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SMOOTH FUNCTIONS AND CONVERGENCE OF SINGULAR INTEGRALS II

BY

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0. Introduction and statement of the main result

Throughout this paper we shall keep the notation and definitions introduced in [2]. Given $f \in L^1(R^n)$, its L^1-modulus of continuity $w(t)$ is defined by

$$w(f, t) = w(t) = \sup_{(h;\, |h| \leq t)} \int_{R^n} |f(x+h) - f(x)|\, dx. \tag{0.1}$$

As in [2], we shall be concerned here with singular kernels satisfying

$$K(\lambda x) = \lambda^{-n} K(x); \quad \lambda > 0,\, x \neq 0. \tag{0.2}$$

If $K(x)$ is odd then

$$\int_{|x|=1} |K(x)| d\sigma < \infty \tag{0.3}$$

where $d\sigma$ stands for the "area" element of the unit sphere. If $K(x)$ is not odd then

$$\int_{|x|=1} K(x)\, d\sigma = 0; \qquad \int_{|x|=1} |K(x)| \log^+ |K(x)|\, d\sigma < \infty. \tag{0.4}$$

Similarly, we shall introduce the L^1-modulus of continuity of the kernel K as

$$w_K(t) = \sup_{h;\, |h| \leq t} \int_{2<|x|<4} |K(x+h) - K(x)|\, dx, \quad 0 < t \leq 1. \tag{0.5}$$

We shall assume that

$$\int_0^1 w_K(t) \frac{dt}{t} = \infty$$

and introduce

$$\phi(t) = \int_t^1 w_K(s) \frac{ds}{s}, \quad 0 < t \leq 1. \tag{0.6}$$

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[2] Note that $\phi(d^n_{k,j}) \sim \phi(d_{k,j})$.

Let E be a bounded Lebesgue-measurable set. We define the ϕ-Entropy of the set E to be

$$(0.7)\qquad |E|_\phi = \inf_{G \supseteq E} \sum_k |I_k| \phi(|I_k|)$$

where $G = \bigcup_1^\infty I_k$; $\mathring{I}_k \cap \mathring{I}_j = 0$, $k \neq j$; $|I_K| \leq \frac{1}{2}$ for all k. We clearly have

$$(0.8)\qquad E_1 \subset E_2 \Rightarrow |E_1|_\phi \leq |E_2|_\phi.$$

Let f be Lebesgue measurable and supported on the cube Q. We define the ϕ-Entropy of f on the cube Q by

$$(0.9)\qquad -\int_0^\infty y\, d\,|\,|f| > y|_\phi = J_\phi(f)$$

The above integral is understood in the Riemann-Stieltjes sense.

THEOREM A. *Let $K(x)$ be a singular kernel satisfying* (0.2) *and* (0.3) *or* (0.2) *and* (0.4). *Let $w_K(t)$ be its modulus of continuity as defined in* (0.5). *Let $\phi(t)$ be the function defined in* (0.6). *If $f \in L^1(R^n)$ and the restriction of f to the cube Q has finite ϕ-Entropy on Q then*

$$(0.10)\qquad \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} K(x-y) f(y)\, dy \text{ exists a.e. on } Q$$

1. Proof of Theorem A

Without loss of generality we may assume that f is supported on Q and non-negative. In fact, if $f \in L^1(R^n)$ and it is supported in the complement of Q the singular integral (0.11) converges a.e. in Q.

Let E_k be the set where $2^{k-1} < f \leq 2^k$; and $E_0 = Q \cap \{f \leq 1\}$. The fact that $J_\phi(f) < \infty$ implies

$$(1.1)\qquad \sum_{k=0}^\infty \sum_{j=1}^\infty 2^k |Q_{k,j}| \phi(|Q_{k,j}|) \leq C J_\phi(f)$$

for a family of cubes $\{Q_{k,j}\}$ satisfying

(1.2) (i) $|Q_{k,j}| \leq \frac{1}{2}$,

(ii) $\mathring{Q}_{k,j} \cap \mathring{Q}_{k,j} = 0, \quad i \neq j$,

(iii) $\bigcup_{j=1}^\infty Q_{k,j} \supset E_k$.

