ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 3, Fall 1980

BOUNDARY BEHAVIOR OF AUTOMORPHIC FORMS AND TRANSITIVITY FOR THE MODULAR GROUP

BY

MARK SHEINGORN[1]

Introduction

This paper deals with a method of analyzing the boundary behavior (for nontangential approach) of the modulus of certain meromorphic automorphic forms on the modular group. The method is also applied to questions of transitivity for the modular group. The idea of the method may be simply stated: the line $\{z \mid z = \xi + iy,\ y > 0\}$ (or a Stolz angle about it) may be decomposed into a series of intervals each one of which is "essentially" mapped into the standard fundamental region for the modular group by a specific transformation. Both the decomposition and the accompanying transformations are given in terms of the continued fraction representation of ξ. Computations involving boundary values and transitivity are thus made more tractable; in the first instance using the functional equation, and in the second, certain geometric and measure theoretic considerations.

Section 1

Let $\Gamma(1)$ be the modular group and q a positive integer. Then $f(z)$ is said to be a meromorphic automorphic form (or simply, a form) on $\Gamma(1)$ of weight q (dimension $-2q$) if f satisfies

$$f(Vz)V'^q(z) = f(z) \tag{1}$$

for all $z \in H = \{z\colon \operatorname{Im}(z) > 0\}$ and all $V \in \Gamma(1)$. Let $H^+ = H \cup \{\infty\}$. Let ξ be an irrational point on the real axis with simple continued fraction representation $\xi = [a_0;\ a_1,\ a_2,\ a_3, \ldots]$. Let p_n/q_n be the *nth* convergent of ξ, so that $p_n/q_n = [a_0;\ a_1, \ldots, a_n]$.

Our principal results on boundary behavior (stated for positive q) are as follows.

THEOREM 1. *If ξ is such that for some $\varepsilon > 0$,*

$$a_{n+1} \geq \frac{2q + \varepsilon}{\pi\mu} \log q_n$$

Received July 31, 1978.

[1] Research partially supported by PSC-BHE Grant No. 11673.

for infinitely many n, then the modulus of any meromorphic form of weight q with a zero of order $\mu \geq 1$ at ∞ has no nontangential limit at ξ. (Indeed our proof shows that the cluster set of this modulus is $[0, \infty]$.)

Conversely:

THEOREM 2. *If ξ satisfies*

$$\left(\frac{2q}{\pi\mu T_{q,\mu}} \log q_n - a_{n+1}\right) \to \infty \quad as \quad n \to \infty,$$

then the modulus of any meromorphic form of weight q with a zero of order $\mu \geq 1$ at ∞ and vanishing nowhere else in H^+, approaches ∞ for nontangential approach to ξ. (Here $T_{q,\mu} > 1$ is an explicity computable constant depending only on q and μ and not ξ.)

Some condition on the nonvanishing of the form away from ∞ is essential for the method to work in Theorem 2. However, these theorems suffice to give a rather complete discussion of $|\Delta|$, where

$$\Delta(z) = e^{2\pi i z} \prod_{m=1}^{\infty} (1 - e^{2\pi i m z})^{24}, \quad \text{Im } z > 0.$$

COROLLARY 1 (Boundary behavior of $|\Delta|$). *If ξ is irrational and*

$$a_{n+1} \geq \frac{12 + \varepsilon}{\pi} \log q_n$$

for some $\varepsilon > 0$ and some infinite sequence of n's, then $|\Delta|$ has no nontangential limit at ξ. Conversely, if

$$\frac{12}{\pi T_{6,1}} \log q_n - a_{n+1} \to \infty \quad \text{as} \quad n \to \infty,$$

then $|\Delta(z)| \to \infty$ as $z \to \xi$ nontangentially.

Theorems 1 and 2 correct the assertion of D. Rosen appearing in the Duke Math. J., vol. 25 (1958), pp. 373–380 (Theorem 6), as well as in Lehner [6, p. 333]. The difficulty with Rosen's proof is elucidated in Lemma 2 below. We observe that the rate of growth $a_{n+1} \approx \log q_n$ arises in connection with one of the important theorems of Khinchin to the effect that almost real numbers display this growth [5, p. 69]. We are not in a position to interpret this fact, however. The proofs of these theorems are contained in section two. As the reader will see in that section, partial results are possible with weaker hypotheses. Also, analogous results hold for negative q.

