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A COUNTING FUNCTION FOR ORIENTATION REVERSING MAPS

BY

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1. Introduction

In this article we count the number of conjugacy classes in the diffeomorphism group, of orientation reversing self-diffeomorphisms of order $2p$, where p is a prime, which act on an orientable compact surface of genus n. To calculate this number we rely heavily on previous work of the author [5], [6] and [7]. We give some notation which is fixed throughout the entire article. Let X be a compact smooth surface of genus $n \geq 2$ and let g be an orientation reversing self-diffeomorphism of X of order $2p$. Let $f = g^2$, $X' = X/\langle f \rangle$, and let g' be the map induced by g on X'. Clearly g' is orientation reversing of order two. If p is odd then g^p is also orientation reversing of order two. We let c (resp. pd) denote the number of loops on X which are fixed pointwise by g^p and fixed by f (resp. permuted by f). It follows by [5] that f has an even number $2a$ of fixed points. By the Riemann-Hurwitz formula $n - 1 = p(m - 1) + a(p - 1)$, where m is the genus of X'. If X is given a conformal structure so that g is anti-conformal, then X may be embedded in $\mathbf{R}^3$ so that f becomes the restriction of a rotation. We denote the angle of rotation of f by $\alpha(f)$ and normalize by requiring $0 < \alpha(f) < 2\pi$.

We denote by $\phi(n, p)$ the number of conjugacy classes in the diffeomorphism group of X, of orientation reversing self-diffeomorphisms g which act on X. We first calculate $\phi(n, p)$ in the case $p = 2$. When p is odd we consider separately three cases. We say g of type one if g' has fixed points and $X'/\langle g' \rangle$ is orientable, g is of type two if g' has fixed points and $X'/\langle g' \rangle$ is non-orientable, and g is of type three if g' has no fixed points. If g is of type three we necessarily have that $X'/\langle g' \rangle$ is non-orientable. Also if p is odd then X'/g' is orientable if and only if X/g^p is orientable and g' has fixed points if and only if g^p does.

Our main result is the following.

THEOREM 1.1. *If X is a compact surface of genus n, then the number of conjugacy classes in the diffeomorphism group of X, of orientation reversing maps on X of order $2p$, where p is a prime, is given by the formula*

$$\phi(n, 2) = [(n + 1)/2]$$

$$\phi(n, p) = \phi_1(n, p) + \phi_2(n, p) + \phi_3(n, p), \quad p > 2,$$

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where $\phi_i(n, p)$, *the number of conjugacy classes of type i, are given in* 3.4, 4.3 *and* 5.3.
Note. Conjugacy classes are always considered in the diffeomorphism group of X.

2. $p = 2$

LEMMA 2.1. *The number* $\phi(n, 2)$ *is the number of pairs* (m, a) *which satisfy the equations* (1) $n = 2m + a - 1$ *and* (2) $m - a \equiv 1 \bmod 2$, *where* $m \geq 0, a \geq 0$, *and* $a + m > 0$.

Proof. Using the notation of Section 1, if g is an orientation reversing map of order four, then m and a must satisfy (1) and (2) by 1.1 and 2.1 of [5]. Conversely for each pair of numbers m and a which satisfy (1) and (2) above, we construct an orientation reversing map g of order four such that g^2 has $2a$ fixed points and $X/\langle g^2\rangle$ has genus m. By 1.1 of [5] the conjugacy class of g is determined by a, so this is sufficient to prove the lemma. To construct a map g we let Y be a surface of genus $n - m$ and let F be the hyperelliptic involution. By (1) and (2), $n - m$ is even and thus by [3], F has an orientation reversing square root G. We let p_i and $p'_i = G(p_i)$, $i = 1, 2, \ldots, m$ be a set of distinct fixed points of F and let D_i and D'_i be discs about p_i and p'_i respectively, such that $F(D_i) = D_i$ and $G(D_i) = D'_i$. Then by the same argument used in 2.2 of [5] there are maps W_i: $\partial D_i \to \partial D'_i$ such that if we identify $x \in \partial D$ to $W_i(x) \in \partial D'_i$, the map G induces an orientation reversing map g on the resulting surface X. Clearly X has genus n, g^2 has $2a$ fixed points, and $X/\langle g^2\rangle$ has genus m.