The inequality (1.1) is a consequence of the definition (0.10). Call f_k the restriction of f to E_k and define the mean values $\mu_{k,j}$ by

$$(1.3)\qquad \mu_{k,j} = \frac{1}{|Q_{k,j}|} \int_{Q_{k,j}} f_k\, dy$$

Clearly, we have $0 \leq \mu_{k,j} \leq 2^k$.

Let $\Psi_{k,j}$ be the characteristic function of $Q_{k,j}$ and define $\bar{f}$ by

$$\bar{f} = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} \mu_{k,j} \Psi_{k,j}. \tag{1.4}$$

If we let $\bar{w}(t)$ denote the L^1-modulus of continuity of $\bar{f}$ then

$$\int_0^1 \bar{w}(s) w_K(s) \frac{ds}{s} < \infty. \tag{1.5}$$

The convergence of the above integral is a consequence of (1.1), the estimate $\mu_{k,j} \leq 2^k$ and the following two estimates:

$$\text{(i)} \int_{R^n} |\Psi_{k,j}(x+h) - \Psi_{k,j}(x)|\, dx \leq C(d_{k,j})^{n-1}|h| \quad \text{if} \quad |h| \leq \frac{d_{k,j}}{10}, \tag{1.6}$$

$$\text{(ii)} \int_{R^n} |\Psi_{k,j}(x+h) - \Psi_{k,j}(x)|\, dx \leq 2|Q_{k,j}| \quad \text{if} \quad |h| > \frac{d_{k,j}}{10},$$

where $d_{k,j} = \text{diam}\,(Q_{k,j})$ and $y_{k,j}$ denotes the center of $Q_{k,j}$. In fact, let $w_{k,j}$ be the L^1-modulus of continuity of $\Psi_{k,j}$. Then, by (i) and (ii),

$$\begin{aligned} &\int_0^1 w_{ij}(t) w_K(t) \frac{dt}{t} \\ &\leq C d_{ij}^{n-1} \int_0^{d_{ij}} w_K(t)\, dt + 2|Q_{ij}| \int_{d_{ij}}^1 w_K(t) \frac{dt}{t} \\ &\leq C_1 |Q_{ij}|[1 + \phi(d_{ij})] \\ &\leq C_2 |Q_{ij}|[1 + \phi(|Q_{ij}|)]. \end{aligned} \tag{1.7}$$

The last inequality above follows from the fact that $\phi(d_{ij}) \sim \phi(|Q_{ij}|)$ (see Lemma C) at the end of this section). Let us decompose f in the following way:

$$f = \bar{f} + \sum_{k=0}^{N} \sum_{j=1}^{\infty} (f_k - \mu_{k,j})\Psi_{k,j} + \sum_{N}^{\infty} \sum_{j=1}^{\infty} f(k - \mu_{k,j})\Psi_{k,j} \tag{1.8}$$

Also define the exceptional set

$$E_N = \bigcup_{k=N}^{\infty} \bigcup_{j=1}^{\infty} 2Q_{k,j} \tag{1.9}$$

where $2Q_{k,j}$ denotes the dilation of $Q_{k,j}$ two times about its center $y_{k,j}$.

Let m_0 be the mean value $(\int_{|x|=1} K(x)|\, d\sigma)S^{-1}$, where S stands for the "area" of the unit sphere. Consider also the kernel

$$K^*(x) = |K(x)| - m_0|x|^{-n}. \tag{1.10}$$

Clearly, we have $w_{K*}(t) \leq w_K(t) + C|t|$ where C is a constant independent of t.

