $P \in X$, then $M_P(X)$ denotes set of irreducible components of codimension 1 of Sing (X) passing through P.

(2.7) Proposition. *Let $\mathfrak{O}$, $\mathfrak{O}'$ be analytic embedded rings*

$$(x) = (x_1, \ldots, x_r), \quad (x') = (x'_1, \ldots, x'_r)$$

systems of local parameters of $\mathfrak{O}$, $\mathfrak{O}'$ respectively; assume that (x) is generic for $\mathfrak{O}$. Suppose there is an isomorphism $\phi\colon (\tilde{\mathfrak{O}})_x \to (\tilde{\mathfrak{O}}')_{x'}$, such that $\phi(x_i) = x'_i$, $i = 1, \ldots, r$. Then (x') is a system of generic parameters of $\mathfrak{O}'$.

Proof. We may assume that the algebraic situation of the statement is induced (by taking germs) by the following diagram of complex spaces:

(2.7.1)
$$\begin{tikzcd}
\bar{V} \arrow[rr, "g", "\sim"'] \arrow[dr, "q_0"] \arrow[dd, "q"'] & & \bar{V}' \arrow[dr, "q'_0"] \arrow[dd, "q'"] & \\
& \tilde{V}(\pi) \arrow[rr, "\phi", "\sim"'] \arrow[dl, "p"] & & \tilde{V}'(\pi') \arrow[dl, "p'"] \\
V \arrow[rr, dashed, "f"] \arrow[dr, "\pi"'] & & V' \arrow[dl, "\pi'"] & \\
& H & &
\end{tikzcd}$$

Here V and V' are hypersurfaces, defined near the origin 0 of $\mathbf{C}^{r+1}$, $\mathfrak{O} = \mathcal{O}_{V,0}$, $\mathfrak{O}' = \mathcal{O}_{V',0}$, we use the notations of (1.7) and, as before, $q\colon \bar{V} \to V$, $q'\colon \bar{V} \to V'$ are the normalization morphisms. All the solid arrows are morphisms of analytic spaces, $f = p'gp^{-1}$ is a homeomorphism (but not necessarily everywhere holomorphic). Moreover, we may assume that $x_i' = x_i$, $i = 1, \ldots, r$ and that for any $W \in M_0(V)$ (resp. $W' \in M_0(V')$) (cf. (2.6)), $\pi(W)$ (resp. $\pi(W')$) is defined by a single global equation. Let $W' \in M_0(V')$, $Z = \pi'(W')$, Z defined by $\xi \in \Gamma(H, \mathcal{O}_H)$. Assume $q'^*(W)$ has components $W_1', \ldots, W_l'$. Given $i = 1, \ldots, l$, there is an open dense $G \subset \bar{V}' - \mathrm{Sing}\,(\bar{V}')$ such that each $x \in G \cap W_i'$ is the center of a coordinate neighborhood on which, in local coordinates, W_i' is given by $z_1^{h(i)} = 0$. We must show that $\xi\pi'q'$ has a zero of order $h(i)$ along $(W_i')_{\mathrm{red}}$. By Theorem (2.4), $W = p\phi^{-1}p'^{-1}(W') \in M_0(V)$ and $\pi(W) = Z$. By the commutativity of (2.7.1), $q^*(W)$ has l components $W_1, \ldots, W_l$, and $(W_i)_{\mathrm{red}} = g^{-1}((W_i')_{\mathrm{red}})$; for points of an open dense, W_i will be given locally by an equation $z_1^{k(i)} = 0$. As $x_1, \ldots, x_r$ are generic parameters for (x), $\xi\pi q$ has, along $(W_i)_{\mathrm{red}}$, a zero of multiplicity $k(i)$, hence by (2.7.1), since g is an isomorphism, $\xi\pi'q'$ has along $(W_i')_{\mathrm{red}}$ a zero of order $k(i)$. Thus we shall prove the statement if we prove $k(i) = h(i)$.

To see this, we shall find another interpretation for these numbers.

First note that the assumptions imply that, near the origin, the branch locus of π and π' coincide (see (3.2)). So we may also assume (since we are really interested in germs of varieties) that $\Delta =$ branch locus of $\pi =$ branch locus of π'. Let $P \in W$ be such that $\pi(P)$ is a non-singular point of Δ, $P' = f(P)$. By Lemma (2.1), we may find open neighborhoods U, U' of P and P' respectively, such that $\pi(U) = \pi'(U')$ is open in H, and the induced morphism $\pi_1\colon U \to \pi(U)$ (resp. $\pi'_{1'}\colon U \to \pi'(U')$) admits a retraction (r, s) (resp. (r', s')). Moreover, from the proof of (2.1) we see that the retractions can be taken compatibly with f (i.e., $fr = r'f$, etc.). By (1.11), if U, U' are small enough,

$$p^{-1}(U) \approx \tilde{U}(\pi_1) \approx C \times (U \cap W) \quad \text{where} \quad C = (rp)^{-1}(P)$$

(and C is isomorphic to the saturation of $r^{-1}(P)$, relative to the morphism $r^{-1}(P) \to s^{-1}(P)$ induced by π). Similarly,

$$p'^{-1}(U) \approx C' \times (U' \cap W) \quad \text{with} \quad C' = (r'p')^{-1}(P')^{\sim}.$$

If, moreover, P' is taken such that $q'^{-1}(P') \in G$, then $q^{-1}(U \cap W)$ is the disjoint union of $W_i^0 = W_i \cap q^{-1}(U)$, $i = 1, \ldots, l$ and W_i^0 is locally defined by $z_1^{k(i)} = 0$. Hence, C will have l branches $C_1, \ldots, C_l$ at $p^{-1}(P)$, and the multiplicity of C_i is $k(i)$. Similarly, C' has branches C_i', $i = 1, \ldots, l$, with multiplicity $h(i)$, $i = 1, \ldots, l$ (respectively) at $p'^{-1}(P)$. But the compatibility of f with the retractions implies $\phi(C_i) = C_i'$, $i = 1, \ldots, l$. Since ϕ is an isomorphism, it follows that the multiplicities of C_i and C_i' at $p^{-1}(P)$, $p'^{-1}(P')$ (respectively) coincide, i.e., $k(i) = h(i)$. This proves the theorem.

(2.8) We recall that Zariski showed ([15], Theorem 8.2) that if (x), (x') are generic local parameters of the embedded analytic ring $\mathfrak{O}$, then $\tilde{\mathfrak{O}}_x = \tilde{\mathfrak{O}}_{x'}$ (as subrings of the normalization $\bar{\mathfrak{O}}$). On the other hand, in [7], Theorem 2, page 15, it is shown that if (x) is generic, then $\tilde{\mathfrak{O}}_x = \tilde{\mathfrak{O}} =$ absolute Lipschitz saturation of $\mathfrak{O}$.

(2.9) COROLLARY. *We keep the assumptions and notations of* (2.7). *Then the absolute saturations of* $\mathfrak{O}$ *and* $\mathfrak{O}'$ *are isomorphic.*

Proof. This is a consequence of (2.7) and (2.8).

3. Statement of the main results

(3.1) In this section we shall be concerned with an analytic hypersurface V, defined near the origin of $\mathbf{C}^{r+1}$ (with coordinates $(x_1, \ldots, x_r, y)$) by an equation

$$(3.11) \qquad y^n + A_1(x)y^{n-1} + \cdots + A_n(x) = 0$$

where $A_i(x) \in \mathbf{C}\{x\}$, $A_i(0) = 0$.

(3.2) In the course of the proof of Theorem 6.1 of [15], Zariski proved the following. Let V' be another hypersurface, given near the origin 0 of $\mathbf{C}^{r+1}$ by an equation

$$(3.2.1) \qquad y^n + B_1(x)y^{n-1} + \cdots + B_n(x) = 0,$$

let π (resp. π') be the map from V (resp. V') into the hyperplane

$$H = \{(x_1, \ldots, x_r, y)/y = 0\}$$

induced by the projection, and let Δ (resp. Δ') the branch locus of π (resp. π'). Assume there is an isomorphism $\phi\colon \tilde{\mathfrak{O}}_x \to \tilde{\mathfrak{O}}'_x$, $\phi(x_i) = x_i$, where $\mathfrak{O} = \mathfrak{O}_{V,0}$, $\mathfrak{O}' = \mathcal{O}_{V',0}$. Then, $\Delta = \Delta'$ near the origin.

We shall prove in this paper the following result in the opposite direction (keeping the notation and conventions of (3.2)).

(3.3) THEOREM. *Let V be the hypersurface in $\mathbf{C}^{r+1}$ given by* (3.1.1). *Then there is a natural number c such that if V' is the hypersurface given by* (3.21) *and*

(a) $A_i \equiv B_i(\text{mod } (x)^{c+1})$, $(x) = (x_1, \ldots, x_r)$

(b) $\Delta' \subset \Delta$ *near the origin,*

then there is an isomorphism $\rho\colon \tilde{\mathfrak{O}}_x \to \tilde{\mathfrak{O}}'_x$ *such that* $\rho(x_i) = x_i$.

(3.4) *Remark.* When the origin 0 is a point of V of dimensionality type 1 (see [14], Definition 4.2), then the Theorem (3.3) is an easy consequence of the results of [4], [7] and [15], by using arguments similar to those of the proof of Theorem (2.4).

(3.5) COROLLARY. *Let V be as in* (3.3), *but now we assume the parameters induced by $x_1, \ldots, x_r$ to be generic. Then there is a natural number c such that if V' is given by* (3.2.1) *and satisfies* (a), (b) *of* (3.3) *then* $\tilde{\mathfrak{O}} = \tilde{\mathfrak{O}}'$, *where* $\tilde{\mathfrak{O}}$, $\tilde{\mathfrak{O}}'$ *denote the absolute saturations of the local rings of V and V′ at the origin, respectively.*

Proof. This is a direct consequence of (3.3) and (2.9).

Theorem (3.3) will be a direct consequence of Propositions (3.6) and (3.8) stated below (cf. (1.3), (1.4) and (1.6)).

(3.6) PROPOSITION. *Let V be as in Theorem* (3.3). *Then, there is a natural number c such that if V' is given by* (3.2.1) *and* (a), (b) *of Theorem* (3.3) *hold* (*for this c*), *then there is a neighborhood U of the origin of $\mathbf{C}^r$ such that:*

(i) *A_i, B_i converge on U, $i = 1, \ldots, n$ and $\Delta' \subset \Delta$* in *U;*

(ii) *there is an isomorphism $\phi: \pi^{-1}(U - \Delta) \cong \pi'^{-1}(U - \Delta)$ which commutes with the projections π, π'.*

(3.7) *Remark.* The fact that ϕ commutes with π, π' implies that ϕ has this form: if $Q \in \pi^{-1}(U - \Delta)$ has coordinates $(x_1, x_2, \ldots, x_r, y)$, then $\phi(Q) = (x_1, \ldots, x_r, \alpha(x_1, \ldots, x_r, y))$, where $\alpha: \pi^{-1}(U - \Delta) \to \mathbf{C}$ is a holomorphic function.

(3.8) PROPOSITION. *Let V and c be as in Proposition* 3.6. *Then, there is a number $c_0 \geq c$ such that if V' is given by* (3.2.1) *and $A_i \equiv B_i (\mathrm{mod}\ (x)^{c_0+1})$ then there is a neighborhood U_0 of the origin of $\mathbf{C}^r$ and real numbers $M > m > 0$, such that* (i), (ii) *of Proposition* (3.6) *hold and moreover:*

(iii) *if $x \in U_0 - \Delta$,* and *$\pi^{-1}(x) = \{(x, y_1), \ldots, (x, y_n)\}$ then for all pairs $i \neq j$ we have*

$$m < |(y_i - y_j)/(\alpha(x, y_i) - \alpha(x, y_j))| < M,$$

where α is the function of Remark (3.7).

We shall prove Propositions (3.6) and (3.7) in Sections 6 and 7, respectively. In Sections 4 and 5 we present some auxiliary results and concepts, necessary for our proof.

4. An analytic form of Hensel's lemma

(4.1) In this section we prove a "convergent analytic" version of Hensel's lemma. This is probably well known, but we include the proof since we could not find it in the literature, and moreover, we shall need later certain auxiliary functions that occur in it. In this section, if $x = (x_1, \ldots, x_r) \in \mathbf{C}^r$, and δ is a real number then $|x| < \delta$ means $|x_i| < \delta$, $i = 1, \ldots, r$. Let $A = \mathbf{C}\{x_1, \ldots, x_n\}$ (cf. (1.2)), M its maximal ideal; $\hat{A} = \mathbf{C}[[x_1, \ldots, x_r]]$ the ring of formal power series, M_0 its maximal ideal. Let

(4.1.1) $$f(y) = y^n + c_1 y + \cdots + c_n$$

be a polynomial with coefficients in M, and $a \in A$ such that

(4.1.2) $$f(a) = f'(a)^2 m, \quad , \in M.$$

Then we may form the Newton sequence

(4.1.3) $$a_0 = a; \quad a_{h+1} = a_h - f(a_h)[f'(a_h)]^{-1}.$$

It is well known that (4.13) is a sequence of elements of $\hat{A}$ which converges, in the M_0-adic topology to the unique solution $a_\infty \in \hat{A}$ of (4.1.1) satisfying $a_\infty \equiv a (\mathrm{mod}\, f'(a)M_0)$. But note that all terms of (4.1.3) are also in A, and we have:

(4.2) PROPOSITION. *We keep the notations and assumptions of* (4.1). *Then, there is a real number* $\delta > 0$, *such that:* (i) *the* c_i *and all terms of* (4.13) *are convergent on* $G = \{x \in \mathbf{C}^r / |x| < \delta\}$; (ii) *the sequence* (4.1.3) *converges uniformly on G to an holomorphic function* α, *satisfying*

(4.2.1) $\quad f(\alpha) = 0 \quad$ *and* $\quad \alpha = a + f'(a)mg, \quad$ *g holomorphic on G.*

Note that by the uniqueness in the formal case we must obtain $\alpha = a_\infty$.

(4.3) We prove (4.2) in (4.3), (4.4) and (4.5). Let

$$b = f'(a) = na^{n-1} + (n-1)c_1 a^{n-2} + \cdots + c_{n-r}.$$

It is immediate to check that a, b are in M. We shall see, by induction, that for any non-negative integer h,

(i) $f(a_h) = b^2 m^{h+1} d_h$, $d_h \in A$

(ii) $f'(a_h) = b\varepsilon_h$, ε_h a unit.

For $h = 0$ this is the hypothesis. Assume true for the index h, we shall check it for $h + 1$. We have, by induction,

$$a_{h+1} = a_h - f(a_h)[f'(a_h)]^{-1} = a_h - bm^{h+1} d_h \varepsilon_h^{-1}.$$

Write

(4.3.1) $$\beta_h = bm^{h+1} d_h \varepsilon_h^{-1} = f(a_h)[f'(a_h)]^{-1}.$$

By using Taylor's formula and (4.3.1), we get

(4.3.2) $$f(a_{h+1}) = f(a_h - \beta_h) = \beta_h^2 Q(a_h, \beta_h),$$

where

(4.3.3) $$Q(U, V) = \tfrac{1}{2} f''(U) + VQ_1(U, V)$$

is a polynomial with coefficients in $\mathbf{Q}[c_1, \ldots, c_r]$, independent of h. Hence

(4.3.4) $$f(a_{h+1}) = b^2 m^{h+2} (m^h d_h^2 \varepsilon_h^{-2} Q(a_h, \beta_h)$$

and we have the inductive step for (i), with

(4.3.5) $$d_{h+1} = m^h d_h^2 \varepsilon_h^{-2} Q(a_h, \beta_h).$$

To see (ii), note that by Taylor's formula,

(4.3.6) $$f'(a_{h+1}) = f'(a_h - \beta_h) = f'(a_h) - \beta_h R(a_h, \beta_n),$$

where $R(U, V)$ has coefficients in $\mathbf{Q}[c_1, \ldots, c_r]$ and is independent of h. Then, $f'(a_{h+1}) = b\varepsilon_h - bm^{h+1} d_h \varepsilon_h^{-1} R(a_h, \beta_h) = b\varepsilon_{h+1}$, where ε_{h+1} is a unit in A. This proves (ii).