We will now describe our results on transitivity which appear in Section 3. Let R_0 be the standard fundamental region for $\Gamma(1)$, i.e.,

$$R_0 = \{z\colon |z| > 1, -1/2 < \text{Re}(z) < 1/2\}.$$

For any irrational number ξ, consider the image of $\{z\colon z = \xi + iy, y > 0\}$ under $\Gamma(1)$ in R_0 (this is a set of h-arcs in R_0).

DEFINITIONS. (i) ξ is called a *point of approximation* if the above image is not nowhere dense (see Beardon-Maskit [2]). Call this set of ξ's T_a.

(ii) ξ is called *point-transitive* if the above image is dense in R_0. Call this set of ξ's T_p.

(iii) ξ is called *line-transitive* if any h-line in R_0 may be approximated arbitrarily closely by some arc of the above image. (One way of making this specific is to require that the given arc and the approximating arc, when extended to the real-axis, fall at nearly coincident points on that axis.) Call this set T_l.

Clearly $T_a \supset T_p \supset T_l$. Artin [1] showed that T_l consists of those real numbers whose continued fraction representation contains each finite sequence of integers. Myrberg shows, first for $\Gamma(1)$ and later for much more general Fuchsian groups, that $\tilde{T}_l$ (the complement of T_l) has measure zero [7], [8]. We shall make use of both of these facts. (Interestingly, Myrberg's proof in the general case proceeds by introducing a kind of continued fraction representation for points in the limit set.) These papers of Artin and Myrberg began a deep study of ergodic properties of certain geodesic flows (see Hedlund [3], and Hopf [4]).

The distinction between T_p and T_l has been obscured in the literature. Lehner [6, p. 321] informally describes T_p and then gives T_l as his formal definition of the transitive set. Nicholls uses T_l in [10] and T_p in [9], calling them both the transitive set. It may be, of course, that T_p and T_l are the same set. However, this is not known, one way or the other, for a single Fuchsian group.[2] Indeed at this time the only known method of proving a point is in T_p is to prove it is in T_l.

Theorem 3 below starts with a set of $\xi \notin T_l$ (not satisfying Artin's criterion) and proves that such $\xi \notin T_p$. Thus it is a step in the direction of proving $T_p = T_l$ for $\Gamma(1)$. However, this theorem suffices to show that point-transitivity is not connected with the above-mentioned boundary value problem for automorphic forms (Corollary 3). There is such a connection for automorphic *functions* ($q = 0$ in (1)); Lehner [6, p. 331].

If T_p does not equal T_l for $\Gamma(1)$, then perhaps T_p is characterized as a class of continued fractions with number-theoretic significance. This is the case for T_a, where the Beardon-Maskit characterization reduces (for $\Gamma(1)$) to the statement $\xi \in T_a$ if and only if ξ is irrational.

Section 2

This section contains the proofs of Theorems 1 and 2. We begin with:

LEMMA 1. *Let f be a form of weight q on $\Gamma(1)$. Assume ξ has unbounded partial*

---

[2] The author has recently shown that $T_p - T_l = \emptyset$ in the case of the modular group (to appear in Proc. Cambridge Philos. Soc.).

denominators (*the* a_i). *Then there exists a sequence* $z_n = \xi + iy_n$ *with* $|f(z_n)| \to \infty$.

Proof (Epstein-Lehner [6, p. 332]). Khinchin [5, p. 36] contains the formula

$$(*)\qquad \frac{1}{q_k^2(a_{k+1}+2)} < \left|\xi - \frac{p_k}{q_k}\right| \le \frac{1}{q_k^2 a_{k+1}}.$$

We see from this that the assumption on ξ is equivalent to assuming that $q_{n_i}^2 |\xi - p_{n_i}/q_{n_i}| \to 0$ on some sequence $n_i \to \infty$. Call $n = n_i$ and set $z_n = \xi + i/cq_n^2$, $c > 0$ to be determined later. Let $V_n \in \Gamma(1)$ be the matrix

$$\begin{pmatrix} \cdot & \cdot \\ q_n & -p_n \end{pmatrix}.$$

We have

$$\operatorname{Im}(V_n(z_n)) = \frac{1/cq_n^2}{|q_n(\xi + i/cq_n^2) - p_n|^2} = \frac{1/c}{q_n^2(\xi - p_n/q_n)^2 + 1/c^2} \to c.$$