Since f is supported on Q we may take K_0 instead of K, where

$$(1.11)\qquad \begin{aligned} K_0(x) &= K(x) \quad \text{if} \quad |x| \leq 2, \\ &= 0 \qquad\ \ \text{if} \quad |x| > 2. \end{aligned}$$

Without loss of generality we may assume that $0 < \varepsilon < 1/8$. Since

$$\sum_{k=0}^{N} \sum_{j=1}^{\infty} (f_k - \mu_{k,j})\Psi_{k,j}$$

belongs to $L^2(R^n)$ the convergence problem reduces to the analysis of

$$\sum_{k=N}^{\infty} \sum_{j=1}^{\infty} (f_k - \mu_{k,j})\Psi_{k,j}$$

Estimates for

$$\sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K_0(x-y) \left(\sum_{N}^{\infty} \sum_{j=1}^{\infty} [f_k - \mu_{k,j}]\Psi_{k,j} \right) dy \right|.$$

Consider $x \in R^n - E_N$ and $\varepsilon > 0$.

Let us designate by $Q_{k,j}^{\varepsilon}$ the cubes that intercept the sphere of radius ε about x. For those cubes,

$$\left| \int_{|x-y|>\varepsilon} K(x-y) \left(\sum_{Q_{k,j}^{\varepsilon}} [f_k - \mu_{k,j}]\Psi_{k,j}(y) \right) dy \right|$$

$$(1.12)\qquad \leq \int_{\varepsilon/2<|x-y|<2\varepsilon} |K(x-y)| \sum_{Q_{k,j}^{\varepsilon}} (f_k - \mu_{k,j})\Psi_{k,j}\, dy$$

$$+ \int_{\varepsilon/2<|x-y|<2\varepsilon} |K(x-y)| \sum_{N}^{\infty} \sum_{j=1}^{\infty} 2\mu_{k,j}\Psi_{k,j}(y)\, dy$$

Note that $Q_{k,j}^{\varepsilon} \subset \{y; \varepsilon/2 < |x-y| < 2\varepsilon\}$. Let g_N be the function defined by the sum

$$(1.13)\qquad \sum_{k=N}^{\infty} \sum_{j=1}^{\infty} \mu_{k,j}\Psi_{k,j}(y).$$

Let Γ_N be the function

$$(1.14)\qquad \sum_{k=N}^{\infty} \sum_{j=1}^{\infty} \int_{R^n} |K_0(x-y) - K_0(x-y_{k,j})| (f_k + \mu_{k,j})\Psi_{k,j}(y)\, dy.$$

We shall use the maximal operators

$$(1.15)\qquad \sup_{\varepsilon>0} \bar{\varepsilon}^n \int_{|x-y|<\varepsilon} |f|\, dy = M_1(f)(x),$$

and

$$(1.16)\qquad \sup_{\varepsilon>0} \left| \int_{|x-y|>\varepsilon} K^*(x-y)f(y)\, dy \right| = M_2(f)(x).$$

It can be readily seen that the right hand member of (1.12) is dominated by

$$C_0[\Gamma_N(x) + M_1(g_N)(x) + M_2(g_N)(x)] \tag{1.17}$$

provided that $x \in R^n - E_N$. To see this, note that $f_k - \mu_{k,j}\Psi_{k,j}$ has mean value zero over $Q^{\varepsilon}_{k,j}$ and consequently

$$\begin{aligned} &\int_{\varepsilon/2<|x-y|<2\varepsilon} |K(x-y)| \sum_{Q_{k,j}^{\varepsilon}} (f_k - \mu_{k,j})\Psi_{k,j}(y)\, dy \\ &= \int \{|K_0(x-y)| - |K_0(x-y_{k,j})|\} \sum_{Q_{k,j}^{\varepsilon}} (f_k - \mu_{k,j})\Psi_{k,j}(y)\, dy \\ &\leq \Gamma_N(x). \end{aligned} \tag{1.18}$$

Also the last integral in (1.12) can be written as

$$2\int_{\varepsilon/2<|x-y|<2\varepsilon} K^*(x-y)g_N(y)\, dy + 2m_0 \int_{\varepsilon/2<|x-y|<2\varepsilon} |x-y|^{-n} g_N(y)\, dy \tag{1.19}$$

We bound the first integral above by the operator (1.16) and the second one by (1.15).