(4.4) Now we shall show the existence of positive real numbers δ, $K > 1$ such that c_i, $i = 1, \ldots, n$, a_h, $d_h (h \geq 0)$, b and m are defined on $G = \{x \in \mathbf{C}^r / |x| < \delta\}$, $|a| < K/2$ on G, and

(a) $|d_h| \leq 1$, $h = 0, 1, \ldots$

(b) $|\varepsilon_h| > |\varepsilon_{h-1}| - (\frac{1}{2})^{h+1}$, $h = 1, 2, \ldots$

(c) $|a_h| < K(\frac{1}{2})^{h+1}$, $h = 1, 2, \ldots$

To define such numbers, consider the function of $r + 1$ variables $\frac{1}{2}f''(u) = F(x, u)$. As $F(0, 0) = 0$, we can choose $0 < K < 1$, $0 < \delta_1$, such that $|F(x, u)| < (\frac{1}{2})^2$ for $|u| < K$, $|x| < \delta_1$; moreover we may assume that a, $c_i (i = 1, \ldots, r)$ and m are defined for $|x| \leq \delta_1$ and that $|a| < K/2$. Consider the polynomials $Q_1(U, V) = \sum_{i,j} \beta_{ij}(x) U^i V^j$ and $R(U, V) = \sum_{t,s} \gamma_{ts}(x) U^t V^s$ of (4.3). Note that the functions $\beta_{ij}(x)$, γ_{ts} are defined for $|x| \leq \delta_1$. and put

$$M = \sum_{i,j} \sup_{|x| \leq \delta_1} |\beta_{ij}(x)|), \tag{4.4.1}$$

$$M' = \sum_{t,s} \sup_{|x| \leq \delta_1} |\gamma_{ts}(x)|). \tag{4.4.2}$$

Since $b(0) = m(0) = 0$, we find $\delta \leq \delta_1$ such that $|b| < \min((4M)^{-1}, (4M')^{-1}, (4K)^{-1}, 1)$ and $|m| < 1/8(M' + 1) < \frac{1}{2}$, we claim that K, δ satisfy the conditions stated above. We have $|m| < \frac{1}{2}$ and $|a| < K/2$ by the choice of the numbers. We prove (a), (b), (c) by induction. They are trivially true for $h = 0$, assume them true for $i \leq h$. Note that this induction hypothesis implies

(4.4.3) $|a_h| < K$,

(4.4.4) $|(\varepsilon_h)^{-1}| < 2$,

(4.4.5) $|\beta_h| < |b|(\frac{1}{2})^h < 1$.

In fact,

$$\begin{aligned} |\varepsilon_h| &> |\varepsilon_{h-1}| - (\tfrac{1}{2})^{h+1} \\ &> |\varepsilon_{h-2}| - (\tfrac{1}{2})^h - (\tfrac{1}{2})^{h+1} > \cdots > |\varepsilon_0| - (\tfrac{1}{2})^2 - \cdots - (\tfrac{1}{2})^{h+1} > 1 - \tfrac{1}{2} = \tfrac{1}{2}, \end{aligned}$$

which proves (4.4.4). Similarly,

$$|a_h| < |a_{h-1}| + K(\tfrac{1}{2})^{h+1} < \cdots < K(\tfrac{1}{2} + \cdots + (\tfrac{1}{2})^{h+1}) < K,$$

which proves (4.3.3). (4.4.5) is trivial.

To show (a), use the expression (4.3.5) of d_{h+1}; since $|m^h d_h \varepsilon_h^{-2}| < 2$ on G, we must show that $Q(a_h, \beta_h) \leq \frac{1}{2}$ on G. But $Q = F + VQ_1$ (cf. (4.4.3)); since on G, $|a_n| < K$ and $|\beta_h| < 1$ (and $\delta \leq \delta_1$), we get $|F(x, a_h)| < (\frac{1}{2})^2$ there. On the other hand,

$$|\beta_h Q_1(a_h, \beta_h)| \leq |\beta_h| \sum_{i,j} \sup |\beta_{i,j}(x)| \, |a_h|^i |\beta_h|^j < |b| M (\tfrac{1}{2})^h < (\tfrac{1}{2})^2.$$

Thus, $|Q| \leq \frac{1}{2}$ on G, and (a) follows.

We now prove (b).

$$|\varepsilon_{h+1}| = |\varepsilon_h - m^{h+1} d_h \varepsilon_h^{-1} R(a_h, \beta_h)| \geq |\varepsilon_h| - |m^{h+1} d_h \varepsilon_h^{-1} R(a_h, \beta_h)|.$$

But

$$\begin{aligned}|m|^{h+1}|d_h|\,|\varepsilon_h^{-1}|\,|R(a_h, \beta_h)| &< [(\tfrac{1}{8}(1+M'))^{h+1}2\sum \beta_{ij}(x)|a_h|^t|\beta_h|^s] \\ &< (\tfrac{1}{4}(1+M'))^{h+1}M' \\ &< (\tfrac{1}{2})^{h+2},\end{aligned}$$

and (b) is proved.

For (c), we have

$$|a_{h+1}| = |a_h - \beta_h| \leq |a_h| + |\beta_h| < |a_h| + |b|(\tfrac{1}{2})^h < |a_h| + k(\tfrac{1}{2})^{h+2},$$

and (c) is proved.

(4.5) With this, now we prove convergence. The convergence of $\{a_h\}$ is equivalent to the convergence of $\sum_{h=0}^{\infty}(a_h - a_{h-1})$. (Recall $a_{-1} = 0$.) We claim that this series normally converges. In fact,

$$|a_h - a_{h-1}| = |\beta_h| < (\tfrac{1}{2})^{h+1},$$

so that our series is majorized by $\sum_{h=1}^{\infty}(\frac{1}{2})^h$, which converges to 1. Hence, $\{a_h\}$ converges to a function α, holomorphic on G, which clearly is a root of (4.1.1). We shall see that α satisfies (4.3.1). In fact, we have $a_h = a_0 + \sum_{i=0}^{h-1}\beta_i$, and the series

$$\sum_{i=0}^{\infty}\beta_i = \sum_{h=0}^{\infty} bm^{h+1}d_h\varepsilon_h^{-1} = bm\sum_{h=0}^{\infty} m^h d_h \varepsilon_h^{-1},$$

where the last series is majorized by $\sum_{h=0}^{\infty}(\frac{1}{2})^h$. Passing to the limit, we get $\alpha - a_0 = bmg$, for some g holomorphic on G. Proposition (4.2) is proved.

(4.5) Note that the polynomials F, Q_1, R introduced in (4.3) can be formally defined for any polynomial (4.1.1), either if (4.1.2) holds for some a or not. They depend on f, and we shall denote them by $F(f)$, $Q_1(f)$, $R(f)$ respectively.

(4.6) Let us summarize the results about convergence of the sequence (4.1.3) (which will be important in (6.7), a crucial step of the proof of Theorem (3.3)). According to the proof of (4.2), given f and a as in (4.1), we can insure that all terms of (4.13) are defined on a polydisk $D \subset \mathbf{C}^r$, and the sequence converges there uniformly to the root α of (4.2.1) if the following hold.

(i) |the coefficients of f can be continuously extended to the closure of D;

(ii) there is a positive real number $K < 1$ such that $|F(x, u)| < \frac{1}{4}$, for $x \in D$, $|u| < K$ (here $F = (f)$ is the polynomial of (4.5));

(iii) $|b| < \min((4M)^{-1}, (4M')^{-1}, (4K)^{-1}, 1)$, where M, M' are defined in (4.4.1), (4.4.2) (with sup taken over $x \in D$), using $Q_1(f)$, $R(f)$ (cf. (4.5)). Note that M, M' exist by (i);

(iv) $|m| < \frac{1}{8}(1 + M')$ on D.

(v) $|a| < K/2$ on D.

5. Local families of curves

(5.1) DEFINITION. An m-dimensional local family of curves on a complex space X is a system $\mathcal{N} = (U, \phi, k, X)$ where $U = U_1 \times \cdots \times U_{m+1}$ is a polydisk in $\mathbf{C}^{m+1}$, $U_i = \{z_i / z_i - z_i^0 | < \varepsilon_i\}$, $i = 1, \ldots, m+1$, $\phi \colon U \to X$ is a holomorphic mapping, $1 \leq k \leq m+1$. These data are subject to the condition:

(L) ϕ induces an isomorphism of $U'_k = \{z \in U / z_k \neq z_k^0\}$ with an open of X, disjoint from $\phi(\{z \in U / z_k = z_k^0\})$.

Note that if $\phi(U'_k)$ is an open of X, disjoint from $\phi(\{Z \in U / Z_k = Z_k^0\})$ consisting of non-singular points, and $\phi \mid U'_k$ is injective, then (L) is automatically satisfied.

(5.2) The set $N = \phi(U'_k)$ (which is an open submanifold of X) is called the open carrier of the family, $N_0 = \phi(U)$ the carrier of the family. Given an m-tuple $a = (a_i)$, $a_i \in U_i$, $i \neq k$, the set $\Lambda(a) = \{x \in X / x = \phi(z), z_i = a_i \text{ for } i \neq i_k, z_k \in U_k\}$ is called the curve of the family corresponding to a, the point $x_0 = \phi(a')$, where $a'_i = a_i$ for $i \neq k$, $a'_k = z_k^0$, is called the origin of $\Lambda(a)$.

(5.3) Given two families (of dimension m) on X,

$$\mathcal{N} = (U_1 \times \cdots \times U_{m+1}, \phi, k, X) \quad \text{and} \quad \mathcal{N}' = (U'_1 \times \cdots \times U'_{m+1}, \phi', k', X),$$

we say that $\mathcal{N}$ is a subfamily of $\mathcal{N}'$ (written $\mathcal{N} \subset \mathcal{N}'$) if $k = k'$ and for all j, the disk U_j is contained in U'_j, with U_k and U'_k concentric, and ϕ is the restriction of ϕ' to U. If $U_j \neq U'$ for some $j \neq k$ we say that $\mathcal{N}$ is a strict subfamily of $\mathcal{N}'$, and we write $\mathcal{N} < \mathcal{N}'$.

(5.4) We say that a family $\mathcal{N}$ can be "extended to the boundary" if $\mathcal{N}$ is a subfamily of some family $\mathcal{N}'$, such that (with the notations of (5.3)) U_k is properly contained in U'_k. In all of our applications the families will be extensible to the boundary.

(5.5) It is not difficult to see that if X is reduced, then what we called the curves of the family are actually locally closed analytic subspaces of dimension 1 of X. Moreover, if Λ is the curve of $\mathcal{N}$ (the family of (5.1)) corresponding to

$$(c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_{m+1}),$$

then $\phi_c \colon B(\varepsilon_k) \to \Lambda$ given by

$$\phi(z) = (c_1, \ldots, c_{k-1}, z, c_{k+1}, \ldots, c_{k+1})$$

is a parametrization of Λ. Given this situation, when we say that "Λ is parametrized by $z_k \in B(\varepsilon_k)$" we mean that we use the map ϕ just described to parametrize the curve.

(5.6) In the sequel, we shall keep the notation of (5.1), i.e., local families will be denoted with capital script letters, the open carrier by the corresponding capital letter, the same letter with a subscript "zero" denotes the carrier. The expression "neighborhood of x" means a set containing an open set containing x, but the neighborhood itself could be not open. The closure of a set A is denoted cl (A). If the origin of all the curves of $\mathcal{N}$ is the same point $x_0 \in X$, we say that $\mathcal{N}$ is a "family through x_0" or "is composed of curves through x_0".

(5.7) PROPOSITION. *Let Δ be an analytic set in an open G of $\mathbf{C}^r$, $x_0 \in \Delta$, d a positive integer. Then, there is a positive integer e, a positive real number $\delta^{(0)}$ and for each pair (λ, δ), $\lambda = 1, \ldots, d$, $\delta \le \delta^{(0)}$ a collection of local families $\mathcal{L}_1^\lambda(\delta), \ldots, \mathcal{L}_e^\lambda(\delta)$ on G and through x_0 such that if $G^\lambda(\delta) = \bigcup_{b=1}^e L_b^\lambda(\delta)$ then:*

(a) *For all λ, $\delta \le \delta^{(0)}$, $G^\lambda(\delta) \subset G - \Delta$;*

(b) *$G^\lambda(\delta) \cup \Delta$ is a neighborhood of x_0, for $\lambda = 1, \ldots, d$, $\delta \le \delta^{(0)}$;*

(c) *If $\delta \le \delta' \le \delta^0$ then $\mathcal{L}_b^\lambda(\delta) \subset \mathcal{L}_b^\lambda(\delta')$, $\lambda = 1, \ldots, d$, $b = 1, \ldots, e$;*

(d) *$\mathcal{L}_b^\lambda(\delta) < \mathcal{L}_b^{\lambda-1}(\delta)$, $\lambda = 2, \ldots, d$, $\delta \le \delta^{(0)}$, $b = 1, \ldots, e$;*

(e) *Given any open neighborhood G' of x there is a $\delta \le \delta^0$ such that for all λ, $G^\lambda(\delta) \subset G'$.*

Moreover, these families are extensible to the boundary.

(5.8) We present the proof of (5.7) in (5.8)–(5.11). First we shall find an open neighborhood $T \subset G$ of x_0, a manifold F and a proper surjective morphism q: $F \to T$ such that the following hold.

(i) $E(\Delta) = q^{-1}(\Delta) = E_1 \cup \cdots \cup E_s$, where E_i is non-singular, codim $(E_i) = 1$, and the E_i have only normal crossings, i.e., the equation of $q^{-1}(\Delta)$ is given locally by $z_1 \cdots z_i = 0$, $i \le r$ for suitable local coordinates (z_i);

(ii) $E(x_0) = q^{-1}(x_0) = E_1 \cup \cdots \cup E_\mu$, $\mu \le s$.

To do this we use the following result of Hironaka (see [4], Lemma 7): let X be a complex space, $Y \subset X \times \mathbf{P}^n$, Y non-singular, π: $Y \to X$ induced by the first projection, J a coherent sheaf of ideals on Y, $x \in X$. Then, there is an open neighborhood U of x in X and a finite succession of monoidal transforms $\{f_i$: $V_{i+1} \to V_i\}$, with $V_0 = \pi^{-1}(U)$, centers D_i in V_i, with the following properties.

(i) D_i is non-singular, contained in supp $(\mathcal{O}_{V_i}/J_i)$, where $J_0 = J|_{V_0}$ and $J_{i+1} = f_i^{-1}(J_i)$, for all i;

(ii) J_r is of the form $\prod_{i=1}^\alpha \mathcal{P}_i^{r(i)}$, where $\mathcal{P}_i$ is the ideal of a nonsingular irreducible complex subspace E_i of V_r of codimension 1 and $\bigcup_{i=1}^\alpha E_i$ has only normal crossings.

We use this result in this way: first we blow up the point x_0 of G; we get a submanifold $T_1 \subset G \times \mathbf{P}^{r-1}$, and a morphism q_0: $T_1 \to G$ induced by the first projection. Let J be the ideal of $q_0^{-1}(\Delta)$. By Hironaka's result (applied to (T_1, J)) there is a neighborhood T of x_0, a non-singular space F and a morphism q with the required properties. Note that $q^{-1}(x_0) = E_1 \cup \cdots \cup E_\mu$ is compact, since q is proper.