PROPOSITION 2.2. $\phi(n, 2) = [(n + 2)/2]$.

Proof. By 2.1 we need only count the number of solutions to (1) and (2). By considering separately the cases in which n is even and n is odd, it is easy to show that $\phi(n, 2) = (n + 2)/2$ and $\phi(n, 2) = (n + 1)/2$, respectively. Hence in general $\phi(n, 2) = [(n + 2)/2]$.

3. g is of type one, i.e. g' has fixed points and X'/g' is orientable

We calculate the function $\phi_1(n, p)$ of all conjugacy classes of orientation reversing maps g of type one. Before doing this, however, we need a combinatorial lemma which we will also use in subsequent sections.

LEMMA 3.1. *Let* $a = a_0$, $m = m_0$ *be the solution of* $n - 1 = p(m - 1) + a(p - 1)$ *for which a is smallest, and m is largest. Then the values of* a_0 *and* m_0 *are described in the table below. The number of solutions to the above equation is* $[m_0/(p - 1)] + 1$. *Also the solutions* (a, m) *are given by* $a = a_i = a_0 + ip$ *and* $m_i = m_0 = i(1 - p)$, $i = 0, 1, \ldots, [m_0/(p - 1)]$.

n	a_0	m_0
$n \equiv 1 \bmod p$	0	$(n+p-1)/p$
$n \equiv 0 \bmod p$	1	n/p
$n \not\equiv 0$ or $1 \bmod p$	$p(1+[n/p])+1-n$	$(1+[n/p])(1-p)+n$

Proof. Let q and r be integers such that $n/p = q + r/p$, $0 \le r < p$. Then

$$(n + a - 1)/p = q + (r + a - 1)/p.$$

The given equation implies that $p \mid (n + a - 1)$, so that we must have $p \mid (r + a - 1)$. If $r = 0$ or $r = 1$ then the smallest possible values for a are $a = 1$ and $a = 0$, respectively. This gives the first two lines of the above table. If $n \not\equiv 0$ or $1 \bmod p$ then $p > r > 1$ and the smallest value of a occurs when $r + a - 1 = p$, so that $a = p + 1 - r = p + 1 - p(n/p - [n/p])$. It is an easy matter to calculate the corresponding values of m.

If we list the pairs (a, m) which satisfy $n - 1 = p(m - 1) + a(p - 1)$ in order of increasing values of a, then we obtain a finite sequence (a_i, m_i), $i = 0, 1, 2, \ldots, k$. Since $p \mid (a_i + r - 1)$ we must have $a_{i+1} = a_i + p$. Also $m_{i+1} = m_i + 1 - p$. Thus $a_i = a_0 + ip$ and $m_i = m_0 + i(1 - p)$. The largest value of i such that $m_i \ge 0$ is thus $[m_0/(p - 1)]$. Hence there are $[m_0/(p - 1)] + 1$ solutions.

LEMMA 3.2. *Let $\alpha(f) = \alpha$ be fixed. Then the number of conjugacy classes of orientation reversing maps g of type one with $g^2 = f$ is the number $\psi_1(n, p)$ of 4-tuples (a, m, c, d) which satisfy:*

(1) $n - 1 = p(m - 1) + a(p - 1)$, $m \ge 0$;
(2) $m + 1 \equiv c + d \bmod 2$;
(3) $n + 1 \equiv c + pd \bmod 2$;
(4) $a + c \equiv 0 \bmod 2$;
(5) $c \ge 0, d \ge 0, c + d > 0, m \ge c + d - 1, n \ge c + pd - 1$.

Proof. Assume g is a map of type one and let a, m, c, d be as defined in Section 1. Equation (1) is immediate from the Riemann-Hurwitz formula; (2) and (3) follow from the fact that X' and X are doubles of surfaces with $c + d$ and $c + pd$ boundary components; and (4) follows by 2.3 of [7].