Also,

$$|V_n'^q(z_n)| = \frac{1}{q_n^2(\xi - p_n/q_n)^2 + 1/c^2 q_n^2} \to \infty.$$

Choose c so that f has no zero of the form $x + ic$. We may also choose V_n so that $-1/2 \le \operatorname{Re}(V_n(z_n)) \le 1/2$. Then, by the above, we can get a subsequence (again called n) such that $V_n(z_n) \to z^*$ and $f(z^*) \ne 0$. Now

$$|f(z_n)| = |f(V_n z_n)|\,|V_n'^q(z_n)| \to \infty.$$

LEMMA 2. *Assume ξ is such that there exists $\varepsilon, \delta > 0$ with*

$$q_n^2 |\xi - p_n/q_n| < \frac{1}{\left(\dfrac{(2+2\delta)q + \varepsilon}{\pi\mu}\right)\log q_n}$$

for infinitely many n where f is holomorphic at ∞ having a zero of order $\mu \ge 1$ there. Then there exists a sequence $z_n \to \xi$ nontangentially with $f(z_n) \to 0$.

Proof. Let $z_n = \xi + i|\xi - p_n/q_n|$, where $\{n\}$ is the sequence described in the hypothesis. Let V_n be as in Lemma 1. Now

$$(2)\qquad \operatorname{Im}(V_n(z_n)) = \frac{1}{2q_n^2|\xi - p_n/q_n|} \to \infty$$

and

$$(3)\qquad |V_n'^q(z_n)| = \frac{1}{[2q_n^2|\xi - p_n/q_n|^2]^q}.$$

Because of (1) and (2) we have

$$|f(z_n)| = O\left[\frac{\exp\left(\dfrac{-\pi\mu}{q_n^2\,|\xi - p_n/q_n|}\right)}{q_n^{2q}\,|\xi - p_n/q_n|^{2q}}\right],$$

where the O constant does not depend on n and the estimate holds only for n sufficiently large, i.e., Im $V_n(z_n)$ sufficiently large.

It is at this point that Rosen [12] alleges (p. 379, Lines 2–3) that the O term approaches 0 with n if $a_n \to \infty$. That this is not the case without further hypothesis may be seen by taking

$$a_{n+1}\left(\approx \frac{1}{q_n^2\,|\xi - p_n/q_n|}\right) \approx \log\log q_n.$$

We now split into two cases:

(i) $|\xi - p_n/q_n| < 1/q_n^{2+\delta}$ for infinitely many n. Thus $q_n^2|\xi - p_n/q_n| < |\xi - p_n/q_n|^{\delta/(2+\delta)}$ and

$$|f(z_n)| = O\left[\frac{\exp\left(\dfrac{-\pi\mu}{|\xi - p_n/q_n|^{\delta/(2+\delta)}}\right)}{q_n^{2q}\,|\xi - p_n/q_n|^{2q}}\right] = o(1).$$

(ii)

$$\frac{1}{\dfrac{(2+2\,\delta)q+\varepsilon}{\pi\mu}\log q_n} > q_n^2\,|\xi - p_n/q_n| \geq 1/q_n^{\delta}$$

holds for infinitely many n. Thus $q_n^2\,|\xi - p_n/q_n|^2 > 1/q_n^{2+2\,\delta}$ and

$$|f(z_n)| = O\left[\frac{\exp\left(\dfrac{(2+2\,\delta)q+\varepsilon}{\pi\mu}\log q_n(-\pi\mu)\right)}{q_n^{(-2-2\,\delta)q}}\right]$$
$$= O(q_n^{(2+2\,\delta)q}q_n^{-[(2+2\,\delta)q+\varepsilon]}) = o(1).$$

Because ε and δ may depend on q we have shown Lemma 2 and with it Theorem 1.

Clearly the same techniques yield analogous results for forms of negative weight.

We now wish to establish a converse to Theorem 1. Since our approach will be based on Equation (1), we must have some knowledge of the zeros at f. That is, as in Lemma 1, the growth of $|V_n'^q(z_n)|$ is to determine the behavior of $|f(z_n)|$. But for that we must know $|f(V_n(z_n))| \not\to 0$ too rapidly. As will be seen, at ∞ this is not a problem. But off a neighborhood of ∞, there is no à priori reason that $f(V_n(z_n))$ is not equal to zero for all n. Thus, for the remainder of this section, we will assume that *f has no zeros in H*. As the reader will see, weaker but less natural assumptions are possible.