In a similar manner we obtain

$$\sum_{Q_{k,j} \cap \{|x-y|\leq\varepsilon\}=0} \left| \int_{|x-y|>\varepsilon} K_0(x-y)(f_k - \mu_{k,j})\Psi_k(y)\, dy \right| \leq C_0\Gamma_N(x); \tag{1.20}$$

$$x \in R^n - E_N.$$

Using the fact that

$$\int_{|x|>2t,\ |h|<t,\ (0<t<1/4)} |K_0(x+h) - K_0(x)|\, dx \leq C[\phi(t) + 1],$$

we obtain

$$\int_{R^n - E_N} \Gamma_N(x)\, dx < C \sum_{k=N}^{\infty} \sum_{j=1}^{\infty} 2^k |Q_{k,j}| (\phi(|Q_{k,j}|) + 1). \tag{1.21}$$

From Theorem A in [2] we obtain

$$|Q \cap E(M_2(g_N) > \delta)| < \frac{c}{\delta} \sum_{k=N}^{\infty} \sum_{j=1}^{\infty} \mu_{k,j} |Q_{k,j}| (1 + \phi(|Q_{k,j}|). \tag{1.22}$$

From Hardy-Littlewood maximal theorem we have

$$|E(M_1(g_N) > \delta)| < \frac{c}{\delta} \sum_{k=N}^{\infty} \sum_{j=1}^{\infty} \mu_{k,j} |Q_{k,j}|. \tag{1.23}$$

Using the estimates (1.17) to (1.23) we obtain

$$\overline{\lim_{\varepsilon\to 0}} \int_{|x-y|>\varepsilon} K(x-y) \left[\sum_{k=N}^{\infty} \sum_{j=1}^{\infty} (f_k - \mu_{k,j})\Psi_{k,j} \right] dy < 4\,\delta \tag{1.24}$$

in Q except for a set whose measure does not exceed

$$\frac{C_0}{\delta} C_1 \sum_{k=N}^{\infty} \sum_{j=1}^{\infty} 2^k |Q_{k,j}| [1 + \phi(|Q_{k,j}|)] \tag{1.25}$$

where C_0 and C_1 do not depend on N or δ. Once $\delta > 0$ has been fixed (1.25) can be made arbitrarily small by choosing N large enough.

The convergence of $K_\varepsilon \times \bar{f}$ follows from Theorem A in [2], and that of

$$K_\varepsilon \left[\sum_{k=0}^{N} \sum_{j=1}^{\infty} (f_k - \mu_{k,j}) \Psi_{k,j} \right]$$

follows from the fact that the function between brackets is in $L^2(R^n)$.

LEMMA C. *Let $\phi(t)$ be the function defined in* (0.6). *Then,*

$$\phi(s^n)/n \le \phi(s) \le \phi(s^n), \quad n > 0, \quad 0 < s < 1. \tag{1.26}$$

Proof. If $0 < s < 1$ then

$$\int_{s^n}^{1} w_k(t) \frac{dt}{t} \ge \int_{s}^{1} w_k(t) \frac{dt}{t}. \tag{1.27}$$

On the other hand, a change of variables shows

$$\int_{s^n}^{1} w_k(t) \frac{dt}{t} = n \int_{s}^{1} w_k(t^n) \frac{dt}{t}. \tag{1.28}$$

Also

$$n \int_{s}^{1} w_k(t^n) \frac{dt}{t} \le n \int_{s}^{1} w_k(t) \frac{dt}{t}. \tag{1.29}$$

Now, (1.27) and (1.29) give the thesis.