(5.9) Next, consider the subsets S^i of $E(x_0) = q^{-1}(x_0)$ where for $i = 1, \ldots, r$,

$$S^i = \{y \in E(x_0)/\text{at } y \text{ there are at least } i \text{ components of } E(\Delta)\}.$$

We have $s^r \subset S^{r-1} \subset \cdots \subset S^1 = E(x_0)$, and these sets are closed in $E(x_0)$, therefore compact. Now we fix a covering of $E(x_0)$ by opens $M^0(i, j)$, $i = 1, \ldots, r$ and $j = 1, \ldots, \omega_i$, with the following properties.

(a) For all i, j there is a polydisk $H^0(i, j) = \prod_{k=1}^r H^0(i, j, k)$, where

$$H^0(i, j, k) = B(\varepsilon^0(i, j, k))$$

(cf. (1.2) (*a*), and an isomorphism $\theta(i, j)$: $H^0(i, j) \to M^0(i, j)$, such that $\theta(i, j)(x) \in S^i$, where c is the center of $H^0(i, j)$, $\theta(i, j)^{-1}(E(\Delta))$ corresponds to

$$\{z \in H^0/(i, j)/z_1 \cdots z_i = 0\}$$

and $\theta(i, j)^{-1}(E(x_0))$ to

$$\{z/z_1 \cdots z_{i'} = 0 \text{ for some } i' \leq i\}$$

(i' depends on i, j).

(b) $M^0(i, j) \cap S^l = \emptyset$ if $l > i$.

It is easy to construct such a covering by descending induction, using the facts that $E(x_0)$ is compact, F is non-singular and the components of $E(\Delta)$ have only normal crossings.

(5.10) Now, by elementary topological considerations we may choose, for each triple i, j, k, d numbers

$$\varepsilon^d(i, j, k) < \cdots < \varepsilon^1(i, j, k) < \varepsilon^0(i, j, k)$$

such that if, for $\lambda = 1, \ldots, d$ we write

$$H^\lambda(i, j, k) = B(\varepsilon^\lambda(i, j, k)), \quad H^\lambda(i, j) = \prod_{k=1}^{r} H^\lambda(i, j, k)$$

and

$$M^\lambda(i, j) = \theta(i, j)(H^\lambda(i, j)),$$

then $\{M^\lambda(i, j)\}$ is a covering of $E(x_0)$.

Now, for any triple (i, j, k), fix d collections of disks $D^\lambda(i, j, k; s)$, $\lambda = 1, \ldots, d$ and $s = 1, \ldots, s(i, j, k)$, such that:

(i) cl $(D^\lambda(i, j, k; s) \subset H^{\lambda-1}(i, j, k)$;

(ii) $0 \notin$ cl $(D^\lambda(i, j, k; s))$, $s > 4$;

(iii) $D^\lambda(i, j, k; s) \supset D^{\lambda-1}(i, j, k; s)$, $\lambda = 2, \ldots, d$;

(iv) $\bigcup_{s=1}^{s(i,j,k)} D\lambda(i, j, k; s) \supset H^\lambda(i, j, k) - \{0\}$.

For instance, let $D^\lambda(i, j, h, s)$, $\lambda = 1, \ldots, d$ be the disk of center $\gamma_s(\exp(\frac{1}{2}i\pi s)$ and radius a real number γ_s, $0 < \gamma_s < \frac{1}{2}\varepsilon^\lambda(i, j, k)$ (here i is the imaginary unit), for $s > 4$ the choice is clear.

(5.11) Let

$$J' = \{(i, j, k)/i = 1, \ldots, r, \quad j = 1, \ldots, \omega_i, \quad k = 1, \ldots, r\}$$

(cf. (5.9));

$$J = \{(i, j, k) \in J'/k \leq i'\}$$

(recall that on $H^0(i, j)$, the set $E(x_0)$ has an equation $z_1, \ldots, z_{i'} = 0$ (cf. (5.9));

$$I = \{(i, j, k; s_1, \ldots, s_{k-1}, s_{k+1}, \ldots, s_i)/(i, j, k) \in J;\ \ s_t = 1, \ldots, s(i, j, t), \text{ for all } t\}$$

(cf. (5.10)). Sometimes we just write $(i, j, k; (s))$ to denote an element of I. Let e be the cardinal of I. Fix a bijection τ: $\{1, \ldots, e\} \to I$; define v: $I \to J$ by $v(i, j, k; (s)) = (i, j, k)$.

Let $\delta(0) < \varepsilon^0(i, j, k)$, for all $(i, j, k) \in J$. If $\delta \leq \delta^{(0)}$, define (for $b = 1, \ldots, e$, $\lambda = 1, \ldots, d$) the polydisk $K_b^\lambda(\delta)$ as follows. Let

$$\tau(b) = (i, j, k; s_1, \ldots, \hat{s}_k, \ldots, s_i)$$

(where ^ means that the corresponding index is omitted). Then

(5.11.1) $K_b^\lambda(\delta) = \{z \in \mathbf{C}^r/z_t \in D^\lambda(i, j, t; s_t)$ for $t = 1, \ldots, \hat{k}, \ldots, i$, $|z_k| < \delta$, and $|z_t| < \varepsilon^\lambda(i, j, t)$, $t > i\}$,

We call θ_b to the restriction of $\theta(i, j)$ to $K_b^\lambda(\delta)$, $q_b = q\theta_b$, and we define the local families $\mathscr{L}_b^\lambda(\delta)$ by

$$\mathscr{L}_b^\lambda(\delta) = (K_b^\lambda(\delta), q_b, k, G). \tag{5.11.2}$$

We claim that the number $\delta^{(0)}$ and these families satisfy the conditions of (5.7). It is straightforward to check that these are local families through x_0, extensible to the boundary by construction. Next, note that the union of the open sets (of F)

$$P_b^\lambda(\delta_b) = \theta_b(K_b^\lambda(\delta_b)) \tag{5.11.3}$$

and $E(\Delta)$ form a neighborhood of $E(x_0)$ in F. In fact, if, for $\varepsilon < \varepsilon^d(i, j, k)$, $(i, j, k) \in J'$ we put

(5.11.4) $M^\lambda(i, j, k; \varepsilon) = \{y \in M^\lambda(i, j)/\theta(i, j)^{-1}(y) = (z_1, \ldots, z_n)$ with $|z_k| < \varepsilon\}$,

then clearly, for ε small enough, $E(x_0) \subset \cup\ M^\lambda(i, k, k; \varepsilon) \subset (\bigcup_{b=1}^e P_b^\lambda(\delta_b) \cup E)$. From this and the fact that q is proper, (b) follows. (a), (c) and (d) are obvious from the constructions, and (e) is an easy consequence of the compactness of $E(x_0)$. This proves Proposition (5.7).

(5.12) *Remark.* We can also define, as is clear from the proof given above, the following families of curves (on F) which will be useful later. If $1 \leq b \leq e$ and $\tau(b) = (i, j, k; (s))$, we set, for $\delta \leq \delta^{(0)}$,

$$\mathscr{P}_b^\lambda(\delta) = (K_b^\lambda(\delta), \theta_b, k, F). \tag{5.12.1}$$

In the following proposition we shall keep the notation of Proposition (5.7) (and its proof).

(5.13) PROPOSITION. *Let V be a purely r-dimensional analytic set, defined in the polydisk*

$$D = \{z/|z_i| < \xi_i,\ i = 1, \ldots, n\} \subset \mathbf{C}^n, \quad G = \{z/|z_i| < \xi_i,\ i = 1, \ldots, r\},$$

V containing the origin. Assume that the morphism $\pi\colon V \to G$ induced by the projection on the first r coordinates is finite; let $\Delta \subset G$ be the branch locus of π. Fix a positive integer d and a system of families $\{\mathscr{L}_b^\lambda(\delta)\}$ in G, $\lambda = 1, \ldots, d$, $b = 1, \ldots, e$, $\delta \leq \delta^{(0)}$, as in Proposition (5.7). *Then, there is a real number $\delta^{(1)} \leq \delta^{(0)}$ and for each triple (λ, δ, b), $\lambda = 1, \ldots, d$, $\delta \leq \delta^{(1)}$, $b = 1, \ldots, e$, a collection $\mathscr{N}_{b,1}^\lambda(\delta), \ldots, \mathscr{N}_{b,\alpha(b)}^\lambda(\delta)$ of local families of curves on V through the origin, satisfying the following.*

(a) *For all b, λ, δ, $\bigcup_{\alpha=1}^{\alpha(b)} N_{b\alpha}^\lambda(\delta) = \pi^{-1}(L_b^\lambda(\delta))$.*

(b) *For any pair (b, α), $b = 1, \ldots, e$, $\alpha = 1, \ldots, \alpha(b)$, there is a natural number $d(b, \alpha)$ such that (using* (5.11.1))

$$\mathscr{N}_{b\alpha}^\lambda(\delta) = (K_b^\lambda(\rho_\alpha), q_{b\alpha}, k, V), \quad \lambda = 1, \ldots, d,\ \delta \leq \delta^{(1)},$$

where k is the third coordinate of $\tau(b)$ (cf. (5.11)), *$(\rho_\alpha)^{d(b,\alpha)} = \delta$, and $q_{b\alpha}$ is a suitable function.*

(c) *For any λ, b, α, δ (as in* (b)) *there is a commutative diagram*

$$(5.13.1)\qquad \begin{array}{ccccc} K_b^\lambda(\rho_\alpha) & \xrightarrow{q_{b\alpha}} & (N_{b\alpha}^\lambda(\delta))_0 & \longrightarrow & \pi^{-1}(T) \\ \downarrow{\scriptstyle \pi_{b\alpha}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ K_b^\lambda(\delta) & \xrightarrow{q_b} & (L_{b\alpha}^\lambda(\delta))_0 & \longrightarrow & T \end{array},$$

where we use the notations of (b) *and* (5.2), *T is the open of Proposition* (5.7) *and $\pi_{b\alpha}(z_1, \ldots, z_r) = (z'_1, \ldots, z'_r)$, where $z'_i = z_i$ if $i \neq k$, $z'_k = z_k^{d(b,\alpha)}$*

(5.14) *Proof.* Let the families $\{\mathscr{L}_b^\lambda(\delta)\}$ be given. Take an index b, $b = 1, \ldots, e$, let $\tau(b) = (i, j, k; (s))$, and $\Lambda_b(c, \delta)$ denote the curve in $\mathscr{L}_b^\lambda(\delta)$ corresponding to the $(r-1)$-tuple $c = (c_1, \ldots, \hat{c}_k, \ldots, c_r)$ (cf. (5.2)). Let

$$(5.14.1)\qquad \beta_c\colon B(\delta) \to \{(z) \in K_b^\lambda(\delta)/z_t = c_t,\ t \neq k\}$$

be the natural isomorphism, $\beta_c(z_k) = (c_1, \ldots, z_k, \ldots, c_r)$. Let $a = (a_t)$, $t = 1, \ldots, r$, where a_t is the center of $D^\lambda(i, j, t, s_t)$ if $t \leq i$, $t \neq k$ (cf. (5.10)) and the center of $H^\lambda(, j, t)$ for $t > i$, $\Lambda' = \Lambda_b(a, \delta^{(0)})$. Consider $\pi^{-1}(\Lambda') = \Gamma$; since Λ' meets Δ only at the origin x_0, the morphism $\pi'\colon \Gamma \to \Lambda'$ induced by π ramifies only at x_0. Let $\alpha(b)$ be the number of irreducible components of Γ at x_0. We claim that for some $\delta^{(1)} \leq \delta^{(0)}$ we get commutative diagrams

$$\begin{array}{ccc} \{\eta \in C/|\eta| < \rho_\alpha^{(1)}\} & \xrightarrow{\theta_{b\alpha}} & \Gamma_\alpha \\ \downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle \pi} \\ B(\delta^{(1)}) & \xrightarrow{q'_b} & \Lambda = \Lambda_b(a, \delta^{(1)}) \end{array}$$

where $q'_b = q_b \beta_a$ (cf. (5.11.12) and (5.14.1)) and the following hold:

(1) Γ_α, $\alpha = 1, \ldots, \alpha(b)$ are the irreducible components of $\pi^{-1}(\Lambda)$.

(2) $\theta_{b\alpha}$ is holomorphic, and an isomorphism for $\eta \neq 0$ (i.e., a parametrization of the curve Γ_α).

(3) $\pi_\alpha(\eta) = \eta^{d(b,\alpha)}$, for some integer $d(b, \alpha)$ such that $(\rho^{(1)}\alpha^{d(b,\alpha)}) = \delta^{(1)}$.
In fact, let $\{\Gamma'_\alpha\}$ be the irreducible components of Γ and C_α the fiber product

$$\begin{tikzcd} C_\alpha \arrow[r, "p_2"] \arrow[d, "p_1"'] & \Gamma'_\alpha \arrow[d, "\pi"] \\ B(\delta^{(0)}) \arrow[r, "q'_b"'] & \Lambda' \end{tikzcd}$$

(where $q'_b = q_b \beta_a$, cf. (5.14.2)). Then, C_α is again a curve, it is irreducible at x_0, and p_2 is an isomorphism outside the origin. Consider a parametrization

$$B(\rho^{(1)}_\alpha) \xrightarrow{f} C_\alpha.$$

Clearly we may choose it in such a way that $p_1 f(\eta) = \eta^{d(b,\alpha)}$, for all $\eta \in B(\rho^{(1)}_\alpha)$, where $d(b, \alpha)$ is the degree of the covering map $\Gamma_\alpha - \{x_0\} \to \Lambda - \{x_0\}$. Now it is obvious that our claim about (5.14.1) holds, with $\delta^{(1)} = (\rho^{(1)}_\alpha)^{d(b,\alpha)}$.

It is also clear that the diagrams (5.14.1) induce similar diagrams, with $\delta^{(1)}$ replaced by $\delta < \delta^{(1)}$ (and ρ_α accordingly).

Now, we construct the family $\mathcal{N}^\lambda_{b\alpha}(\delta)$. If $b = 1, \ldots, e, \alpha = 1, \ldots, \alpha(b), \delta \leq \delta^{(1)}$,

$$\mathcal{N}^\lambda_{b\alpha}(\delta) = (K^\lambda_b(\rho_\alpha), k, q_{b\alpha}, V) \tag{5.14.2}$$

where $K^\lambda_b(\rho_\alpha)$ is as in (5.12), $(\rho_\alpha)^{d(b,\alpha)} = \delta$, and $q_{b\alpha}$ is defined as follows. To compute $q_{b\alpha}(z_1, \ldots, z_r)$, with $z_k \neq 0$, consider $Q = \theta_{b\alpha}(z_k) \in \Gamma_\alpha$ (cf. (5.14.1)). Let

$$D = \{(w) \in K^\lambda_b(\delta)/w_k = z_k\}, \quad D = q(D).$$

Since D' is contractile, the lifting from Q of any real arc γ joining $\pi(Q)$ and $q(z')$, where $z'_t = a_t$, $t \neq k$, $z'_k = (z_k)^{d(b,\alpha)}$, will end at a well defined point $q_{b\alpha}(z_1, \ldots, z_r) \in V$. We complete the definition of $q_{b\alpha}$ by putting $q_{b\alpha}(z) = x_0$ if $z_k = 0$. It is readily checked that, using the coordinates of $\mathbf{C}^n$

$$q_{b\alpha}(z) = (x_1, \ldots, x_r, y_1, \ldots, y_v),$$

$r + v = n$, where $(x) = q_b(z')$ and $z'_t = z_t$ for $t \neq k$, $z'^{d(b,\alpha)}_k$, and $y_1, \ldots, y_v$ are holomorphic functions of $z_1, \ldots, z_r$. To see the latter statement one uses the theory of analytic continuation and Riemann's extension theorem. Proposition (5.13) is proved.