Lastly, in what follows we will work with the line $\xi + iy$ rather than a Stolz angle at ξ in order to simplify computations. All results hold for the Stolz angle, however.

PROPOSITION 1. *Let ξ be an irrational number. Then there exists $\varepsilon(=\varepsilon_\xi)$ such that if $\varepsilon > y > 0$ there is an n with*

$$\frac{|\xi - p_n/q_n|}{10a_{n+1}} \leq y < 1/q_n^2. \tag{4}$$

Proof. To show this, we must merely show that

$$1/q_{n+1}^2 \geq \frac{|\xi - p_n q_n|}{10a_{n+1}} \quad \text{or} \quad 10 \geq \frac{q_{n+1}^2|\xi - p_n q_n|}{a_{n+1}}.$$

Now

$$\begin{aligned}\text{RHS} &= \frac{(q_n a_{n+1} + q_{n-1})^2|\xi - p_n/q_n|}{a_{n+1}} \\ &= \frac{(a_{n+1} + q_{n-1}/q_n)^2 q_n^2|\xi - p_n/q_n|}{a_{n+1}} \\ &\leq \frac{(a_{n+1}+1)^2}{a_{n+1}^2},\end{aligned}$$

by(∗). We are done since a_{n+1} is a positive integer.

PROPOSITION 2. *Choose $y > 0$ and let $z = \xi + iy$. Select n so that* (4) *is satisfied. Then* $\operatorname{Im}(V_n(z)) \geq 1/20$ *(V_n as in Lemma* 1).

Proof.

$$\operatorname{Im}(V_n(z)) \geq \min\left\{\operatorname{Im}(V_n(\xi + i/q_n^2), \ \operatorname{Im}\left(V_n\left(\xi + \frac{i|\xi - p_n/q_n|}{10a_{n+1}}\right)\right)\right\}.$$

First,

$$\operatorname{Im}(V_n(\xi + i/q_n^2)) = \frac{1}{q_n^4|\xi - p_n/q_n|^2 + 1} \geq \frac{1}{(1/a_{n+1})^2 + 1} \geq 1/2,$$

by (∗).

Second,

$$\begin{aligned}\operatorname{Im}\left(V_n\left(\xi + \frac{i|\xi - p_n/q_n|}{10a_{n+1}}\right)\right) &= \frac{1}{10a_{n+1}q_n^2|\xi - p_n/q_n|(1 + 1/a_{n+1}^2)} \\ &\geq \frac{1}{20a_{n+1}q_n^2|\xi - p_n/q_n|} \\ &\geq 1/20,\end{aligned}$$

by (∗).

We will need the following computational result:

PROPOSITION 3. *Let* $A = A_n = q_n$, $B = B_n = |\xi - p_n/q_n|$, $C = C_n = AB$, $D = D_n = a_{n+1}$, *and*

$$g(y) = g_n(y) = \frac{\exp\left(\dfrac{-\mu y}{C^2 + A^2y^2}\right)}{(C^2 + A^2y^2)^q}.$$

Then the minimum of $g(y)$ *for*

$$(4') \qquad B/10D \leq y \leq 1/A^2$$

occurs either at $1/A^2$ *on in the interval* $[B, TB]$ *for some constant* $T = T_{q,\mu}$.

Proof. A simple calculation shows

$$g'(y) = \left\{(C^2 + A^2 + y^2)^{-q-2} \exp\left(\frac{-\mu y}{C^2 + A^2y^2}\right)\right\} \cdot [-2A^4qy^3 + \mu A^2y^2 - 2A^2C^2qy - \mu C^2].$$

The term in braces is positive. Call the term in brackets $P(y)$. $P(B/10D)$ is negative. $P(y)$ is negative if y is large. Thus $P(y) = 0$ cannot have three roots in the interval $[B/10D, 1/A^2]$. It is easily seen that the minimum of g in this interval occurs at $1/A^2$ or the smaller root of $P(y) = 0$.

Claim. Assume $P(y) = 0$ has a root in (4′). Then there exists a constant $T = T_{q,\mu}$ such that (for all n) $P(y)$ has its smallest zero (in (4′)) in the interval $[B, TB]$.