2. Remarks on entropy and smoothness

The proof of Theorem A shows that if f has finite ϕ-Entropy on Q, then there exists a smooth function g such that

$$|f| < g \quad \text{a.e. in } Q, \tag{2.1}$$

$$\int_{0}^{1} w(s) w_K(s) \frac{ds}{s} < \infty, \tag{2.2}$$

where $w(s)$ stands for the L^1-modulus of continuity of g. Without loss of generality we may assume that $f \geq 0$. We are going to take $g(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{\infty} 2^k \Psi_{k,j}(x)$ where the $\Psi_{k,j}(x)$ are the characteristic function of the cubes $Q_{k,j}$ defined in (1.2). As is readily seen, (2.1) and (2.2) follow from (1.1) and (1.7). We have also the following:

3. THEOREM B. *Suppose that f is supported on a cube Q and its L^1-modulus of continuity satisfies the Dini condition with respect to the weight $w_K(t)$, namely*

$$\int_0^1 w(s) w_K(s) \frac{ds}{s} < \infty \tag{3.1}$$

Then its ϕ-Entropy over Q is finite; moreover we have

$$J_\phi(f) \leq C\left(\|f\|_1 + \int_0^1 w(s) w_K(s) \frac{ds}{s} \right) \tag{3.2}$$

where C depends on Q and on w_K but not on f.

The proof of this result follows step by step the corresponding one in [3] and [4]. We are going to prove Theorem B in three cases:

(i) $f(x)$ is the characteristic function of a set of finite measure.
(ii) $f(x)$ is a simple function taking positive dyadic values only.
(iii) $f(x)$ is a simple function.

The general case will follow from (iii) by a density argument.

Case (i). Let $f(x)$ be the characteristic function of E, $|E| < \infty$. Let $w(t)$ be the L^1-modulus of continuity of f. We assume that

$$\int_0^1 w(t) w_K(t) \frac{dt}{t} < \infty. \tag{3.3}$$

Let $\bigcup_1^\infty Q_k$ be a covering of E by cubes in the following sense:

(α) $E \subset \bigcup_1^\infty Q_k$.
($\alpha\alpha$) If $x \in R^n$, then x belongs to at most $(12)^n$ different cubes Q_k.
($\alpha\alpha\alpha$) $|Q_k \cap E|/|Q_k| = (1/10)^n$, $k = 1, 2, \ldots$.
(αv) If Q is any cube containing a Q_k, then $|Q \cap E|/|Q| \leq (2/5)^n$.

By cube we mean cube with edges parallel to the coordinate axes.

For a prove of this type of lemma see [2, lemma 2.3]. Let $T(|x|)$ be a non-increasing function coinciding with $w_K(|x|)|x|^{-n}$ if $0 < |x| \leq 1$ and such that

$$\int_{|x|>1} T(|x|)\, dx < \infty, \quad T(|x|) \geq 0. \tag{3.4}$$

Let CE be the complement of E. Then

$$\iint_{R^n \times R^n} |f(x)-f(y)|\, T(|x-y|)\, dx\, dy$$

$$\geq \int_E f(y) \int_{CE} T(|x-y|)\, dy \tag{3.5}$$

$$\geq \frac{1}{(12)^n} \sum_1^\infty \int_{Q_k} f(y)\, dy \int T_K(|x-y|)\Psi(x)\, dx$$

where $\Psi(x)$ is the characteristic function of CE, and $T_K(s) = T(s)$ if $s > 4 \operatorname{diam} Q_k$, $T_k(s) = T(4 \operatorname{diam} Q_k)$ if $s \leq 4 \operatorname{diam} (Q_k)$.

By (αv) and Lemma (2.1) in [2] we have

$$\int_{Q_k} f(y)\, dy \int T_K(|x-y|)\Psi(x)\, dx$$

$$\geq C_n[1 - (\tfrac{2}{5})^n] \int_{Q_k} f(y)\, dy \int_{C|Q_k|^{1/n}}^1 T_k(s) s^{n-1}\, ds \tag{3.6}$$

$$= C_n[1 - (\tfrac{2}{5})^n](\tfrac{1}{10})^n |Q_k| \phi(|Q_k|^{1/n}).$$

The estimates in (3.5) and (3.6) give

$$|E|_\phi \leq \sum_1^\infty |Q_k| \phi(|Q_k|)$$

$$\leq C \iint_{R^n \times R^n} T(|x-y|)|f(x)-f(y)|\, dx\, dy \tag{3.7}$$

$$\leq C\left(|E| + \int_0^1 w(t) w_K(t) \frac{dt}{t}\right)$$

Case (ii). Suppose that $f(x) = \sum_{k=1}^N 2^k \varphi_k(x)$, where the $\varphi_k(x)$ are the characteristic functions of the sets E_k, $E_k \cap E_j = 0$, $k \neq j$.