6. Proof of Proposition (3.6)

(6.1) In this section we prove Proposition (3.6). The details of the proof are rather technical; we remind the reader that an informal sketch of it can be seen in the introduction. We are given a hypersurface V, defined near the origin P of

$\mathbf{C}^{r+1}$ (with coordinates $x_1, \ldots, x_r, y$) by the equation (3.1.1). We identify $\mathbf{C}^r$ with $\{(x, y) \in \mathbf{C}^{r+1}/y = 0\}$. Fix a polydisk G in $\mathbf{C}^r$, centered at P, such that $A_i(x)$, $i = 1, \ldots, n$ (cf. 3.1.1)) is defined on G. The branch locus Δ of π, the projection of V on G parallel to the y-axis, is a hypersurface in G. We apply Proposition (5.7) to this situation; with $d = 3$ (for reasons which will be clear in (6.9) and (6.10)). Thus, we get, for $\lambda = 1, 2, 3$; $b = 1, \ldots, e$; $\delta \leq \delta^{(0)}$ local families $\mathscr{L}_b^\lambda(\delta)$ on G through P, with the properties (a) to (e) of (5.7).

Next we apply Proposition (5.13), taking our V as the variety of (5.13) and as the families $\{\mathscr{L}_b^\lambda\}$ of (5.13) the ones we just took. We get families $\mathscr{N}_{b\alpha}^\lambda(\delta)$ on V, $\delta \leq \delta^{(1)} \leq \delta^{(0)}$, satisfying properties (a), (b) of (5.13). Moreover, we shall use families constructed as those of the proofs of propositions (5.7) and (5.12). In particular, we shall use the notations introduced during those proofs, as well as in the remainder of Section 5.

(6.2) We shall determine the number c of (3.6). Let δ be the discriminant of the polynomial $f(y)$ (cf. (3.1.1)). Consider the morphism $q\colon F \to T \subset G$ (cf. (5.8)) and the coordinate neighborhoods $M^0(i, j)$ of (5.9). Then, (keeping the notations of (5.9)) for each pair i, j where $i = 1, \ldots, r$ and $j = j = 1, \ldots, \omega_i$, the composition $\delta_{ij} = \delta q \theta(i, j)$ is an holomorphic function of $z \in H^0(i, j)$. It vanishes on the subspace of $H^0(i, j)$ defined by $z_1 \cdots z_i = 0$. By the Nullstellensatz, there is a number $c(i, j)$ such that

$$(6.2.1) \qquad (z_1 \cdots z_i)^{c(i, j)} = \delta_{ij} h_{ij},$$

where h_{ij} is an holomorphic function (and we may assume it is defined on $H^0(i, j)$). We set

$$(6.2.2) \qquad c = \max\{2c(i, j_i)\}, \quad i = 1, \ldots, r; \quad j = 1, \ldots, \omega_i.$$

We claim that this number c satisfies the conditions of (3.6).

(6.3) Now consider another hypersurface V', of equation (3.2.1), satisfying (i), (ii) of (3.6) with the number c defined in (6.2). We shall use the notations of (3.6). We want to define an isomorphism $\phi\colon \pi^{-1}(U - \Delta) \to \pi'^{-1}(U - \Delta)$, ($U$ a suitable neighborhood of P), commuting with the projections π, π'. First of all, choose an open neighborhood U_1 of P in $\mathbf{C}^r$ such that $U_1 \subset T$ and the series B_i, $i = 1, \ldots, n$, converges there. Using property (e) of (5.7), we find a positive number $\delta^{(2)} \leq \delta^{(1)}$ such that $G^\lambda(\delta) \subset U_1$ for $\delta \leq \delta^{(2)}$.

(6.4) We are going to define holomorphic mappings

$$(6.4.1) \qquad a_{b\alpha}^\lambda(\delta)\colon N_{b\alpha}^\lambda(\delta) \to \mathbf{C}$$

for $\lambda = 1, 2, 3$; $b = 1, \ldots, e$; for b fixed $\alpha = 1, \ldots, \alpha(b)$; $\delta \leq \delta^{(2)}$ (cf. (5.2), (5.13)). These maps will be compatible, in the sense that if $\delta' \leq \delta$, $a_{b\alpha}^\lambda(\delta')$ is obtained from $a_{b\alpha}^\lambda(\delta)$ by restriction; and they will satisfy the following:

$$(6.4.2) \quad \text{If } (x, y) \in N_{b\alpha}(\delta) \subset V, \quad \text{then} \quad (x, a_{b\alpha}^\lambda(\delta)(x, y)) \in V'.$$

In (6.10) we shall see that there is a real number $\delta^{(4)} \leq \delta^{(2)}$, such that for $\delta \leq \delta^{(4)}$, the maps $a_{b\alpha}^{3}(\delta)$ agree on the intersections and thus we obtain (by (6.4.2)) a morphism $\phi\colon \pi^{-1}(U - \Delta) \to \pi'^{-1}(U - \Delta)$ (for some small open U) as desired. To define these maps we shall use Lemma (6.5) given below. To clarify its statement, consider the following commutative diagram (cf. (5.13.1))

$$\text{(6.4.3)} \qquad \begin{tikzcd} K_{b\alpha} \arrow[r, "q_{b\alpha}"] \arrow[d, "\pi_{b\alpha}"] & (N_{b\alpha})_0 \arrow[r] \arrow[d] & V \arrow[dr] & & V' \arrow[dl] \\ K_b \arrow[r, "q_b"] & (L_b)_0 \arrow[rr] & & U_1 & \end{tikzcd}$$

where, to simplify, we wrote $V = \pi^{-1}(U_1)$, $V' = \pi'^{-1}(U_1)$, $L_b = L'_{b\alpha}(\delta^{(2)})$, $K_{b\alpha} = K'_{b\alpha}(\rho_\alpha)(\rho_\alpha^{d(b\alpha)} = \delta^{(2)})$, $K_b = K_b^1(\delta^{(2)})$, and we use the notations of (5.2) and (5.13). For any open U in $\mathbf{C}^n$ let $\Gamma(U) = \Gamma(U, \mathcal{O}_U)$ (cf. (1.2)). Let $B_{b\alpha i} = B_i q_b \pi_{b\alpha} \in \Gamma(K_{b\alpha})$, $B_{bi} = q_b B_i \in \Gamma(K_b)$. Write

$$\text{(6.4.4)} \qquad g_b = Y^n + \sum_{i=1}^{n} B_{bi} Y^{n-i} \in \Gamma(K_b)[Y]$$

$$\text{(6.4.5)} \qquad g_{b\alpha} = Y^n + \sum_{i=1}^{n} B_{b\alpha i} Y^{n-i} \in \Gamma(K_{b\alpha})[Y].$$

Then, we have:

(6.5) LEMMA. *Consider the polynomial $g_{b\alpha}$ (see (6.4.5)), let*

$$\tau(b) = (i, j, k; (s))$$

(cf. (5.11), $y\colon V \to \mathbf{C}$ the map induced by the projection $(x, y) \to y$, $w_{b\alpha} = yq_{b\alpha}$. Then there is a unique root $a_{b\alpha} \in \Gamma(K_{b\alpha})$ of $g_{b\alpha}$ with the following property: there is a neighborhood $\mathcal{G}_0$ of $X_k = \{z \in K_{b\alpha}/z_k = 0\}$ and $m_0 \in \Gamma(\mathcal{G}_0)$, vanishing on X_k, such that

$$\text{(6.5.1)} \qquad a_{b\alpha} - w_{b\alpha} = [g'_{b\alpha}(a_{b\alpha})]m_0 \quad \text{on} \quad \mathcal{G}_0.$$

Moreover, for points $z \in \mathcal{G}_0$,

$$\text{(6.5.2)} \qquad a_{b\alpha}(z) = \lim_{j \to \infty} w_j$$

where $w_0 = w_{b\alpha}(z)$ and $w_{j+1} = w_j - g(x, w_j)/g'(x, w_j)$ with $x = q_b \pi_{b\alpha}(z)$.

In order not to break the main proof we postpone the proof of (6.5) to (6.12).

(6.6) Accepting the lemma, define

$$\text{(6.6.1)} \qquad a_{b\alpha}^{1}(\delta^{(2)}) = a_{b\alpha} q_{b\alpha}^{-1}\colon N_{b\alpha}^{1}(\delta^{(2)}) \to \mathbf{C}.$$

By taking the suitable restrictions we define $a_{b\alpha}^{\lambda}(\delta)$, $\lambda = 1, 2, 3$, $\delta \leq \delta^{(2)}$. It is clear that these maps have the property (6.4.2).

Note the following facts, which will be useful later. Using the notations of (5.11.3) and (5.11.4), for δ small enough,

$$M^\lambda(i, j, k; \delta) - E \subset \bigcup_{b \in S_{ijk}} P_b^\lambda(\delta^{(2)}) \subset F \tag{6.6.2}$$

(where $S_{ijk} = \{b \in I/v\tau(b) = (i, j, k)\}$) for $\lambda = 1, 2, 3$, $(i, j, k \in J'$ (cf. (5.11)). Given $\delta \leq \delta^{(2)}$, define, for λ, i, j, k as in (6.6.2),

$$R^\lambda(i, j, k; \delta) = \{Q \in V/\pi(Q) \in q(M^\lambda(i, j, k; \delta)) - \Delta\}. \tag{6.6.3}$$

Then, from Lemma (6.5), it immediately follows that the maps $a_{b\alpha}^\lambda(\delta)$, $b \in S_{ijk}$, patch together to give a well defined holomorphic map

$$a^\lambda(i, j, k; \delta)\colon R^\lambda(i, j, k; \delta) \to \mathbf{C}, \quad \lambda = 1, 2, 3. \tag{6.6.4}$$

Moreover, if Λ is a curve in one of the systems $\mathscr{L}_b^\lambda(b \in S_{ijk})$, then there is some real $\rho > 0$ (depending on Λ), such that if $Q = (x, y) \in V$ and $x \in \Lambda$ corresponds to a value z_k of the parameter of Λ, with $0 < |z_k| < \rho$ (cf. (5.5)) then

$$a_{b\alpha}^\lambda(\delta)(Q) = \lim w_j \tag{6.6.5}$$

where $w_0 = y$ and $w_{j+1} = w_j - g(x, w_j)/g'(x, w_j)$.

(6.7) Next we want to show that the maps $a_{b\alpha}^3(\delta)$ agree on the intersections of their domains, if δ is small enough. To see that, first we define a number $\delta^{(3)}$ as follows.

Consider the polynomials $F(g)(U)$, $Q_1(g)(U, V)$, $R(g)(U, V)$ associated to $g(y)$, introduced in (4.5). Since $B_i(0) = 0$, $i = 1, \ldots, n$ for some $0 < K < 1$, $\delta_1 > 0$ we have $|F(g)(x, u)| < 1/4$ for $|x| < \delta_1$, $|u| < K$. Now we compute the numbers M, M' of (4.4.1), (4.4.2), with our data and δ_1. Next, for some $\delta_2 \leq \delta_1$, $|g'(x, y)| < \min((4M)^{-1}, (4M')^{-1}, (4K)^{-1}, 1)$ for $|x| < \delta_2$, $|y| < \delta_2$. Now, by continuity of the algebroid functions, we may choose $\delta_3 \leq \delta_2$ such that $|x| < \delta_3$ implies that all roots $y_i(x)$ of $f(x, y) = 0$ satisfy $|y_i(x)| < \min(\delta_2, K/2)$. By condition (e) of (5.7) we can get a number $\delta^{(3)}$ such that $G^\lambda(\delta^{(3)})$ is contained in

$$U_2 = \{z \in \mathbf{C}^r/|z_t| < \delta_3, t = 1, \ldots, r\}. \tag{6.7.1}$$

Next, we would like to analyze the following situation. Consider one of the coordinate opens $M^1(i, j, k, \delta)$ in F (with $\delta \leq \delta^{(3)}$). Assume $E(P)$ is defined by $z_1 \cdot \ldots \cdot z_{i'} = 0$, $i \geq 2$. For each k, $1 \leq k \leq i'$, we may consider the curves Λ_a: $x_t = a_t$, $t \neq k$, for constants $a_1, \ldots, \hat{a}_k, \ldots, a_r$. We know that for $|x_k|$ small, if $Q = (x, y) \in V$ and $x \in q(\Lambda_a) - \Delta$, then $a^1(i, j, k)(Q)$ is given by the "Newton sequence" (6.5.2). How small should $|x_k|$ be to achieve this? The following lemma says that if a coordinate a_v, $1 \leq v \leq i$, $v \neq k$, is close to zero, then we take $|x_k|$ "not too small". Precisely, we have:

(6.8) Lemma. *With notations as before, consider the coordinate open $M^1(i, j, k; \delta) \subset F$, $i = 1, \ldots, r$, $j = 1, \ldots, \omega_i$, $\delta \leq \delta^{(3)}$, with coordinates $(z_1, \ldots, z_r)$ (cf.*

(5.9), (5.11.4)); *let* $E(P) = q^{-1}(P)$ (*cf.* (6.3)) *be defined by* $z_1 \cdots z_{i'} = 0$, *E by* $z_1 \cdots z_i = 0$, $i \geq i' \geq 2$, *and assume* $1 \leq k$, $v \leq i'$, $v \neq k$. *Let* $Q \in M^1(i, j, k; \delta)$, *with coordinates* $(a_1, \ldots, a_r)$, $a_v = 0$, $a_t \neq 0$ *for* $1 \leq t \leq i, t \neq k$ (a_k *may be zero or not*). *Then, there is a real number* $\eta > |a_k|$, *and positive real numbers* γ_l, $1 \leq l \leq i$, $l \neq k$, *such that for any point* $Q' = (x, y) \in V$ *satisfying:*

(i) $x \in q(M^1(i, j, k, \delta)) - \Delta$ (*cf.* (6.3)),

(ii) $q^{-1}(x)$ *has coordinates* $(z_1, \ldots, z_r)$, *with* $|z_k| < \eta$, $|z_t - a_t| < \gamma_t$, $t = 1, \ldots, i$, $t \neq k$,

we have

$$a^1(i, j, k)(Q') = \lim_{m \to \infty} w_m$$

where $w_0 = y$ *and* $w_{m+1} = w_m - g(x, w_m)/g'(x, w_m)$.

The proof of this lemma will be presented in Sections 6.14 to 6.16.

(6.9) Now we define a number $\delta^{(4)} \leq \delta^{(3)}$ such that $\{a^3_{b\alpha}(\delta)\}$ will agree on the intersection of their domains, for $\delta \leq \delta^{(4)}$.

For that, fix real numbers

$$0 < \rho_3 < \rho_2 < \rho_1 = \delta^{(3)}. \tag{6.9.1}$$

We define, for any pair (b, c) of integers, $1 \leq b < c \leq e$ (where, say, $\tau(b) = (i, j, k; (s))$, $\tau(c) = (i', j', l; (s'))$) a real number $\varepsilon(b, c)$ as follows. Consider the families (on F) $\mathscr{P}^3_b(\rho_3)$, $b = 1, \ldots, e$ (cf. (5.12).

(a) If $(P^3_b(\rho_3))_0 \cap (P^3_c(\rho_3))_0 = \emptyset$; set $\varepsilon(b, c) = \rho_3$ (recall that P_0 is the carrier of a family $\mathscr{P}$ cf. (5.2)).

(b) Assume this intersection is non-empty. First note the following facts.

(i)

$$\{Q \in P^\lambda_c(\rho_3)/l\text{-coordinate of } \theta_c^{-1}(Q) \text{ is zero}\}$$

(where $\theta_c^{-1}\colon P^\lambda_c(\rho_3) \to K^\lambda_c(\rho_3)$ is the coordinate isomorphism of (5.12)) is contained in exactly one irreducible component of $E(P)$, say E_0.

(ii) For $\delta \leq \rho_3$, $(P^3_c(\delta))_0$ can be viewed as a disjoint union of curves of the form $\Lambda = \theta_c(\{(z_i) \in K^3_c(\delta)/z_i$ is a fixed number for $i \neq t$, $|z_l| < \delta\}$.