Clearly $P(y) < 0$ for $0 \leq y \leq B$. Say n is such that $a_{n+1} \geq 20q/3\mu$. We have $P(2B) = A^2B^2[3\mu - 20qA^2B] \geq 0$ since $A^2B = q_n^2|\xi - p_n/q_n| \leq 1/a_{n+1}$, by (*). Thus in this case the claim is valid with $T = 2$. For those n with $a_{n+1} \leq 20q/3\mu$, we have by hypothesis a $t > 1$ with $P(tB) \geq 0$. This means that

$$(t^2 - 1)/(t^3 - t) \geq 2qA^2B/\mu$$

which, for these n, implies $1/t \geq q/(10q + 3\mu)$ using the LHS of (*). This establishes the claim and with it Proposition 3.

We are now ready to prove our converse. Let $z_m = \xi + iy_m$, $y_m \to 0$. By Proposition 1, to each y_m there is an $n = n(m)$ so that (4) is satisfied. By Proposition 2, Im $(V_n(z_m)) \geq 1/20$. Using (1), the fact that f does not vanish in H, and the fact that f has a zero of order $\mu \geq 1$ at ∞ we get

$$(5) \quad |f(z_m)| = |f(V_n(z_m))|\,|V_n'^q(z_m)| \geq \delta \exp(-2\pi\mu \operatorname{Im}(V_n(z_m)) \cdot |V_n'^q(z_m)|$$

for some positive δ independent of m. The RHS of (5) is equal to

$$\frac{\exp\left[\dfrac{-2\pi\mu y}{C^2 + A^2y^2}\right]}{(C^2 + A^2y^2)^q} = g(y).$$

(Here the notation is that of Proposition 3, $2\pi\mu$ replaces μ, and we have dropped the subscript m.) Since $y = y_m$ is in the range (4), which is the same as the range (4′), the minimum of $g(y)$ occurs at $1/A^2$ or in the interval $[B, TB]$, by Proposition 3. It follows from the proof of Lemma 1 that $g(1/A^2) = g(1/q_n^2) \to \infty$ as $n \to \infty$. (Take $c = 1$ and the fact that $q_n^4 \,|\, \xi - p_n/q_n|^2$ is bounded). Thus we must calculate $g(t\,|\,\xi - p_n/q_n|)$ for $1 \le t \le T$:

$$\begin{aligned} g(t\,|\,\xi - p_nq_n|) &= \frac{\exp\left(\dfrac{-2\pi\mu t}{q_n^2\,|\,\xi - p_n/q_n|\,(1 + t^2)}\right)}{\{q_n^2\,|\,\xi - p_n/q_n|^2(1 + t^2)\}^q} \\ &\ge q_n^{2q}a_{n+1}^{2q}(1 + T^2)^{-q}\exp\left(-\pi\mu T(a_{n+1} + 2)\right) \\ &\ge q_n^{2q}e^{-\pi\mu T a_{n+1}} \cdot (1 + T^2)^{-q}e^{-2\pi\mu T} \end{aligned}$$

where the first inequality uses (*). The log of this expression is

$$K_1\left[\frac{2q}{\pi\mu T}\log q_n - a_{n+1}\right] - K_2,$$

where $K_1, K_2 > 0$. Thus we have proved Theorem 2.

Again, analogous results hold for forms of negative weight.

COROLLARY 1. (Boundary behavior of $|\Delta|$). *If ξ is irrational and*

$$a_{n+1} \ge \frac{(12 + \varepsilon)}{\pi}\log q_n$$

for some $\varepsilon > 0$ and some infinite sequence of n's, then $|\Delta|$ has no nontangential limit at ξ. If

$$\frac{12}{\pi T_{6,1}}\log q_n - a_{n+1} \to \infty,$$

then $|\Delta(z)| \to \infty$ as $z \to \infty$ nontangentially.

Proof. Δ is a nonvanishing (in H) modular form of weight 6 with a zero of order 1 at ∞ (Rademacher [11, p. 136]).

Section 3

The aim of this section is to prove:

THEOREM 3. *Let $\xi = [a_0;\, a_1, a_2, \ldots]$. If there exists a sequence $\{n_i\}$ and a constant K such that $\{a_{n_i}\} \to \infty$ and $a_m \le K$, $m \notin \{n_i\}$, then $\xi \notin T_p$.*

We begin with some results needed for the proof:

LEMMA 3. $q_{n-1}/q_n = [0; a_n, a_{n-1}, \ldots, a_1]$.