The following inequality holds:

$$\iint_{R^n \times R^n} |f(x)-f(y)|\, T(|x-y|)\, dx\, dy$$

$$\geq \frac{1}{2} \sum \int_{E_j} 2^j\, dx \int_{C(E_j)} T(|x-y|)\, dy. \tag{3.8}$$

To see this, observe that $|f(x)-f(y)| \geq \frac{1}{2} f(x)$ for $x \in E_i$, $y \in E_j$, $i \neq j$.

Now, we apply to each E_j the covering argument of Case (i) and get

$$\int_{E_j} 2^j\,dx \int_{C(E_j)} T(|x-y|)\,dy \tag{3.9}$$
$$\geq \sum_k (12)^{-n} \int_{E_j \cap Q_k} 2^j\,dx \int T(|x-y|)\Psi_j(y)\,dy$$

where $\bigcup_1^\infty Q_k$ is a covering of E_j in the sense described in Case (i) and CE_j stands for the complement of E_j. As is readily seen, (3.9) directly gives

$$C \int_{E_j} 2^j\,dx \int_{CE_j} T(|x-y|)\,dy \geq C2^j |E_j|_\phi. \tag{3.10}$$

Now, combining (3.8) and (3.10) we obtain the thesis in this case.

Case (iii). Let $\varphi_k(x)$ be the characteristic functions of the measurable sets E_k, $E_i \cap E_j = 0$, $i \neq j$. Consider the simple function $\sum_k \alpha_k \varphi_k(x)$, $\alpha_k > 0$. We are going to construct a simple function $f^*(x)$ taking dyadic values only, such that

(β) $\frac{1}{2}f(x) \leq f^*(x) \leq 2f(x)$
($\beta\beta$) Let $\|g\|_D = \iint_{R^n \times R^n} |g(x)-g(y)|\,T(|x-y|)\,dx\,dy$; then

$$\|f^*\|_D \leq C\|f\|_D.$$

The construction of f^* is going to be accomplished in successive steps. We will modify f within the range of values $2^j < f \leq 2^{j+1}$. Once that modification is carried out we go to the next range $2^{j+1} < f \leq 2^{j+2}$ and so on.

We shall illustrate the basic step only. Consider the range of values

$$2^k < f(x) \leq 2^{k+1}. \tag{3.10}$$

Let $\alpha_{k_1} < \alpha_{k_2} < \cdots < \alpha_{k_m}$ be the values of $f(x)$ within the above range, namely

$$2^k < \alpha_{k_1} < \alpha_{k_2} < \cdots < \alpha_{k_m} \leq 2^{k+1}. \tag{3.11}$$

We construct a new function $\bar{f}_{k,1}$ defined in the following way: $\bar{f}_{k,1}(x) = f(x)$ if $f(x) \leq 2^k$ or $f(x) > 2^{k+1}$; $\bar{f}_{k,1}$ takes the value α_{k_2} or 2^k on E_{k_1} depending on whether

$$\sum_{\alpha_j > \alpha_{k,1}} \int_{E_{k,1}} \int_{E_j} T(|x-y|)\,dx\,dy \quad \text{or} \quad \sum_{\alpha_j < \alpha_{k,1}} \int_{E_{k,1}} \int_{E_j} T(|x-y|)\,dx\,dy \tag{3.12}$$

is larger. On the sets $E_{k_2}, E_{k_3}, \ldots, E_{k_m}$, $\bar{f}_{k,1}$ takes the same values as f. The construction gives

$$\|\bar{f}_{k,1}\|_D \leq \|f\|_D. \tag{3.13}$$

Our next step will be to modify $\bar{f}_{k,1}$. We have $\bar{f}_{k,2} = \bar{f}_{k,1}$ on $R^n - E_{k_2}$. On E_{k_2}, $\bar{f}_{k,2}$ is going to take the value α_{k_3} or 2^k depending on whether

$$(3.14)\qquad \sum_{\alpha_j>\alpha_{k,2}}\int_{E_{k,2}}\int_{E_j} T(|x-y|)\,dx\,dy \quad\text{or}\quad \sum_{\alpha_j<\alpha_{k,2}}\int_{E_{k,2}}\int_{E_j} T(|x-y|)\,dx\,dy$$

is larger.