Conversely, one may construct a surface X with an orientation reversing map g such that $f = g^2$ has $2a$ fixed points, $\alpha(f) = \alpha$, g^p fixes $c + pd$ loops pointwise, c of which are fixed by f and pd of which are permuted by f, and X' has genus m. We first let Y be a surface of genus $n + 1 - (c + pd)$ with no boundary components. This surface has an embeddable map F of order p with $a + c$ fixed points. One may see this by observing figures 1 and 2 of [2]. Let $\alpha = 2\pi j/p$, $1 \le j < p$. If j is even then some power of F, say H has $\alpha(H) = 2\pi j/2p$. If j is odd, then some power of F, say H has $\alpha(H) = 2\pi(j + 1)/2p$. Thus in both cases $\alpha(H^2) = 2\pi j/p = \alpha$. Again by observing Figures 1 and 2 of [2] it is easy to see that Y may be embedded so that Y is invariant under a reflection K in the

$x - y$ plane which fixes pd loops which are permuted by H. Now we let p_i and $K(p_i) = p_i'$, $i = 1, 2, \ldots, a$ be fixed points of H and let D_i and D_i' be open discs about p_i and p_i', respectively, with the property that $H(D_i) = D_i$, $H(D_i') = D_i'$, $K(D_i) = D_i'$ and $K(D_i') = D_i$. We now remove D_i and D_i', $i = 1, \ldots, a$, and glue ∂D_i to $\partial D_i'$ by identifying $x \in \partial D_i$ with $K(x) \in \partial D_i'$. We then obtain a surface X of genus n. The maps K and H induce maps K' and H' on X and $g = H'K'$ has the desired properties.

LEMMA 3.3. *The number $\psi_1(n, p)$ of 4-tuples (a, m, c, d) which solve (1)–(5) of 3.2 is given by*

$$\psi_1(n, p) = \begin{cases} \sum_{i=0}^{k} \left(\sum_{e=1}^{k_1} [(e+2)/2] + \sum_{e=1}^{k_{i+1}} [(e+1)/2] \right) - j, & a_0 \text{ even}, \\ \sum_{i=0}^{k} \left(\sum_{e=1}^{k_{i+1}} [(e+2)/2] + \sum_{e=1}^{k_i} [(e+1)/2] \right) - j, & a_0 \text{ odd}, \end{cases}$$

where $k = [m_0/(p-1)]$, $k_i = [(m_0 + i(1-p) + 2)/2]$, a_0 and m_0 are obtained from 3.1, $j = 1$ if $n \equiv 1 \bmod p$ or $n \equiv 0 \bmod p$ and $j = 0$ otherwise.

Proof. We first remark that (3) is redundant. We let $m + 1 = c + d + 2j$, where $j \geq 0$, and if we substitute for m in (3) we obtain

$$n + 1 = c + pd + 2(p(j-1) + 1 + (a+c)(p-1)/2.$$

By (4), $(a+c)(p-1)/2$ is an integer, so that $n + 1 \equiv c + pd \bmod 2$.

To count the number of solutions we fix a solution (m, a) of (1) and count the number of pairs (c, d) which satisfy (2), (4) and (5). Let $e = c + d$. If we require that $e > 0$ then there are $[(m+2)/2]$ solutions to the equations $m + 1 \equiv e \bmod 2$, as can be seen by considering the cases m even and m odd separately. For each such value e there are $e + 1$ ordered pairs (c, d), with $c \geq 0$ and $d \geq 0$, such that $e = c + d$. Thus for a fixed value of e, a and m there are $(e+2)/2$ pairs (c, d) which satisfy (2) and (4) if a is even, and $(e+1)/2$ pairs if a is odd. Now fix a solution (a, m) of (1). Then there are $\sum_{e=1}^{k} [(e+2)/2]$ or $\sum_{e=1}^{k} [(e+1)/2]$, $k = [(m+2)/2]$, pairs (c, d) which satisfy (2) and (4), depending on whether a is even or odd. We remark that if a_i is even then a_{i+1} is odd. The condition (5) will be satisfied provided we do not have both $a + c \leq 2$ and $c + d = m + 1$. This can only arise if $a = 0$, $c = 0$, $d = m + 1$, or $a = 1$, $c = 1$, $d = m - 1$, which in turn only occurs when $n \equiv 1 \bmod p$ or $n \equiv 0 \bmod p$. The formula now follows directly.