Proof.

$$\frac{q_{n-1}}{q_n} = \frac{q_{n-1}}{a_n q_{n-1} + q_{n-2}} = \frac{1}{a_n + \dfrac{q_{n-2}}{q_{n-1}}}$$

and proceed by induction.

LEMMA 4. *If* $[0; a_1^{(i)}, a_2^{(i)}, \ldots] \to [0; \alpha_1, \alpha_2, \ldots]$ *as* $i \to \infty$, *then* $a_n^{(i)} \to \alpha_n$ *as* $i \to \infty$ *for each* n.

Proof. This can be done by induction. We omit the details.

LEMMA 5. *Let* $S = S_\xi$ *be the derived set of* $\{q_{n-1}/q_n\colon n = 1, 2, \ldots\}$. *Then* $m(S) = 0$ *or* 1. (S *obviously lies in* $[0, 1]$.)

Proof. Assume $m(S) \neq 0$. Since S is closed, we must show $S = [0, 1]$. Since $m(T_l) = 1$, we have that $S \cap T_l \neq \emptyset$. Let $\tau \in S \cap T_l$, $\tau = [0; \alpha_1, \alpha_2, \ldots]$. Take any $\beta \in [0, 1]$, $\beta = [0; \beta_1, \beta_2, \ldots]$. Choose $L > 0$. The sequence $\beta_1, \ldots, \beta_L$ appears in the expansion of τ (infinitely often) since $\tau \in T_l$. Say $\alpha_{N+j-1} = \beta_j$, $1 \le j \le L$. Since $\tau \in S$ we may select a sequence $q_{n_i-1}/q_{n_i} \to \tau$. By Lemma 4, for all i sufficiently large, the expansion of q_{n_i-1}/q_{n_i} begins with $\alpha_1, \ldots, \alpha_{N+L-1}$. Now Lemma 3, applied to q_{n_i-1}/q_{n_i}, shows that the sequence $\beta_L, \ldots, \beta_1$ appears in the expansion of ξ. Again by Lemma 3 there exists an s_L such that

$$(6) \qquad q_{s_L-1}/q_{s_L} = [0; \beta_1, \ldots, \beta_L, \ldots, a_1].$$

Now let $L \to \infty$. Then the RHS of (6) converges to β. Since β was arbitrary, we have shown $S = [0, 1]$.

The proof shows that $\beta_L, \ldots, \beta_1$ was contained in the expansion of ξ. This implies

COROLLARY 2. *With notation as in Lemma* 3, $m(S_\xi) \neq 0$ *implies* $\xi \in T_l$.

LEMMA 6. *Let*

$$V_n = \begin{pmatrix} q_{n+1} & -p_{n+1} \\ q_n & -p_n \end{pmatrix},$$

and $\xi = [a_0; a_1, a_2, \ldots,]$. *Then*

$$(7) \qquad -V_n(\xi) = [0; a_{n+2}, a_{n+3}, \ldots]$$

Proof. Let $r_n = [a_{n+1}; a_{n+2}, a_{n+3}, \ldots]$ and

$$\tau_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Then (see Khinchin [5, (16)] $\tau_n(r_n) = \xi$. Now

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tau_{n+1}^{-1} = V_n$$

and

$$V_n(\xi) = \frac{-1}{\tau_{n+1}^{-1}(\xi)} = \frac{-1}{r_{n+1}} = \frac{-1}{[a_{n+2}; a_{n+3}, \ldots]} = -[0; a_{n+2}, a_{n+3}, \ldots].$$

We will apply this lemma when n is even, and thus $V_n \in \Gamma(1)$. When n is odd, defining

$$W_n = \begin{pmatrix} -q_{n+1} & p_{n+1} \\ q_n & -p_n \end{pmatrix}$$

gives

$$(7') \qquad W_n(\xi) = [0; a_{n+2}, a_{n+3}, \ldots].$$

LEMMA 7. *The derived sets of* $\{V_n(\xi)\}_{n \text{ even}}$ *and* $\{W_n(\xi)\}_{n \text{ odd}}$ *have measure* 0 *or* 1.