(iii) Each such curve meets E_0 only at the point corresponding to $z_l = 0$ (i.e., the origin).

Now, we claim that for certain $\rho' \leq \rho_3$, all such curves Λ (parametrized now by $z_l \in B(\rho')$) have these properties: (1) if Λ meets $(P^3_b(\rho_3))_0$, then the origin of Λ belongs to $M^2(i, j, k; \rho_2)$; (2) if the origin of Λ is in $M^2(i, j, k; \rho_2)$, then Λ is entirely contained in $M^1(i, j, k; \rho_1)$.

To see (1), consider the compact set $E_0 - M^2(i, j, k; \rho_2)$. It is disjoint from the closure of $(P^3_b(\rho_3))_0$ (since $M^2(i, j, k; \rho_2) \supset \text{cl}\,(P^3_b(\rho_3))$), hence the distance (with respect to any metric inducing the topology of F) between them is positive. Then, it is clear that for some $\rho_4 \leq \rho_3$, any curve Λ of the family $\mathscr{P}^3_c(\rho_4)$ with origin outside $M^2(i, j, k; \rho_2)$ does not meet $(P^3_b(\rho_3))_0$. To see (2), consider the compact cl $(M^2(i, j, k; \rho_2)) \cap E_0$, disjoint from the closed $F - M^1(i, j, k; \rho_1)$, hence at positive distance. Again, it is easy to see that for some $\rho' \leq \rho_4$, any Λ in $\mathscr{P}_c^3(\rho')$ with origin in $M^2(i, j, k; \rho_2)$ is completely contained in $M^1(i, j, k; \rho_1)$.

Take $\varepsilon(b, c) = \rho'$, and define

(6.9.2) $$\delta^{(4)} = \min \varepsilon(b, c), \quad 1 \leq b < c \leq e.$$

(6.10) Now we are in conditions to prove that the functions $a_{b\alpha}^3(\delta)$ (cf. (6.6)) coincide on the intersections of their domains, for $\delta \leq \delta^{(4)}$. Let $Q \in N_{b\alpha}^3(\delta^{(4)}) \cap N_{c\beta}^3(\delta^{(4)})$, $1 \leq b < c \leq e$, $1 \leq \alpha \leq \alpha(b)$, $1 \leq \beta \leq \alpha(c)$ (cf. (5.2) and (5.13)). We want to see that

(6.10.1) $$a_{b\alpha}^3(Q) = a_{c\beta}^3(Q).$$

To simplify, we shall write $\varepsilon = \delta^{(4)}$, $a^3 = a_{c\beta}^3(\delta^{(4)})$. First, note that there is a unique curve, say Λ, in the family $\mathscr{N}_{c\beta}^3(\varepsilon)$ passing through Q. But (we claim) $\Lambda - \{P\} \subset R^1(i, j, k; \rho_1)$ (cf. (6.6.3) and (6.9.1)). In fact, the projection $\Lambda_0 = \pi(\Lambda) \subset \mathbf{C}^r$ is a curve in $\mathscr{L}_c^3(\varepsilon)$; on the other hand $\Lambda_0 = q(\Lambda')$ is a curve in $\mathscr{P}_c^3(\varepsilon)$, having at least one point (namely, $q^{-1}\pi(Q)$) in common with $P_b^3(\varepsilon)$. By (1) and (2) of (6.9), $\Lambda' \subset M^1(i, j, k; \rho_1)$, hence by Definition (6.6.3), $\Lambda - \{P\} \subset R^1(i, j, k; \rho_1)$, as claimed.

Let $\tau(b) = (i, j, k; (s))$. In the sequel, we write $a^1 = a^1(i, j, k; \rho_1)$ (cf. (6.6.4)). Note that if we find a point $Q_1 \in \Lambda$ such that $a^3(Q_1) = a^1(Q_1)$, then $a^3(Q) = a^1(Q)$. In fact, the functions a^3 and a^1 defined by the formulas

(6.10.2) $$(x, y) \to (x, a^i(x, y)), \quad i = 1, 3,$$

are two mappings from $\Lambda - \{P\}$ to $\pi'^{-1}(\Lambda_0 - \{P\})$, commuting with the projections on $\Lambda_0 - \{P\}$. Since these spaces are etale over the connected space $\Lambda_0 - \{P\}$, if the maps (6.10.2) agree at a point they agree everywhere, hence we shall obtain $a^3(Q) = a^1(Q)$. But also note (use (6.6)) that $a^1(Q) = a_{b\alpha}^3(\varepsilon)(Q)$. Hence, if such point Q_1 exists, (6.10.1) would follow.

To show the existence of such Q_1, consider $\Lambda_0 = \pi(\Lambda)$, parametrized (using the map q_c, cf. (5.11.2) and (5.5)) by $\{z \in \mathbf{C}/|z| < \varepsilon\}$. Then, by (6.6.4), there is some $\rho > 0$ such that, if $\Lambda_0(\rho) = \{R \in \Lambda_0/R$ corresponds to $z \in B(\rho)\}$, and $Q' \in \Lambda$ verifies $\pi(Q') \in \Lambda_0(\rho)$, then $a^3(Q') = \lim_{s\to\infty} w_j$ (where w_j is defined as in (6.6.5)). On the other hand, consider Λ' (the curve in $\mathscr{P}_c^3(\varepsilon)$ such that $q(\Lambda') = \Lambda_0$). Let Q_0 be its origin, $Q_0 \in E(P)$. Then, there is a neighborhood $\mathscr{U}$ of Q_0, such that for any $R \in V$, with $\pi(R) \in q(\mathscr{U}) - \Delta$, we have $a^1(R) = \lim_{j\to\infty} w_j$, where w_j is defined by (6.6.5). In fact, let the coordinates of Q_0 (in the coordinate neighborhood $M^1(i, j, k; \delta^{(1)})$ be (a_γ). Since $Q_0 \in E(P)$ (defined by $z_1 \cdots z_{i'} = 0$), $a_t = 0$ for some t, $1 \leq t \leq i'$. If $t = k$, the existence of $\mathscr{U}$ follows immediately from Lemma (6.5), if $t \neq k$, from (6.8). If $\Lambda'(\rho) \subset F$ is defined by the condition $q(\Lambda'(\rho)) = \Lambda_0(\rho)$, then clearly $\mathscr{V} = (\mathscr{U} - E) \cap \Lambda'(\rho) \neq \emptyset$. Then, for any $Q_1 = (x, y) \in \Lambda$ such that $\pi(Q_1) \in q(\mathscr{U})$, clearly $a^1(Q_1) = a^3(Q_1) = (x, y_0)$, where $y_0 = \lim_{j\to\infty} w_j$, and w_j is defined by (6.6.5). The proof of (6.10.1) is complete.

(6.11) Now we prove that the morphism

$$\phi\colon \pi^{-1}(U - \Delta) \to \pi'^{-1}(U - \Delta), \tag{6.11.1}$$

(where $U \subset G^3(\delta^{(4)}) \cup \Delta$ is an open neighborhood of P, cf. (5.7) for the notation) given by

$$(x, y) \mapsto (x, a^3_{b\alpha}(x, y)) \in V' \tag{6.11.2}$$

is an isomorphism. As before, let $\delta^{(4)} = \varepsilon$. Since $\pi^{-1}(U - \Delta)$ and $\pi'^{-1}(U - \Delta)$ are non-singular, it is enough to show that ϕ is bijective, and by (a) of (5.13), it suffices to show that for any curve Λ in some family $\mathscr{L}^3_b(\varepsilon)$, the map

$$\phi(\Lambda)\colon \pi^{-1}(\Lambda - \{P\}) \to \pi'^{-1}(\Lambda - \{P\})$$

induced by ϕ is bijective. Since these spaces are etale over $\Lambda - \{P\}$ and $\phi(\Lambda)$ commutes with π, π', it suffices to check this for points of some deleted neighborhood of the origin.

So let Λ be a curve of certain $\mathscr{L}^3_b(\varepsilon)$, parametrized by

$$\psi(z) = (\psi_1(z), \ldots, \psi_n(z)), \quad z \in D = B(\varepsilon),$$

where $\psi\colon D \to \Lambda$ is naturally induced by $q\theta_b\colon K^3_b(\varepsilon) \to U$ (cf. (5.11.2), (5.5)). Consider the pull-back (by ψ) of $\pi^{-1}(\Lambda)$ and $\pi'^{-1}(\Lambda)$ to D; we get a diagram, with cartesian squares:

$$\begin{tikzcd}
 & C' \arrow[rr, "h'"] \arrow[dd] & & \pi'^{-1}(\Lambda) \arrow[ddl] \\
C \arrow[rr, "h"] & & \pi^{-1}(\Lambda) \arrow[d] & \\
 & D \arrow[r, "\psi"'] & \Lambda &
\end{tikzcd} \tag{6.11.3}$$

Then, C, C' are the plane curves given by the equations

$$\tilde{f}(z, y) = y^n + A_1(\psi(z))y^{n-1} + \cdots + A_n(\psi(z)) = 0$$

and

$$\tilde{g}(z, y) = y^n + B_1(\psi(z))y^{n-1} + \cdots + B_n(\psi(z)) = 0$$

respectively, and h, h' are homeomorphisms. We claim that there is an isomorphism $\rho\colon C \to C'$, defined near the origin, such that the following diagram commutes:

$$\begin{tikzcd}
C - \{P_0) \arrow[r, "h"] \arrow[d, "\rho"'] & \pi^{-1}(\Lambda) - \{P\} \arrow[d, "\phi(\Lambda)"] \\
C' - \{P_0\} \arrow[r, "h'"] & \pi'^{-1}(\Lambda) - \{P\}
\end{tikzcd} \tag{6.11.4}$$

(P_0 is the origin in the (z, y)-plane, and C, C', Λ really are the restrictions of the corresponding curves to a suitable neighborhood of the origin.) The bijectivity of h and h' clearly implies that $\phi(\Lambda)$ is bijective.

To find such a germ of isomorphism ρ is equivalent to find an isomorphism of $\mathbf{C}\{z\}$—algebras

$$\rho': \mathbf{C}\{z\}[Y]/(\tilde{g}(z, Y) \to \mathbf{C}\{z\}[Y]/\tilde{f}(z, Y).$$

By Nakayama's lemma it suffices to find a root a of $\tilde{g}(z, Y)$ in A, satisfying $a \equiv y(\text{mod } (z)A)$, where $A = \mathbf{C}\{x, y\}$. But we have

$$(6.11.5) \qquad g(y) \in [g'(y)]^2(zA).$$

We shall prove it in (6.10). By Hensel's lemma there is a unique solution, giving the isomorphism ρ'. It is easy to check (using (4.2)) that the induced isomorphism ρ has, near the origin, the form

$$\rho(z, y) = (z, a(z, y)),$$

where $a(z, y) = \lim_{j\to\infty} w_j$, $w_0 = y$ and $w_{j+1} = w_j - g(z, w_j)/g'(z, w_j)$. From this, using the definition of ϕ, it follows that (6.11.4) commutes. Hence $\phi(\Lambda)$ is bijective; and we saw that this implies that ϕ (cf. (6.11.1)) is an isomorphism.

(6.12) To finish the proof of Proposition (3.6), we must show Lemmas (6.5), (6.6) and the equality (6.11.5).

Proof of Lemma (6.5). The indices b, α will remain fixed throughout the proof, so we shall write $K_{b\alpha} = K$, $K_b = K'$, $w_{b\alpha} = w$ (cf. (6.4.3)). We also use notation $H_i = B_{b\alpha i}$, $G_i = A_{b\alpha i} = A^i q_b \pi_{b\alpha}$;

$$v(Y) = g_{b\alpha}(Y) = Y^n + \sum H_{n-i} Y^i, \quad u(Y) = f_{b\alpha}(Y) = Y^n + \sum G_{n-i} Y^i.$$

We claim that there is an open neighborhood $\mathcal{G}$ in K) of X_k and $m \in \Gamma(\mathcal{G})$, such that $m(z) = 0$ for $z \in X_k$ and

$$(6.12.1) \qquad v(w) = [v'(w)]^2 m$$

on $\mathcal{G}$. To check this, note that as $y^n + \sum A_i(x)y^{n-i} = 0$, we get

$$(6.12.2) \qquad u(w) = w^n + G_1 w^{n-1} + \cdots + G_n = 0 \quad \text{on} \quad K.$$

Recall that $A_i \equiv B_i \text{ mod } (x)^{c+1}$, with c defined in (6.3.2). Thus

$$A_m(x) - B_m(x) = k_{m,c+1}(x) + k_{m,c+2}(x) + \cdots$$

where $k_{m,i}$ is a form of degree i. We shall consider the pull-back of this equality to $K' = K_b$, via q_b (cf. (6.4.2)). Note that $q_b(z)$, a function from a polydisk to $\mathbf{C}^r$, is of the form

$$q_b(z) = (\bar{q}_1(z), \ldots, \bar{q}_r(z)),$$

where $\bar{q}_i(z)$, $i = 1, \ldots, r$, is holomorphic; and since $q_b^{-1}(P)$ is the analytic set of equation $z_k = 0$, $\bar{q}_i(z)$ has necessarily the form

$$\bar{q}_i(z) = z_k \rho_i(z),$$

for certain $\rho_i \in \Gamma(K')$. Thus, on K',

$$A_{bm}(z) - B_{bm}(z) = k_{m,c+1}(z_k\rho_1(z), \ldots, z_k\rho_r(z)) + \cdots$$

and, using the homogeneity of the k's,

$$A_{bm}(z) - B_{bm}(z) = z_k^{c+1} l_m(z) \tag{6.12.3}$$

for certain $l_m(z) \in \Gamma(K')$. Using (6.2.1) and (6.2.2), we have on $H^0(i, j)$ (cf. (5.9)),

$$(z_1 \cdots z_i)^c = (\delta_{ij})^2 (z_1 \cdots z_i)^{c-2c(ij)} h_{ij}(z).$$

Then, calling $h = (z_1 \cdots z_i)^{c-2c(ij)}(z_1 \cdots \hat{z}_k \cdots z_i)^{-c} h_{ij}$, and noting that on K' we have $z_t \neq 0$, $1 \leq t \leq i$, $t \neq k$, we get, on K',

$$z_k^c = \delta_b^2 h, \quad h \in \Gamma(K') \tag{6.12.4}$$

where δ_b is the restriction of δ_{ij} to K'.

Thus, from (6.12.3) we get

$$A_{bm}(z) - B_{bm}(z) = (\delta_b(z))^2 z_k j_{bm}(z) \tag{6.12.5}$$

for some $j_{bm} \in \Gamma(K')$. Then, by pulling back this equality to K, and recalling how $\pi_{b\alpha}\colon K \to K'$ is defined (cf. (5.13.1)) we get

$$H_m(z) = G_m(z) - \delta_0(z)^2 z_k^d j_m(z) \tag{6.12.6}$$

where $\delta_0 = \delta_b \pi_{b\alpha}$, $j_m = j_{bm}\pi_{b\alpha}$, $d = d(b\alpha)$. Hence, replacing (in $v(Y)$) H_m by (6.12.6), and using (6.12.2), we get

$$v(w) = \delta_0^2 p z^d \tag{6.12.7}$$

for certain $p \in \Gamma(K')$. To finish the proof of the claim about $\mathscr{G}$, we shall relate δ_0 to $v'(w)$.

It is clear that δ_0 is the discriminant of $u(Y)$, hence by a classical formula,

$$\delta_0 = u(Y)E(Y) + u'(Y)D(Y) \tag{6.12.8}$$

for some $E(Y)$, $D(Y)$ in $\Gamma(K)[Y]$. Hence, making $Y = w$ and using (6.13.2) we get

$$\delta_0 = u'(w)D(w). \tag{6.12.9}$$

But using (6.12.6), an elementary calculation yields

$$u'(w) = v'(w) - \delta_0^2 h_0 z^d \tag{6.12.10}$$

for certain $h_0 \in \Gamma(K)$. So, by (6.12.9) and (6.12.10) we get

$$\delta_0 = D(w)v'(w) - \delta_0^2 s,$$

$s \in \Gamma(K)$, and $s(z) = 0$ when $z_k = 0$.