PROPOSITION 3.4.

$$\phi_1(n, p) = \begin{cases} ((p+1)/2)\psi_1(n, p) & \text{if } n \not\equiv 1 \bmod p \\ \sigma_1(n, p) + ((p+1)/2)\tau_1(n, p) & \text{if } n \equiv 1 \bmod p \end{cases}$$

where

$$\sigma_1(n, p) = \sum_{e=1}^{k} [(e+2)/2] - 1, \quad k = [(m_0+2)/2],$$

$$\tau_1(n, p) = \psi_1(n, p) - \sigma_1(n, p) \quad \text{and} \quad m_0 = (n+p-1)/p.$$

Proof. The conjugacy class of an embeddable map f with $2a > 0$ fixed points is determined by $\alpha(f)$. It follows by Nielsen's Theorem [1, p. 53], that f^i and f^j, $1 \leq i < j \leq p$ are conjugate iff $j = p - i$. Thus there are $(p+1)/2$ conjugacy classes for f if f has fixed points. If $n \equiv 1 \bmod p$ then $a_0 \neq 0$ so by 3.3 we conclude that

$$\phi_1(n, p) = \tfrac{1}{2}(p+1)\psi_1(n, p).$$

If $n \equiv 1 \bmod p$ then $a_0 = 0$ and by the argument used in 3.3 for a fixed value of $\alpha(f) = \alpha$, there are $\sigma_1(n, p)$ conjugacy classes of g with no fixed points and $\tau_1(n, p)$ conjugacy classes of g with fixed points. Nielsen's theorem [1] implies that if f is fixed point free, it is conjugate to all of its non-trivial powers. Thus

$$\phi_1(n, p) = \sigma_1(n, p) + \tfrac{1}{2}(p+1)\tau(n, p)$$

in this case.

4. g is of type two, i.e. g' has fixed points and $X'/\langle g' \rangle$ is non-orientable

We first make some preliminary remarks. We note that g' has fixed points iff g^p does (3.3 [7]), and $X'/\langle g' \rangle$ is non-orientable iff X/g^p is (2.1 [7]). Now let Y be a surface and K an orientation reversing map of order two with the property that $Y/\langle K \rangle$ has boundary components. As in Section 3 of [7], we define an annular region for K to be a region A homeomorphic to an annulus, with the property that $A/\langle K \rangle$ is a moebius strip. By 3.4 of [7], we know that there are either one or two annular regions for g^p on X, each of which is fixed by f and hence projects to an annular region for g' on X'. If we remove these annular regions from X and X', then the quotients of the resulting surfaces by the maps induced by g and g', respectively, are orientable. Also, the number of annular regions depends only on the topological type of $X/\langle g^p \rangle$. Thus let $e = 1$ or 2 be the number of annular regions for g^p (and hence also g').

We have the following analogue of 3.2.

LEMMA 4.1. *Let $\alpha(f) = \alpha$ be fixed. Then the number of conjugacy classes of orientation reversing maps of type two is the number of 5-tuples (a, m, c, d, e) which satisfy the following.*

(1) $n - 1 = p(m-1) + a(p-1)$, $a \geq 0$, $m \geq 0$.

(2) $m + 1 \equiv c + d + e \bmod 2$.

(3) $n + 1 \equiv c + e + pd \bmod 2$.

(4) $a + c + e \equiv 0 \bmod 2$.

(5) $c \geq 0$, $d \geq 0$, $2 \geq e \geq 1$, $c + d > 0$, $m \geq c + d + e - 1$, $n \geq c + pd + e - 1$.