Proof. Let $T = T_\xi$ be the derived set of $\{W_n(\xi)\}_{n=1}^{\infty}$. Say $m(T) \neq 0$. Then, as in Lemma 5, there is $\tau \in T_l \cap T$. Let $\tau = [0; \alpha_1, \alpha_2, \ldots]$. (We showed in Lemma 6 that $T \subset [0, 1]$.) Let $b_1, \ldots, b_L$ be an arbitrary sequence of positive integers. $b_1, \ldots, b_L$ appears in the expansion of τ. Say $\alpha_{N+j-1} = b_j$; $j = 1, \ldots, L$. Let $W_n(\xi) \to \tau$. By Lemma 4, if n is sufficiently large $[0; \alpha_1, \ldots, \alpha_{N+L-1}]$ is the $N + L - 1$ convergent of $W_n(\xi)$. By (7′), the sequence $b_1, \ldots, b_L$ appears in the expansion of ξ. Thus $\xi \in T_l$ and $m(T) = 1$, again by (7′).

LEMMA 8. $\operatorname{Im} V_n(\xi + i/q_n^2) \geq 1/3$, $\operatorname{Im} V_n(\xi + i/q_{n+1}^2) \geq 1/5$. *The same result holds for* W_n.

Proof. We have already done the first calculation as part of Lemma 1.

$$\begin{aligned}
\operatorname{Im} V_n(\xi + i/q_{n+1}^2) &= \frac{1}{q_n^2 q_{n+1}^2[(\xi - p_n/q_n)^2 + 1/q_{n+1}^4]} \\
&= \frac{1}{q_{n+1}^2 + q_n^2|\xi - p_n/q_n|^2 + q_n^2/q_{n+1}^2} \\
&\geq \frac{1}{[(a_{n+1} + 1)^2/a_{n+1}^2] + 1} \\
&\geq 1/5,
\end{aligned}$$

by (7) and (∗).

We are now in a position to prove Theorem 3.

Proof. Let $h_n = \{z\colon\ z = \xi + iy,\ 1/q_{n+1}^2 \le y \le 1/q_n^2$. By Lemma 8, $\operatorname{Im} V_n(h_n) \ge 1/5$. (We will now assume that $V_n \in \Gamma(1)$, i.e., that n is odd. If n is even a corresponding argument with W_n must be substituted.) Select Y so large that (i) the set $J = J_Y = \{z\colon z = x + iY\}$ has no points $\Gamma(1)$ equivalent to any point t with $1/5 \le \operatorname{Im}(t) \le 5$ and (ii) $Y > K + 2$. Now if $\xi \in T_p$, $\Gamma(1)$ images of the h_n must be dense in J. $V_n(h_n)$ is an arc of the h-line whose endpoints are $V_n(\xi)$ and q_{n+1}/q_n. Now $|V_n(\xi) - q_{n+1}/q_n| = 1/(q_n^2\,|\xi - p_n/q_n|)$. It follows that for $n \notin \{n_i\}$, $V_n(h_n) \cap J = \emptyset$. On the other hand

$$|V_{n_i}(\xi) - q_{n_i+1}/q_{n_i}| \to \infty \quad \text{as} \quad n_i \to \infty.$$

Relabel the $\{n_i\}$ sequence as $\{n\}$. Then for n sufficiently large,

$$V_n(h_n) \cap J \ne \emptyset.$$

(Here we use the simple facts that $\operatorname{Im}(V_n(\xi + i/q_n^2), V_n(\xi + i/q_{n+1}^2)) \le 9$.) If U is any element of $\Gamma(1)$ with $U(h_n) \cap J \ne \emptyset$, then it follows from this, Lemma 8, and the choice of J, that $U = TV_n$ for some translation T in $\Gamma(1)$ uniquely determined by n. We have already noted that the (euclidean) radius of the h-line of which $V_n(h_n)$ is an arc, approaches ∞. This means the arc(s) between $J \cap V_n(h_n)$ and the real axis given by these h-lines is nearly a vertical line, for n sufficiently large. (That is, this arc lies in a Stolz angle of width ε for $n > N = N(\varepsilon)$.) The same analysis applies, of course, to the arcs determined by the h-lines $T_n V_n(h_n)$.