Hence

$$\delta_0(1 + \delta_0 s) = D(w)v'(w). \tag{6.12.11}$$

Since s vanishes on X_k, we can find a neighborhood $\mathscr{G}$ of X_k such that $|\delta_0 s| < 1$ on $\mathscr{G}$. Hence, on $\mathscr{G}$ the function $(1 + \delta_0 s)$ is invertible and we can write

$$\delta_0 = v'(w)s_1, \quad s_1 \in \Gamma(\mathscr{G}. \tag{6.12.12}$$

The equalities (6.12.7) and (6.12.12) prove our claim. With this, we shall prove the lemma. If $c \in X_k$, we can find a polydisk $D(c) \subset \mathscr{G}$ about c and a unique root $a_c \in \Gamma(D(c))$ of the polynomial

$$v_c(Y) = Y^n + \sum (H_{n-i}|D(c))Y^{n-i}$$

satisfying $a_c - w = v_c(w)m$ on $D(c)$. This is a consequence of (6.12.1), which allows us to apply Hensel's lemma (cf. (4.2)). We can extend analytically a_c to $D^0(c) = \{z \in K/z_k \in D(c)\}$. This is a consequence of the fact that

$$q_b\pi_{b\alpha}\colon D^0(c) - X_k \to U_1 \quad \textit{and} \quad \pi'\colon \pi'^{-1}(U_1 - \Delta) \to U_1$$

are etale, and the morphism

$$\psi\colon D(c) \to \pi'^{-1}(U_1 - \Delta)$$

defined by $\psi(z) = (q_b\pi_{b\alpha}(z), a_c(z))$ satisfies $\pi'\psi = q_b\pi_{b\alpha}$. Let $\bar{a}_c$ be the extension of a_c to $D^0(b)$. Consider a collection of points $c \in X_k$, which are centers of polydisks $D(c)$ as above, such that $\cup D^0(c) = K$.

The maps $\bar{a}_c$ agree on the intersections; in fact, $D^0(c) \cap D^0(c')$ is connected, then using the uniqueness part of (4.2) and the principle of analytic continuation, $\bar{a}_c = \bar{a}_{c'}$ on $D^0(c) \cap D^0(c')$. Thus, we get $a \in \Gamma(K')$, which clearly satisfies $v(a) = 0$. We take $\mathscr{G}_0 = \cup D(c)$, $m_0 = m|\mathscr{G}_0$ and (6.5.1) holds. (6.5.2) is a consequence of Lemma (4.2). Thus, (6.5) is proved.

(6.13) With the same argument, we can show the following result, to be used later. Make the same assumptions as in (6.5), but now we assume $A_i \equiv B_i \bmod (x)^{c+s+1}$, $i = 1, \ldots, n$, where s is a positive integer. Then, there is a function $a_{b\alpha}$ as in Lemma (6.5), but satisfying the stronger condition

$$a_{b\alpha} - w_{b\alpha} = [g'_{b\alpha}(a_{b\alpha})]^2 z_k^s m_0 \tag{6.13.1}$$

on a neighborhood $\mathscr{G}_0$ X_k; $m_0 = 0$ on X_k.

(6.14) *Proof of Lemma* (6.8). We may assume, after changing the indices if necessary, that $k = 1$, $v = 2$. First we show the following lemma, in which we use the notations of Section 5 and (6.5), (6.8) and (6.12).

(6.15) LEMMA. *The assumptions are as in* (6.8) (*with* $k = 1, v = 2$) *but we also assume that* $a_t \in D^1(i, j, t, s_t)$, $t = 3, \ldots, i$ (*cf.* (5.10)). *If* $1 \leq s \leq 4$, *and* γ *is a real number, let* $B_s(\gamma) = B(\gamma \exp(\frac{1}{2}\mathbf{i}\pi s), \gamma)$ (*cf.* (1.2) (a), *here* $\mathbf{i}$ *is the imaginary unit*). *Fix* s_2, $1 \leq s_2 \leq 4$. *Let* $b = \tau^{-1}(i, j, 1; s_2, \ldots, s_i)$, $1 \leq \alpha \leq \alpha(b)$. *Then, there are real numbers* $\eta > |a_1|$, $\gamma_l > 0$, $l = 2, \ldots, r$, *such* that:

(a) $S' = \{(z) \in H^1(i, j, 1)/|z_1| < \eta,\ z_2 \in B_{s_2}(\gamma_2),\ |z_t - a_t| < \gamma_t,\ t = 3, \ldots, r\} \subset K_b^1(\delta)$ (*cf.* (5.12), (5.2)).
(b) *If* $Q' = (x, y) \in V$, $x \in q\theta(ij)(S')$, *then* $a^1(i, j, 1, \delta)(Q') = \lim_{j\to\infty} w_j$, *where* $w_0 = y$, $w_{j+1} = w_j - g(x, w_j)/g'(x, w_j)$.

Proof. We keep the notations of (6.12). In particular, generally we drop the fixed indices b, α. Consider the polydisk $H^0(i, j) = H^0$ (cf. (5.9)) and $\chi = q\theta(ij)$: $H^0 \to \mathbf{C}^r$. Since we assume

$$\{z \in H/z_1 z_2 = 0\} \subset E(P),$$

then $\chi(z) = (z_1 z_2 \chi_i(z))$, $i = 1, \ldots, r$, where each χ_i is an holomorphic function on H^0. Reasoning as in (6.12) we get the following formula, analogous to (6.12.3):

$$A_m\chi(z) - B_m\chi(z) = (z_1 z_2)^{c+1}\psi_m(z) \tag{6.15.1}$$

for all $z \in H = \chi^{-1}(U_2)$ (cf. (6.7.1)); $\psi_m \in \Gamma(H)$. Using (6.2.1), we get, on H,

$$(z_1 z_2)^{c(ij)} = (z_3 \cdots z_i)^{-c(ij)}\delta_{ij}h_{ij} \tag{6.15.2}$$

for $z \in H' = \{z \in H/z_t \in D^1(i, j, t; s_t),\ t = 3, \ldots, i\}$. Write

$$h(z) = (z_3 \cdots z_i)^{-c(ij)}.$$

By the choice of c (cf. (6.2.2)), from (6.15.1) and (6.15.2) we get, on H,

$$A_m\chi - B_m\chi = (z_1 z_2)\delta_1^2 h^2\psi'_m \tag{6.15.3}$$

where $\delta_1 = \delta\chi$, $\psi'_m \in \Gamma(H)$, note that h is not holomorphic (unless $i = 2$), but meromorphic.

Let $d = d(b, \alpha)$, ρ the positive d-root of δ, $K = K'_{b\alpha}(\rho)$, $K' = K'_b(\delta) \subset H$, $p = \pi_{b\alpha}$ (cf. (6.4.3)). Note that $p(z) = z'$, where $z'_i = z_i$ for $i > 1$, $z'_1 = z_1^d$ (since $\tau(b) = (i, j, 1; (s))$). Consider the pull-back of the equalities (6.15.3) under p. We get, on K (using the notation of (6.12)),

$$G_m - H_m = z_1^d z_2 \delta_0^2 h^2 L_m \tag{6.15.4}$$

where $\delta_0 = \delta_1 p$, $L_m = \psi'_m p$, note that h did not change, since it does not involve the variable z_1. Also note that all functions on the right hand side, except h, are pull-backs of functions in $\Gamma(H)$. We may relate δ_0 with $v'(w)$ as in Lemma (6.5) (cf. (6.12.8) ff.), but note that now the coefficients of the polynomial $D(Y)$ (cf. (6.12.8)) are pull-backs (by p) of functions in $\Gamma(H)$. With elementary computations we obtain, as in (6.12.11), the following equality on K:

$$\delta_0(1 + \delta_0 s) = D(w)v'(w), \tag{6.15.5}$$

where

$$s = z_1 z_2 h^2\sigma_1\Theta(w) \tag{6.15.6}$$

and Θ is a polynomial; moreover σ_1 and the coefficients of Θ are pull-backs of functions in $\Gamma(H)$. Let $\sigma_2 = \sigma_1 \Theta w$; then (we claim) σ_2 is bounded on K. In fact, w is obviously bounded, and σ_1 and the coefficients of Θ are bounded, since they are pull-backs (by p) of holomorphic functions on H, which contains the compact set cl $(p(K))$. Hence, $|\sigma_2| < M$ on K, for certain constant M. Consider

$$K' = \{z/|z| < \delta\} \times \prod_{t=2}^{i} D^1(i, j, t; s_t) \times \prod_{t=i+1}^{r} H^1(i, j, t);$$

choose $\eta > |a_1|$, γ_t, $t = 3, \ldots, r$ such that $\eta < \delta$ and $\{z/|z - a_t| \leq \gamma_t\}$ is contained in $D^1(i, j, t, s_t)$ for $t = 3, \ldots, i$ and in $H^1(i, j, t)$ for $t = i + 1, \ldots, r$. Then, since no disk D^1 contains the origin, there is some positive constant M_0 such that $|h(z)| < M_0$ for $z = (z_1, \ldots, z_r)$ with $|z_i - a_t| < \gamma_t$, $t = 3, \ldots, i$. Write, for a real number γ,

$$\begin{aligned}(6.15.7)\quad S'(\gamma) = \{z = (z_1, \ldots, z_r)/z_2 \in B_{s_2}(\gamma),\ |z_1| < \eta;\\ z_t \in B(a_t, \gamma_t),\ t = 3, \ldots, i;\ z_t \in H^1(i, j, t),\ t > i\},\end{aligned}$$

and $S(\gamma) = p^{-1}(S'(\gamma))$. Clearly for $\gamma = \gamma_0$ small enough, $S'(\gamma_0) \subset K'$ and also (using (6.15.6) and the bounds M, M_0) $|\delta_0 s(z)| < \frac{1}{2}$ for $z \in S(\gamma_0) \subset K$. Then, if we restrict the equality (6.15.5) to S, we can invert $(1 + \delta_0 s)$, to get

$$(6.15.8)\qquad \delta_0 = v'(w)r_0, \quad r \in \Gamma(S),$$

where $r_0 = D(w)(1 + \delta_0 s)^{-1}$; note that r_0 is bounded on $S(\gamma_0)$. Using (6.15.3) we get

$$(6.15.9)\qquad G_m - H_m = v'(w)^2 m,$$

where

$$(6.15.10)\qquad m = z_1^d z_2 r_0^2 h^2 L_m.$$

Note that $S'(\gamma)$, for $\gamma \leq \gamma_0$ satisfies condition (a) of the statement of Lemma (6.15). Now using the fact that $\delta \leq \delta^{(3)}$, we shall see that for some $\gamma_2 \leq \gamma_0$, condition (b) is also satisfied. We want to use (4.2) to prove that, for some $\gamma_2 \leq \gamma_0$, the function $f = a^1(i, j, 1; \delta)q_{b\alpha}$ (cf. (6.4.3)) can be computed with the "Newton sequence" (6.5.2), for all $z \in S(\gamma_2)$. Consider the sufficient conditions for convergence (on the polydisk $S(\gamma)$) of the Newton process, listed in (4.6). The choice of $\delta^{(3)}$ implies that all of them, except perhaps (iv) ($|m| < \frac{1}{8}(1 + M')$) are automatically satisfied. But from (6.15.10), clearly for some $\gamma_2 \leq \gamma_0$, $|m| < \frac{1}{8}(1 + M')$ on $S(\gamma_2)$. Then the polydisk $S' = S'(\gamma_2)$ satisfies (a) and (b), and the lemma is proved.

(6.16) Now we conclude the proof of Lemma (6.8). The assumptions imply the existence of integers s_t, $t = 3, \ldots, i$, such that $a_t \in D^1(i, j, t; s_t)$. In fact, for fixed i, j, t, these disks (with s_t varying) cover $H^1(i, j, t)$ (cf. (5.10)). We apply repeatedly Lemma (6.15), for $s_2 = 1, 2, 3, 4$, and for each s_2, for all possible values of α. We get numbers $\gamma_{t,s,\alpha}$, $\mathcal{N}_{s,\alpha}$. Let

$$\gamma_t = \inf_{s,\alpha} (\gamma_{t,s,\alpha}),\ t > 2, \quad \eta = \inf_{s,\alpha} \{\eta_{s,\alpha}\}, \quad \gamma_2' = \inf_{s,\alpha} \{\gamma_{2,s,\alpha}\}.$$

Consider $H^1(i, j, 2)$, it is clear that for γ_2 small enough, the set

$$\{z/0 < |z| < \gamma_2\} \subset \bigcup_{s=1}^{4} B(\gamma_2' \exp(\tfrac{1}{2}i\pi s), \gamma_2').$$

With this choice of η and γ_t, $t = 2, \ldots, r$, Lemma (6.8) is an immediate consequence of (6.15).

(6.19) The proof of the equality (6.11.5) is completely similar to the proof of formula (6.12.1), and we do not repeat it here.

The proof of Proposition (3.6) is complete.

7. Proof of Proposition (3.8)

(7.1) We shall use the notations of Sections 5 and 6. In this section, $\mathbf{R} = \{\text{real numbers}\}$, $\mathbf{R}(+) = \{a \in \mathbf{R}/a \geq 0\}$, $\mathbf{R}(-) = \{a \in \mathbf{R}/a \leq 0\}$.

Let us define the number c_0 of (3.8). Consider the families $\mathcal{L}_b^3(\delta^{(4)})$, which will be denoted here simply by $\mathcal{L}_b$, similarly K_b will denote $K_b^3(\delta^{(4)})$. Throughout this section, the number $\delta^{(4)}$ of (6.9) will be denoted by δ. Fix any $b = 1, \ldots, e$. We let

$$\tilde{K}_b = \{(z) \in K_b/z_k \neq 0\};$$

here we assume $\tau(b) = (i, j, k; (s))$. Note that the fundamental group of $\tilde{K}_b$ is isomorphic to $\mathbf{Z} = \{\text{integers}\}$; let ρ be a generator. Thus, $\tilde{K}_b$ is not simply connected, but we can express it as a union of two simply connected subsets, namely,

$$(7.1.1) \qquad K_b' = \{z \in \tilde{K}_b/z_k \notin \mathbf{R}(+)\}, \qquad K_b'' = \{z \in \tilde{K}_b/z_k \notin \mathbf{R}(-)\}.$$

Using the monodromy theorem, by fixing $Q \in q_b(K_b')$, considering the different roots y_p of $y^n + \sum A_i(Q)y^{n-i}$ and analytic continuation, we find n holomorphic functions

$$(7.1.2) \qquad \eta_p^n\colon K_b' \to \mathbf{C} \qquad i = 1, \ldots, n$$

such that $\eta_p^n + A_1' \eta_p^{n-1} + \cdots + A_n' = 0$, where $A_j' = A_j q_b$ (cf. (5.11.2)). The fundamental group of $\tilde{K}_b$ operates on the set $\{\eta_1, \ldots, \eta_n\}$ in an obvious way. For any $p = 1, \ldots, n$, there is a smallest positive integer r_p so that $\rho^{r_p}\eta_p = \eta_p$. This follows from the definition of η_p, moreover, it is easy to check that r_p is precisely the integer $d(b\alpha)$ of (5.15) (c), if $(Q, y_p) \in N_{b\alpha}$. Let $b = 1, \ldots, e$, $\tau(b) = (i, j, k; (s))$, and consider, for $1 \leq p, q \leq n$, $p \neq q$ the polydisk

$$G_b(p, q) = \{z \in \mathbf{C}^r/z_t \in D^3(i, j, t; s_t) \quad \text{for} \quad i \leq t \leq i, t \neq k,$$
$$|z_k|^{r_p r_q} < \delta,\ z_t \in H^3(i, j, t) \quad \text{for} \quad t > i\}.$$

Define a map $w_1: G_b(p, q) \to K_b$, by $w_1(z) = z'$, where $z'_t = z_t$ if $t \neq k$, $z'_k = z_k^{r_p r_q}$. Then, the open set of $G_b(p, q)$,

$$S_b(p, q) = \{z \in G_b(p, q)/0 < \arg(z_k) < 2\pi/r_p r_q,\ z_k \neq 0\}$$

is isomorphic, by w_1, to K'_b.