Proof. We remark that if g is of type two, then by an argument similar to that used in 3.2 the conditions (1)–(3) and (5) may be verified. One may prove (4) by an argument similar to that used in 2.3 of [7]. Thus to each orientation reversing map of type two we may associate a 5-tuple (a, m, c, d, e) and clearly this 5-tuple is determined by the conjugacy class of g.

Now let (a, m, c, d, e) be an arbitrary 5-tuple which satisfies conditions (1)–(5). By 3.2 we may construct a surface Y of genus n with an orientation reversing map G of order $2p$, such that if $F = G^2$, then $\alpha(F) = \alpha$, F has $2a$ fixed points, and $X/\langle F\rangle$ has genus m. Also there are $c + e$ loops which are fixed pointwise by G^p and fixed by F and pd loops which are fixed pointwise by G^p and permuted by F.

We now construct a surface X and an orientation reversing map g. We choose e loops which are fixed pointwise by G^p and fixed by F. We first consider the case $e \equiv 1$. Thus let γ denote this loop. The surface Y may be cut along γ to obtain a surface Y' with two boundary components on which G^p and F both induce maps K and H, respectively. Clearly K and H commute. Let γ_i: $S^1 \to Y'$, $i = 1, 2$, be parametrizations of these boundary components with the property that

$$K(\gamma_1(\exp i\theta)) = \gamma_2(\exp i\theta) \quad \text{and} \quad H(\gamma_1(\exp i\theta)) = \gamma_1(\exp i(\theta + \alpha)).$$

Then $K(\gamma_2 \exp i\theta)) = \gamma_1(\exp i\theta)$ and $H(\gamma_2(\exp i\theta)) = \gamma_2(\exp i(\theta + \alpha))$. Now identify $\gamma_1(\exp i\theta)$ with $\gamma_2(\exp i(\theta + \alpha))$. We thus obtain a surface X of genus n. The maps K and H induce maps k and f, respectively, on X. If we let $g = kf^j$, $j = (p + 1)/2$, then $g^2 = f$ and $g^p = k$ and there are c loops which are fixed pointwise by g^p and fixed by f and pd loops which are fixed pointwise by g^p and permuted by f. If $e = 2$ then a similar argument may be used. Thus to each 5-tuple (a, m, c, d, e) satisfying (1)–(5) we may associate an orientation reversing map g of type two. This completes the proof.

LEMMA 4.2. *The number of* 5-*tuples* (a, m, c, d, e) *which satisfy* (1)–(5) *is given by*

$$\psi_2(n, p) = \begin{cases} \sum_{i=0}^{k} \left(\sum_{r=2}^{k_i} [r/2] + \sum_{r=2}^{k_{i+1}} [(r + 1)/2] \right) - j, & a_0 \text{ even}, \\ \sum_{i=0}^{k} \left(\sum_{r=1}^{k_i} [r/2] + \sum_{r=1}^{k_{i+1}} [(r + 1)/2] \right) - j, & a_0 \text{ odd}, \end{cases}$$

where $k = [m_0/(p - 1)]$, $k_i = [(m_0 + i(1 - p) + 1)/2]$, a_0, *and* m_0 *are obtained from* 3.1, *and* $j = 2$ *if* $n \equiv 1 \bmod p$, $j = 1$, *if* $n \equiv 0 \bmod p$, and $j = 0$ *otherwise.*

Proof. The proof of this lemma is analogous to that of 3.3. It is similarly true that (3) is redundant. Let $r = c + d + e$ and let (a, m) be a fixed solution of

(1). We first count the number of choices for r. Since $m + 1 = r + 2j$, for some j, and since $r \geq 2$, there are $[(m + 1)/2]$ possible choices for r. For each fixed choice of r we count the possibilities of writing $r = c + d + e$. If $e = 1$, then there are $r = r - 1 + 1$ choices for (c, d), such that (2) holds. Thus for fixed values of a, r, and m there are $[r/2]$ choices for (c, d) which satisfy (2) and (4) if a is even, and $[(r + 1)/2]$ choices if a is odd. If $e = 2$ then there are $r - 1 = r - 2 + 1$ choices for (c, d) such that (2) holds. Similarly, if we fix values for a, r and m, then there are $[r/2]$ choices for (c, d) which satisfy (2) and (4) if a is even, and $[(r + 1)/2]$ choices if a is odd.