The base points of these arcs and their $\Gamma(1)$-equivalents in $[0, 1]$ are easily determined. $0 \ge V_n(\xi) \ge -1$ as we saw in Lemma 6. Also, q_{n+1}/q_n is $\Gamma(1)$ equivalent to q_{n-1}/q_n by the equation $q_{n+1} = a_{n+1}q_n + q_{n-1}$. Thus a complete set of $\Gamma(1)$-inequivalent base points in $[0, 1]$ lies in the set $D = \{x\colon x = V_n(\xi) + 1$, or $x = q_{n-1}/q_n$, for some $n\}$. (There may be repetitions.) By Lemma 5 and 7, $\bar{D}$ has measure zero and is, of course, closed. Thus we may choose a closed interval $[a, b]$, $0 < a < b < 1$ with $[a, b] \cap \bar{D} = \emptyset$. Consider $J[a, b]$, the subset of J whose elements have real part in $[a, b]$. Because $[a, b] \cap \bar{D} = \emptyset$, and these arcs lies in Stolz angles of arbitrarily small width for all n sufficiently large, we have that $J[a, b]$ has no point in common with these arcs for n sufficient large. (Recall that the arcs in question are the segments of $T_n V_n(h_n)$ for height Y, i.e., from their intersection with J, to the real axis.) From this it follows that $J[a, b]$ contains (uncountably many) points not in the closure of

$$\bigcup_n (\Gamma(1)(h_n) \cap J) = \bigcup_n (T_n V_n(h_n) \cap J).$$

There remains the possibility that these points lie in the closure of a sequence of arcs of $\Gamma(1)(h_n)$ not intersecting J (i.e., below it). But all members of this sequence beyond some finite point would have to intersect $J_{Y-\delta}$ for some small δ and the argument above would show that the points $J_Y[a, b]$ do not lie in the closure of this sequence. Thus we have shown

$$R_0 \not\subset \overline{\bigcup_n \Gamma(1)(h_n))}$$

and so $\xi \notin T_p$. This concludes the proof of Theorem 3.

The fact that $a_{n_i} \to \infty$ (and is not just unbounded) is essential to this proof as otherwise the Stolz angle argument fails. But the fact that a_n may approach ∞ arbitrarily slowly allows us to state:

COROLLARY 3. *No analytic condition on the closeness of rational approximation to ξ determines whether or not $\xi \in T_p$. In particular, point-transitivity and the existence of boundary values of automorphic forms are different problems.*

BIBLIOGRAPHY

1. E. ARTIN, *Ein mechanisches System mit quasiergodischen Bahnen*, Abh. Math. Sem. Hamburg, vol. 3 (1923), pp. 170–175.
2. A. BEARDON and B. MASKIT, *Limit points of Kleinian groups and finite sided fundamental polyhedra*, Acta Math., vol. 132 (1974), pp. 1–12.
3. G. A. HEDLUND, *The dynamics of geodesic flows*, Bull. Amer. Math Soc., vol. 45 (1939), pp. 241–260.
4. E. HOPF, *Fuchsian groups and erogodic theory*, Trans. Amer. Math. Soc., vol. 39 (1936), pp. 299–314.
5. A. YA. KHINCHIN, *Continued fractions*, University of Chicago Press, Chicago, 1964.
6. J. LEHNER, *Discontinuous groups and automorphic functions*, Math. Survey no. 8, Amer. Math. Soc., 1964, Providence.
7. P. J. MYRBERG, *Einige Andwendungen der Kettenbrüche in der Theorie der binären quadratischen Formen und der elliptischen Modulfunktionen*, Ann. Acad. Sci. Fenn Ser. AI no. 23, 1934.
8. ———, *Ein Approximationssatz für die Fuchsschen Gruppen*, Acta Math., vol. 57 (1931), pp. 389–409.
9. P. NICHOLLS, *Special limit points for Fuchsian groups and automorphic functions near the limit set*, Indiana Univ. Math. J., vol. 24 (1974), pp. 143–148.
10. ———, *Transitivity properties for Fuchsian groups*, Canad. J. Math., vol. 28 (1976), pp. 805–814.
11. H. RADEMACHER, *Topics in analytic number theory*, Springer-Verlag, New York, 1973.
12. D. ROSEN, *A note on the behavior of certain automorphic functions and forms near the real axis*, Duke Math. J., vol. 25 (1958), pp. 373–380.

BARUCH COLLEGE, CUNY
NEW YORK