Clearly, by the choice of r_p, the functions $\eta_p w_1 \in \Gamma(S_b(p, q))$ can be extended first to an holomorphic function on $\{z \in G_b(pq)/z_k \neq 0\}$ and then, since it is bounded, to a function ϕ^1_{bpq} defined on $G_b(pq)$. Similarly, working with η_q we get functions $\phi^2_{bpq} \in \Gamma(G_b(pq))$. Consider the difference $\phi^1_{bpq} - \phi^2_{bpq}$, let $l = l(1, b, p, q)$ be the order at which it vanishes along $z_k = 0$, i.e.,

$$\phi^1_{bpq} - \phi^2_{bpq} = \lambda \cdot z_k^l, \tag{7.1.3}$$

where $\lambda \in \Gamma(G_b(pq))$ and $z_k = 0$ is not a zero of λ.

Working in a similar fashion with the different sets K''_b we get functions w_2, ϕ^i_{bpq}, and, we find numbers $l(2, b, p, q)$. Take

$$s > \sup\{l(1, b, p, q),\quad l(2, b, p, q)\},$$

where $1 \leq b \leq e$, $1 \leq p, q \leq n$, $p \neq q$. Define

$$c_0 = c + s. \tag{7.1.4}$$

(7.2) By the choice just made, $c_0 > c$, the number of Proposition (3.6). Hence, by that result, if V' is another hypersurface given by equation (3.2.1) with $B_i \equiv A_i(\text{mod } (x)^{c_0+1})$, and $\Delta' \subset \Delta$ near the origin (cf. (3.2)), then there is an isomorphism

$$\phi: \pi^{-1}(U - \Delta) \to \pi'^{-1}(U - \Delta)$$

for certain neighborhood U of the origin; moreover, if $(x, y) \in V$, then $\phi(x, y) = (x, \alpha(x, y)) \in V'$, $\alpha \in \Gamma(\pi^{-1}(U - \Delta))$.

(7.3) LEMMA. *Fix an index b, $b = 1, \ldots, e$ and let $\tau(b) = (i, j, k; (s))$, Λ_a be the curve in $\mathscr{L}_b$ corresponding to $a \in Y = \{a \in K_b/a_k = 0\}$ (cf. (5.2)); this curve is parametrized by $z_k \in B(\delta)$ cf. (5.5)). Then, there is a subvariety Y' of Y, of dimension $< r - 1 = \dim Y$, with the following property: given ε real positive, for any $a \in Y - Y'$ there is a number $\delta(a)$, such that for any pair (x, y_p), (x, y_q) in V, with $x \in \Lambda_a$ and corresponding to a value z_k of the parameter satisfying $|z_k| < \delta(a)$, we have*

$$|(y'_p - y'_q/y_p - y_q) - 1| < \varepsilon \tag{7.3.1}$$

where $y'_i = \alpha(x, y_i)$, $i = p, q$.

Proof. We know by (6.4), that α is induced, for points in $\pi^{-1}(L_b \cap U)$, by the functions $a_{b\alpha} = a^3_{b\alpha}(\delta) \in \Gamma(N_{b\alpha})$, where $N_{b\alpha} = N^3_{b\alpha}(\delta)$. We may work on $\pi^{-1}(L_b) = \bigcup_\alpha N_{b\alpha}$, since only such points are involved in the lemma. We also omit the index b, since it does not vary during the proof.

Fix a point $z_0 \in K' \cap K''$ (cf. (7.1.1)). Consider a pair $p \neq q$, $1 \leq p, q \leq n$, and assume (cf. (5.13.1))

$$(q(z_0), \eta_p(z_0)) \in N_\alpha, \tag{7.3.2}$$

$$(q(z_0), \eta_q(z_0)) \in N_\beta. \tag{7.3.3}$$

Let η'_p: $K' \to K_\alpha$ $(K_\alpha = K^3_{b\alpha}(\rho^{(4)}),\ (\rho^{(4)})^{d(b\alpha)} = \delta)$ be given by $\eta'_p(z) = q^{-1}_{b\alpha}(q(z), \eta_p(z))$ (cf. (5.13.1)). Then $\pi_\alpha \eta'_p(z) = z$, for $z \in K'$ ($\pi_\alpha = \pi_{b\alpha}$, recall we are omitting the index b). It is readily checked that if we define j: $G(p, q) \to K_\alpha$ by $j(z) = z'$, where $z'_t = z_t$ for $t \neq k$, $z'_k = \mu z_k^{r_q}$, μ a conveniently chosen r_p-root of 1 (and r_p, r_q as in (7.1)), then we have a commutative diagram

$$\begin{tikzcd}
G(p, q) \arrow[rr, "j"] \arrow[drr, "w_1"'] & & K_\alpha \arrow[d, "\pi_\alpha"] \\
& & K \\
S(p, q) \arrow[uu] \arrow[r, "\sim"'] & K' \arrow[uur, "\eta'_p"] \arrow[ur] &
\end{tikzcd} \tag{7.3.4}$$

This is an immediate consequence of the definitions, using the remark that $r_p = d(b\alpha)$.

Consider now the function $a_\alpha = a_{b\alpha}$ of (6.14). Since $A_i \equiv B_i (\text{mod } (x)^{c+s+1})$, we get an equality

$$a_\alpha - w_\alpha = [g'_\alpha(a_\alpha)]^2 z_k^s m,$$

on certain neighborhood $\mathcal{G}'_p$ of $X_k = \{z \in K_\alpha / z_k = 0\}$. If we pull-back this equality to $G(p, q)$ via j, we find

$$a_p - \phi^1_{pq} = z_k^s m_p, \tag{7.3.5}$$

where $a_p = a_\alpha j$, and m_p is holomorphic in some neighborhood $\mathcal{G}_p$ of $Y_{pq} = \{z \in G(p, q)/z_k \neq 0\}$. This easily follows from diagram (7.3.4).

Similarly, interchanging the roles of p and q, we get an equality

$$a_q - \phi^2_{pq} = z_k^s m_q, \tag{7.3.6}$$

valid on a neighborhood $\mathcal{G}_q$ of Y_{pq}. From (7.3.5) and (7.3.6) we get, on $\mathcal{G} = \mathcal{G}_p \cap \mathcal{G}_q$,

$$(a_p - a_q/\phi^1_{pq} - \phi^2_{pq}) - 1 = z_k^s (m_p - m_q)(\phi^1_{pq} - \phi^2_{pq})^{-1}. \tag{7.3.7}$$

But $\phi^1_{pq} - \phi^2_{pq} = z_k^t \lambda$, $t < s$ and Y_{pq} is not a zero of λ (cf. (7.1.3)). Then, $\lambda = b_0(z_1, \ldots, \hat{z}_k, \ldots, z_r) + z_k b_1(z)$, $b_0 \neq 0$. Hence

$$(a_p - a_q/\phi^1_{pq} - \phi^2_{pq}) - 1 = z_k m_{pq}/b_0 + z_k b_1 \tag{7.3.8}$$

where $m_{pq} \in \Gamma(\mathcal{G})$, $b_i \in \Gamma(G(p, q))$, $i = 0, 1$; b_0 does not depend on z_k.

Now it is clear that given $a = (a_1, \ldots, a_r)$, $a \in \mathcal{G}$, $b_0(a) \neq 0$ and a real number

$\varepsilon > 0$, if in (7.3.8) we fix $z_t = a_t$, $t \neq k$ and consider values of z_k with $|z_k| < \delta_a$ small enough, then

$$|(a_p - a_q)/(\phi^1_{pq} - \phi^2_{pq}) - 1/ < \varepsilon. \tag{7.3.9}$$

So far, we worked with a fixed pair (p, q), hence to be precise we should write throughout an index (p, q). For instance, $b_0 = b'_{opq}$, $\mathscr{G} = \mathscr{G}_{pq}$, etc. Also, we have used in this process the manifold K'_b of (7.1.1). Clearly, we may apply to $K'' = K''_b$ of (7.1.1) a similar procedure (with obvious changes), to get functions b''_{pq}, $\bar{a}_p$, $\bar{a}_q$, $\bar{\phi}_{pq}$, w_{2pq}, etc., with properties analogous to those of a_p, a_q, etc.

Apply this process to all possible pairs p, q (with both K' and K''); in particular we get a collection $\{b'_{pq}, b''_{pq}\}$ of functions of $r - 1$ variables, analogous to the function b_0 defined above. Consider the different morphisms w_{ipq}: $G(p, q) \to K$, $i = 1, 2$, $1 \leq p < q \leq n$, and the various images of the subspaces of equation $z_k = 0$, $b_{ipq} = 0$. This way we have a finite collection of subspaces of K (w_{ipq} is proper), let Y' be its union. We claim that this is the set Y' of (7.3) In fact, given $\varepsilon > 0$, if $a \in Y - Y' \subset K$ and Λ_a is the corresponding curve in $\mathscr{L}$, then the inequality

$$|(y'_i - y'_j/y_i - y_j) - 1| < \varepsilon$$

is equivalent (for suitable p, q) to

$$|(a_p - a_q)/(\phi'_{pq} - \phi^2_{pq}) - 1| < \varepsilon \tag{7.3.10}$$

(or, if we had to work with a w_{2pq}, the same with $\bar{a}_p$, $\bar{\phi}^i_{pq}$ instead of a_p, ϕ'_{pq} respectively). But (7.3.10) follows from (7.3.9). This proves Lemma 7.3.

(7.4) Now we shall prove the existence of real numbers, m, M and a set G', such that $G' \cup \Delta$ contains an open neighborhood U_0 of P, $U_0 \subset U$, and if $x \in G'$, (x, y_p), (x, y_q) are in V, and $\alpha(x, y_p) = y'_p$, $\alpha(x, y_q) = y'_1$, then

$$m < |(y'_p - y'_q)/(y_p - y_q)| < M. \tag{7.4.1}$$

This will prove Proposition (3.8), and hence Theorem (3.3).

Consider (using the notations of (5.9) and (5.10)) the polydisk

$$K(ijk) = \{z \in H^3(ij)/|z_k| < \delta\};$$

let $M(ijk) = \phi(ij)(K(ijk))$. Let

$$M'(ijk) = M(ijk) - E, \quad K'(ijk) = K(ijk) - \{z/z_1 \cdots z_i = 0\}$$

where $\phi(ij)^{-1}(E)$ is defined by $z_1 \cdots z_i = 0$. $K'(ijk)$ is not simply connected, but it can be expressed as the union of 2^i simply connected open subsets. In fact, $K(ijk) = \prod_{t=1}^r H(ijt)$, where $H(ijt) = H^3(ijt)$ if $t \neq k$,

$$H(ijk) = \{z \in H^3(ijk)/|z| < \delta\}$$

(cf. (5.10)), then for $t \leq i$, let

$$H^{(1)}(ijt) = \{z \in H(ijt)/z \notin \mathbf{R}(-)\},$$
$$H^{(2)}(ijt) = \{z \in H(ijt)/z \notin \mathbf{R}(+)\}.$$

Then, $K'(ijk)$ is the union of the 2^i possible products

$$\prod_{\tau=1}^{i} H^{(\lambda(\tau))}(ij\tau) \times \prod_{\tau=i+1}^{r} H(ij\tau) \subset K(ijk) \tag{7.4.2}$$

where $\lambda(\tau)$ is 1 or 2. These spaces are simply connected. We let $\mathscr{H}_l$, $l = 1, \ldots, 2^i$, be the sets in (7.4.2) (taken in a certain order). Fix v, $v = 1, \ldots, 2^i$; since $\mathscr{H}_v$ is simply connected, by analytic continuation we may find n holomorphic functions $\xi_1, \ldots, \xi_n$ on $\mathscr{H}_v$, such that

$$\xi_p^n + A_{v,1}\xi_p^{n-1} + \cdots + A_{vn} = 0 \quad \text{on} \quad \mathscr{H}_v, \quad p = 1, \ldots, n, \tag{7.4.3}$$

where $A_{v,t} = A_t q\theta(ij)$ (restricted to $\mathscr{H}_v$). Note that $\xi_p(x) \neq \xi_q(x)$ for all $x \in \mathscr{H}_v$, if $p \neq q$. The fundamental group Γ of $K'(ijk)$ is isomorphic to $\mathbf{Z}^i$, and it acts on the set $\{\xi_1, \ldots, \xi_n\}$ in an obvious way. Let $\rho_1, \ldots, \rho_i$ be a free basis of Γ. For $1 \leq t \leq i$, $1 \leq p \leq n$, let $r(t, p)$ be the smallest positive integer such that $\rho_t^{r(t,p)}\xi_p = \xi_p$; it is immediate to check that these numbers exist. For $p \neq q$, $1 \leq p, q \leq n$, let $\mathscr{G}_{pqv} = \prod_{t=1}^{r} \mathscr{G}_v^t$, where $\mathscr{G}_v^t = H(ijt)$ for $t > i$; and for $t \leq i$,

$$\mathscr{G}_v^t = \{z \in \mathbf{C}/ \,|\, z\,|^{r(t,p)r(t,q)} < \varepsilon(t, v)\},$$

where $\pi(t, v) = \varepsilon^3(ijt)$ if $t \neq k$, and $\varepsilon(k, v) = \delta$. Next consider the mapping u_{pqv}: $\mathscr{G}_{pqv} \to K(ijk)$, given by

$$u_{pqv}(z) = z'; \quad z'_t = (-1)^{\lambda(\tau)+1} z_t^{r(tp)r(tq)}, \; t \leq i; \quad z'_t = z_t, \; t > i \tag{7.4.4}$$

($\lambda(\tau)$ is the index in (7.4.2)). Let

$$S_{pqv} = \{z \in \mathscr{G}_{pqv}/0 < \arg(z_t) < 2\pi(r(tp) \cdot r(tq))^{-1}, t = 1, \ldots, i\},$$

then S_{pqv} is isomorphic to $\mathscr{H}_v$ under u_{pqv}. Clearly, by our choice of the numbers $r(t, p)$ and Riemann's extension theorem, the functions induced by ζ_p, ζ_q on S_{pqv} can be extended to holomorphic functions ζ'_p, ζ'_q on $\mathscr{G}_{pqv}$.

Now consider the functions

$$a(i, j, k) = a^3(i, j, k; \delta)\colon R^3(i, j, k, \delta) \to \mathbf{C} \tag{7.4.5}$$

(cf. (6.6.3) and (6.6.4)). Let a_p: $S_{pqv} \to \mathbf{C}$ be defined by

$$a_p(x) = a(i, j, k)(q\theta(i, j)u_{pqv}(x), \zeta_p(x)). \tag{7.4.6}$$

Note that a_p can be extended to a function $\tilde{a}_p \in \Gamma(\mathscr{G}_{pqv})$. This follows from the definition of u_{pqv} (cf. (7.4.4)) and the fact that $(x, y) \to (x, a(i, j, k)(x, y))$ is a well-defined morphism from $R^3(i, j, k; \delta) \subset V$ into V', commuting with the projections, and hence $\rho^{r(t,p)}a_p = a_p$ for $t = 1, \ldots, i$, $p = 1, \ldots, n$. Thus, on $\mathscr{G}_{pqv}$

it makes sense to form the quotient

$$(\tilde{a}_p - \tilde{a}_q)/(\zeta'_p - \zeta'_q) = \psi_{pqv}. \tag{7.4.7}$$

(7.5) We claim that this meromorphic function is actually holomorphic and never vanishing on $\mathscr{G}_{pqv}$. We shall prove this by showing that it has neither zeros or poles. In this section the indices v, p, q remain fixed, so we suppress them, i.e., $\psi = \psi_{pqv}$, $\mathscr{G} = \mathscr{G}_{pqv}$, $u = u_{pqv}$, etc. It is clear from the construction that ψ is a unit outside $z_1 \cdots z_i = 0$, hence the only possible zeros or poles could be the subvarieties $z_t = 0$, $t = 1, \ldots, i$.