The condition (5) will be automatically satisfied provided we do not have both $a + c + e \leq 2$ and $c + d + e = m + 1$. This can happen only when $e = 2$, $c = a = 0$, $d = m - 1$, or $e = 1$, $c = 1$, $a = 0$, $d = m - 1$, or $e = 1$, $c = 0$, $a = 1$, $d = m - 1$. The first two cases occur when $n \equiv 1 \bmod p$ and the last occurs when $n \equiv 0 \bmod p$. The formula now follows easily.

PROPOSITION 4.3.

$$\phi_2(n, p) = \begin{cases} \frac{1}{2}(p + 1)\psi_2(n, p) & \text{if } n \not\equiv 1 \bmod p, \\ \sigma_2(n, p) + \frac{1}{2}(p + 1)\psi_2(n, p) & \text{if } n \equiv 1 \bmod p, \end{cases}$$

where $\sigma_2(n, p) = \sum_{r=2}^{k} [r/2] - 2$, *and* $\tau_2(n, p) = \psi_2(n, p) - \sigma_2(n, p)$.

Proof. The proof is analogous to 3.4. We remark that $\sigma_2(n, p)$ is the number of conjugacy classes of maps of type two which have no fixed points.

5. g is of type three, i.e. g' has no fixed points

LEMMA 5.1. *Let $\alpha(f) = \alpha$ be fixed. Then the number of conjugacy classes of orientation reversing maps of type three is the number of pairs (a, m) which satisfy the equation $n - 1 = p(m - 1) + a(p - 1)$, where a is even if m is odd.*

Proof. We set up a one-one correspondence between pairs (a, m) satisfying the above conditions and conjugacy classes of maps of type three. By 1.1 of [7] the conjugacy class of a map g of type three determines the pair (a, m), and by 4.3 [7] a is even if m is odd.

We now construct a map g of type three given a pair (a, m). Let Y be a surface of genus n. By 3.2 we may construct a map G on Y of type one which corresponds to the 4-tuple (a, m, c, d). Here $c = 0$, $d = 1$ if a is even and $c = 1$, $d = 0$ if a is odd. If m is odd and a is even then by 4.1 we may construct a map G of type two on Y corresponding to the 5-tuple $(a, m, 1, 0, 1)$. In both cases we cut Y along the loops which are fixed pointwise by G^p and reglue as was done in 3.2 so that G induces a map g of type three on the resulting surface X.

LEMMA 5.2. *The number of pairs* (a, m) *satisfying the conditions of* 5.1 *is*

$$\psi_3(n, p) = \begin{cases} \left[\dfrac{m_0}{(p-1)}\right] + 1 & \text{if } m_0 \text{ is even,} \\ \left[\dfrac{m_0}{2(p-1)}\right] + 1 & \text{if } m_0 \text{ is odd and } a_0 \text{ is even,} \\ \left[\dfrac{(m_0 + 1 - p)}{2(p-1)}\right] + 1 & \text{if } m_0 \text{ is odd and } a_0 \text{ is odd.} \end{cases}$$

Here m_0 *and* a_0 *are obtained from* 3.1.

Proof. This follows immediately from 3.1 and 5.1.

PROPOSITION 5.3. *If* $\phi_3(n, p)$ *denotes the number of conjugacy classes of maps of type three then*

$$\phi_3(n, p) = \begin{cases} \frac{1}{2}(p+1)\psi_3(n, p) & \text{if } n \not\equiv 1 \bmod p, \\ \frac{1}{2}(p+1)(\psi_3(n, p) - 1) + 1 & \text{if } n \equiv 1 \bmod p. \end{cases}$$

Proof. This is analogous to 3.4 and 4.3. If $n \equiv 1 \bmod p$, then a_0 and by 5.2, there is exactly one conjugacy class with f fixed point free.

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