The next step is the modification of $\bar{f}_{k,2}$ on E_{k_3}. Define $\bar{f}_{k,3}=\bar{f}_{k,2}$ everywhere except at E_{k_3}; $\bar{f}_{k,3}$ takes the values α_{k_4} or 2^k on E_{k_3} depending on whether

$$(3.15)\qquad \sum_{\alpha_j>\alpha_{k,3}}\int_{E_{k,3}}\int_{E_j} T(|x-y|)\,dx\,dy \quad\text{or}\quad \sum_{\alpha_j<\alpha_{k,3}}\int_{E_{k,3}}\int_{E_j} T(|x-y|)\,dx\,dy$$

is larger. In this way we construct the functions

$$(3.16)\qquad \bar{f}_{k,1}, \bar{f}_{k,2}, \ldots, \bar{f}_{k,m}.$$

If $\alpha_{k_m}=2^{k+1}$, we take $\bar{f}_{k,m}=\bar{f}_{k,m-1}$. If $\alpha_{k_m}<2^{k+1}$, we take $\alpha_{k_{m+1}}=2^{k+1}$ in our construction. The conditions (3.12), (3.14), (3.15) and the construction itself give

$$(3.17)\qquad \|f\|_D \ge \|\bar{f}_{k,1}\|_D \ge \|\bar{f}_{k,2}\|_D \ge \cdots \ge \|\bar{f}_{k,m}\|_D.$$

The construction gives $\bar{f}_{k,m}$, that takes the values 2^k or 2^{k+1} on the sets $E_{k_1}, \ldots, E_{k_m}$. This finishes the proof.

4. A Soboleff type of inequality

If $J_\phi(f)<\infty$ over Q, then

$$(4.1)\qquad \int_Q |f|\,\phi\left(\frac{1}{|f|}\right)dx \le CJ_\phi(f)$$

with C depending on ϕ and Q only. In fact, going back to the construction (1.1) and assuming without loss of generality that $J_\phi(f)\le 1/8$ we have

$$(4.2)\qquad \sum_{k=0}^{\infty}\sum_{j=1}^{\infty} 2^k|Q_{k,j}|\,\phi(|Q_{k,j}|) \ge \sum_{k=0}^{\infty}\sum_{j=1}^{\infty} 2^k|Q_{k,j}|\,\phi(2^{-k}).$$

The above inequality follows from the fact that $2^k|Q_{k,j}|\le 4J_\phi(f)\le 1/2$. Inequality (4.2) gives (4.1) for the case $J_\phi(f)\le 1/8$. The general case is obtained by taking

$$f^* = (8J_\phi(f))^{-1}|f| \quad\text{when}\quad J_\phi(f)\ge \tfrac{1}{8}.$$

We have, in this case,

$$(4.3)\qquad \int_Q f^*\phi\left(\frac{1}{f^*}\right)dx \le \tfrac{1}{4}.$$

On the other hand $\phi(1/f^*)\le\phi(1/|f|)$ which directly gives

$$(4.4)\qquad \int_Q |f|\,\phi\left(\frac{1}{|f|}\right)dx \le 2J_\phi(f).$$

By (4.4) and Theorem B in the previous section we immediately obtain

$$\int_Q |f| \phi\left(\frac{1}{|f|}\right) dx \le C\left(\|f\|_1 + \int_0^1 w(s) w_K(s) \frac{ds}{s}\right) \tag{4.5}$$

The above inequality is the corresponding version in this case of the well known Soboleff's inequality.

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