First, we see that ψ takes the value 1 on the subvariety Y given by $z_k = 0$. Assume first it is not a pole. Hence, for an open dense in Y, ψ is defined and holomorphic. Let $Q_0 = (a)$ be a point where it is defined. If we choose any (real) arc γ ending at Q_0, and contained in $\{z/z_1 \cdots z_i \neq 0\}$, except for its end point Q_0, by continuity we have $\psi(Q_0) = \lim \psi(Q)$ for Q tending to Q_0 along γ. In particular, we can choose γ inside

$$\Lambda'_a = \{z \in \mathscr{G}_v/z_t = a_t \quad \text{for} \quad t \neq k\}.$$

If σ is any positive real number, let $\Lambda'_a(\sigma) = \{z \in \Lambda'_a/|z_k| < \sigma\}$. If σ is small enough, it is clear from our constructions that, letting $\theta = q\theta(ij)u$, we have $\theta(\Lambda'_a(\sigma)) \subset \Lambda$, for some curve Λ in some system $\mathscr{L}_b$, and for each $Q \in \Lambda'_a(\sigma)$,

$$\psi(Q) = (y'_s - y'_t)/(y_s - y_t) \tag{7.5.1}$$

(with the notations of (7.3.1)), where (x, y_s), (x, y_t) are suitable points of V lying over $\theta(Q)$. But, by Lemma (7.3), if $Q_0 = (a)$ was chosen outside some lower dimensional subvariety, then the expression (7.5.1) tends to 1 as Q tends to Q_0 along γ.

This worked with the assumption that ψ did not have a pole along Y. If it does, consider ψ^{-1}, and use the above argument with ψ^{-1}, which has a zero along Y. Thus ψ takes the value 1 on an open dense of Y, hence on all of Y. From this is easily follows that ψ has neither zeros nor poles on $\mathscr{G}_v$: were $z_t = 0$ a zero, then the value of ψ at, say, a point z with $z_t = z_k = 0$, $z_s = \varepsilon$, ε real small enough, $s \neq t, k$, should be simuntaneously 1 and 0. Similarly if $z_t = 0$ is a pole, by considering ψ^{-1} (which again takes the value 1 on Y) we get a contradiction.

(7.6) So, is we replace in the definition of $\mathscr{G}_{pqv}$ the numbers $\varepsilon^3(ijt)$, δ by smaller numbers $\varepsilon'(ijt)$, δ', we get a polydisk $\mathscr{G}'_{pqv} \subset \mathscr{G}_{pqv}$ and positive real numbers

$$m_{pqv}(\{\varepsilon'(ijt)\}, \delta'), \quad M_{pqv}(\{\varepsilon'(ijt)\}, \delta') \tag{7.6.1}$$

such that

$$m_{pqv}(\{\varepsilon'(i, j, t)\}, \delta') < |\psi_{pqv}(Q)| < M_{pqv}(\{\varepsilon'(ijt)\}, \delta') \tag{7.6.2}$$

for $Q \in \mathscr{G}'_{pqv}$ (because now, ψ is the restriction of a function defined on a compact). In particular, using the compactness of $E(P)$, we can choose those numbers $\varepsilon'(ijt)$, δ' so that:

(a) $\mathscr{U} = \cup M'(ijk, \delta')$ is a neighborhood of $E(P)$, where

$M'(ijk, \delta') = \{Q \in M(ijk)/Q$ has coordinates (z), $|z_t| < \varepsilon'(ijt), t \neq k$, $|z_k| < \delta'\}$.

(b) $\delta' < \delta$.

(7.6.3) Let $m = \min \{m_{pqv}(\{\varepsilon'(ijt)\}, \delta')\}$, $M = \{M_{pqv}(\{\varepsilon'(ijt)\}, \delta')\}$ (both taken over all possible values of the indices, which is a finite set). Then we claim that $G' = q(\mathscr{U} - E)$, and m, M just defined have the properties stated at the beginning of (7.4).

It is trivial that $G' \cup \Delta = U_0$ is a neighborhood of P, contained in U. To see that (7.4.1) holds, first note that since $G' \subset U - \Delta$,

$$\phi\colon \pi^{-1}(G') \to \pi'^{-1}(G')$$

is well defined, moreover $\phi(x, y) = (x, a_{b\alpha}(x, y))$ for $(x, y) \in \pi^{-1}(G') \cap N_{b\alpha}$, where $a_{b\alpha} = a^3_{b\alpha}(\delta) = a(ijk)$, (cf. (7.4.5)); here $\tau(b) = (i, j, k; (s))$. Let $(x) \in G'$, (x, y_p), (x, y_q) be in V; then, $q^{-1}(x) \in M'(ijk, \delta')$. Choose an index v such that $(x) \in q\theta(ij)(\mathscr{H}_v)$. We shall identify (to simplify notation) x with $[q\theta(ij)]^{-1}(x)$. Choose functions ζ_s, ζ_t, in $\Gamma(\mathscr{H}_v)$ such that $\zeta_s(x) = y_p$, $\zeta_t(x) = y_q$. We have

$$S' \subset \mathscr{G}'_{pqv} \xrightarrow{\xi} G'$$

with $S' = S_{pqv} \cap \mathscr{G}'_{pqv}$, and $\xi = q\theta(ij)u_{pqv}$. Then ξ induces an isomorphism $S' \approx q(M'(ijk, \delta') \cap \mathscr{H}_v)$. Let $z \in S'$ be such that $\xi(z) = x$. Then, it is clear that

$$|y'_p - y'_q/y_p - y_q| = |\tilde{a}_s - \tilde{a}_t/\xi'_s - \xi'_t| = |\psi_{pqv}| \tag{7.6.4}$$

(cf. (7.4.7)). By our choice in (7.6.3), this and the inequality (7.6.2) imply (7.4.1). This concludes the proof of Proposition (3.8), and hence of Theorem (3.3).

8. Some Applications

(8.1) In this section we present some applications of Theorem (3.3) to problems of algebraization of analytic singularities.

We start by recalling some well-known results. Given an algebraic variety V over $\mathbf{C}$, there is a canonically associated analytic variety, denoted V^h, whose underlying set is $V(\mathbf{C})$, the set of closed points of V (see [9]). Given an analytic variety V and a point $P \in V$, we say that V is algebraic at P, or that P is an algebraic point of V, if there is an algebraic variety W and a closed point $Q \in W$, such that $\mathscr{O}_{V,P}$ is isomorphic to the local ring of W^h at Q. This is equivalent to the assertion that V and W^h be locally isomorphic (about P, Q respectively).

Let $A = C[x_1 \cdots x_n]_M$, where $M = (x_1, \ldots, x_n)$, (i.e., the algebraic local ring of the origin of $\mathbf{C}^n$), then

$$h(A) = \text{henselization of } A = \{a \in \mathbf{C}\{x_1, \ldots, x_n\}/a \text{ is algebraic over } A\}.$$

This is a local ring. It is customary to denote the henselization of A as $\tilde{A}$, we use the notation $h(A)$ to avoid confusion with saturations.

(8.2) If a germ of analytic set V, about the origin P of $\mathbf{C}^n$, corresponds to an ideal $I \subset C\{x_1 \cdots x_n\}$ generated by elements $f_1, \ldots, f_m$ in $h(A)$, then V is algebraic at P (see [2], 18.6).

Finally, recall:

(8.3) ARTIN'S APPROXIMATION LEMMA. *Given polynomials*

$$f_i \in \mathbf{C}[x_1, \ldots, x_n, y_1, \ldots, y_m], \quad i = 1, \ldots, s,$$

assume $y_i(x) \in C[[x_1 \cdots x_n]]$ *satisfy*

$$f_i(x, y_i(x), \ldots, y_m(x)) = 0, \quad i = 1, \ldots, s,$$

(*where* $x = (x_1, \ldots, x_n)$). *Fix any integer* d. *Then, there are elements* $\bar{y}_j(x) \in h(A)$ *such that* $y_j(x) \equiv \bar{y}_j(x) (\mathrm{mod}\ (x)^d)$, $j = 1, \ldots, m$ *and*

$$f_i(x, \bar{y}, (x), \ldots, \bar{y}_m(x)) = 0, \quad i = 1, \ldots, s.$$

(See [1], Theorem 1.10.)

(8.4) THEOREM. *Let V be an analytic hypersurface, defined near a point P of $\mathbf{C}^{r+1}$, and let $\mathfrak{O} = \mathcal{O}_{V,P}$. Assume there is a non-singular variety H and a finite, strictly surjective morphism π: $V \to H$* (cf. (1.2) (g), *defined near P, such that $\pi(P)$ is an algebraic point of Δ, the branch locus of π. Then, there is an algebraic hypersurface W, a point $Q \in W$, local parameters (x) of $\mathfrak{O}$, (x') of $\mathfrak{O}'$ (the local ring of W^h at Q (cf. Remark* (1.2 (f)), *and an isomorphism ϕ: $\tilde{\mathfrak{O}}_x \to \tilde{\mathfrak{O}}'_x$ such that $\phi(x_i) = x'_i$ for all i (we use the notation of* (1.4)).

Proof. It is easy to see that, up to isomorphism, the situation may be assumed as follows: V is given near P, the origin of $\mathbf{C}^{r+1}$ (with coordinates $x_1, \ldots, x_r, y$) by the equation

$$y^n + A_1(x)y^{n-1} + \cdots + A_n(x) = 0, \tag{8.4.1}$$

H is the hyperplane $y = 0$, π is induced by the projection $(x, y) \to (x, 0)$ and if δ_0 is the reduced discriminant of (1) with respect to y (i.e., if δ is the discriminant of (8.4.1) $= \prod p_i^{\alpha_i}$, and p_i is not associated with p_j for $i \neq j$, then $\delta_0 = \prod p_i$), then $\delta_0 = \varepsilon P_0$, where ε is a unit and P_0 is a polynomial in $x_1, \ldots, x_r$. Then, since δ and δ_0 have the same prime factors, we must have

$$\delta_0^m = \varepsilon^m P_0^m = \delta F, \quad \text{for some integer } m > 0, \tag{8.4.2}$$

$$\delta = \delta_0 G = \varepsilon P_0 G, \tag{8.4.3}$$

where F, G are power series, convergent near P. Moreover, there is a polynomial $Q(T_1, \ldots, T_n)$ with integral coefficients such that

$$Q(A_1, \ldots, A_n) = \delta. \tag{8.4.4}$$

Now, look at (8.4.2), (8.4.3) and (8.4.4) as polynomial equations in A_i, δ_0, δ, F, G, ε, with coefficients in $\mathbf{C}[x]$ (note that $P_0 \in \mathbf{C}[x]$). The equalities above show that these equations have a solution in $\mathbf{C}\{x\}$. Take $d = c + 1$, where c is the number of Theorem (3.3). By (8.3), we find B_i, δ'_0, F', G', ε' in $h(A)$ congruent modulo $(x)^d$ to A_i, δ_0. etc. Hence

$$\varepsilon'^m P_0^m = \delta' F', \tag{8.4.5}$$

$$\delta' = \varepsilon' P_0 G', \tag{8.4.6}$$

$$Q(B_1, \ldots, B_n) = \delta'. \tag{8.4.7}$$

Consider the hypersurface W given near P by

$$g(y) = y^n + B_1(x)y^{n-1} + \cdots + B_n(x) = 0. \tag{8.4.8}$$

By (8.4.7), the discriminant of g with respect to y is δ'; by (8.4.5) and (8.4.6), δ' and P_0 have the same prime factors and therefore, since P_0 was a power series without multiple factors, P_0 is the reduced discriminant of g. That is, the branch locus of W is again given by $P_0 = 0$. Since $\{B_i(x)\}$ are in $h(A)$, by (8.2), W is algebraic at P. By Theorem (3.3), since $A_i \equiv B_i \bmod (x)^{c+1}$, we have $\mathfrak{O}_x \approx \mathfrak{O}'_x$ (where $\mathfrak{O}$, $\mathfrak{O}'$ are the local rings of V, W at P respectively), by an isomorphism leaving $\{x_1\}$ fixed. This proves the theorem.

(8.5) *Remark.* Note that these two embedded singularities are topologically equivalent (cf. (2.3)).

(8.6) COROLLARY. *Let V be an analytic hypersurface of dimension 2. Then, there is an algebraic hypersurface W, a point $Q \in W$, local parameters (x), (x') of the local rings $\mathfrak{O}$, $\mathfrak{O}'$ of V and W^h at P and Q respectively, and an isomorphism $\phi\colon \tilde{\mathfrak{O}} \to \tilde{\mathfrak{O}}'_{x'}$, such that $\phi(x_i) = x'_i$, for all i.*

Proof. Take a morphism π with the properties of π in Theorem (8.4) (e.g., a "general" projection into a plane in $\mathbf{C}^3$). Then the branch locus Δ is a plane curve, hence it is algebraic at $\pi(P)$ (cf. [4]). So we are automatically in the conditions of Theorem (8.4), and the corollary is proved.

(8.7) *Remark.* If, in the conditions of Theorem (8.4), for certain regular parameters $h_1, \ldots, h_r$ of the local ring of H at $\pi(P)$ the induced parameters $x_i = \pi^*(h_i)$ form a generic system of local parameters of $\mathcal{O}_{V,P}$ (cf. (2.5)), then using (2.7) and (2.9) we see that the absolute saturations of $\mathcal{O}_{V,P}$ and of $\mathfrak{O}'$ are isomorphic. Note that if $\dim V = 2$, then we can get (use suitable coordinates $x_1, \ldots, x_r$, y about P, and project into $y = 0$) a morphism π and parameters x_i which are generic. Hence we get:

(8.8) THEOREM. *Given an analytic hyperface V, of dimension 2, and a point $P \in V$, there is an algebraic hypersurface W and a point $Q \in W$, such that the* *absolute saturations of the respective analytic local rings at P and Q are isomorphic.*

With Theorem (8.8), one can prove the following known result:

(8.9) COROLLARY. *We make the same assumptions as in Theorem* (8.4), *but now we also suppose $\mathfrak{O}$ to be normal. Then, P is an algebraic point of V.*

Proof. Keeping the notations of Theorems (8.4), there is an algebraic variety W such that $\tilde{\mathfrak{O}}_x$ is isomorphic to $\tilde{\mathfrak{O}}'_{x'}$. Then, the normalizations of $\mathfrak{O}$ and $\mathfrak{O}'$ are again isomorphic. But the normalization $\tilde{\mathfrak{O}}$ of $\mathfrak{O}$ is $\mathfrak{O}$, since $\mathfrak{O}$ is normal; and the normalization of $\mathfrak{O}'$ is algebraic, since the normalization of an algebraic variety is algebraic. Hence, the local ring $\mathfrak{O}$ is algebraic, and the corollary follows.

Again, if $\mathfrak{O}$ is embedded, normal of dimension 2, then the conditions of Corollary (8.9) are satisfied and $\mathfrak{O}$ is algebraic. It is not difficult to see, using these techniques and the local parametrization theorem (see [3], Ch. III.A) that we can drop the condition that $\mathfrak{O}$ is embedded. Thus one can get a different proof of the well-known result that any normal two dimensional singularity is algebraic.

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