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TRANSLATION INVARIANT KÖTHE SPACES

BY

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1. The function spaces that this work deals with were first studied by Köthe in [6], [7], [8] when he examined subspaces of the space of all real sequences put in weak duality. The theory was later generalized by Dieudonné in [1] to subspaces of locally integrable functions on a locally compact σ-compact Hausdorff space with a Radon measure. This was further developed by Luxemburg in [11], Lorentz in [9], [10], and Welland in [14], [15] among others. Here we deal with such function spaces defined over a topological group with invariant Haar measure.

2. We need some definitions and facts.

Let E be a non-discrete, locally compact, σ-compact, additive topological group with regular invariant Haar measure μ. There exists a 0-neighborhood base $\mathscr{U}$ containing a sequence $\{U_n\}_{n=1}^{\infty}$ of symmetric open sets that are relatively compact such that $\{0\} = \bigcap_n U_n$ and $\bar{U}_{n+1} \subset U_n$ for any n [4]. Let $\{E_n\}_{n=1}^{\infty}$ be the increasing sequence of compact sets whose union is E.

The set of all locally integrable functions on E is denoted by

$$\Omega = \left\{ f\colon \int_{E_n} |f| \, d\mu < \infty \text{ for each } n \right\}.$$

For any subset Γ of Ω we define the Köthe space associated with Γ by

$$\Lambda = \Lambda(\Gamma) = \left\{ f \in \Omega\colon \int_E f \cdot g \, d\mu < \infty \text{ for all } g \in \Gamma \right\};$$

$\Lambda^* = \Lambda(\Lambda(\Gamma))$ the Köthe dual of Λ. Λ is a complete vector lattice with (Λ, Λ^*) in weak duality by the bilinear form $\langle f, g \rangle = \int_E f \cdot g \, d\mu$. A set of functions $H \subset \Lambda^*$ is normal if $h \in H$ and $|g| \le |h|$ implies $g \in H$. The set Λ is a complete locally convex topological vector space under the strong topology $S(\Lambda, \Lambda^*) = S$ defined by the seminorms $\rho(f) = S_H(f) = \sup_{g \in H} \int |fg| \, d\mu$ as H runs through the weakly bounded subsets of Λ^*. The weak topology or σ-topology on Λ is generated by the sets $\{f\colon |\int f \cdot g \, d\mu| < 1\}$ for each $g \in \Lambda^*$.

The function defined by $f_t(x) = f(x + t)$ for $t, x \in E$ are translates of f. We define a Köthe space Λ to be translation invariant if each translate of f belongs to Λ whenever f does. A semi-norm ρ is translation invariant if $\rho(f_t) = \rho(f)$ for each $t \in E$ and $f \in \Lambda$.

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3. In this article we give some characterization of the strong topology on a translation invariant space together with a number of counter examples on conjectures raised concerning the behavior of translates of functions in such a space. We then give a theorem which is the main result of this article linking the convergence of the translation to the original function and the condition that the Köthe dual be the same as the topological dual under the strong topology. In general, with the strong topology the Köthe dual is not the same as the topological dual Λ'. Witness the fact that (L^∞, L^1) are Köthe duals but $(L^\infty)^* = L^1 \neq (L^\infty)'$.

In addition a sufficient condition is given for the convergence of the translates to the original function.

4. PROPOSITION 1. *A seminorm ρ on Λ is translation invariant if and only if the bounded set $B \subset \Lambda^*$ is translation invariant*, i.e.

$$\bigcup_{t \in E} \{g_t: g \in B\} = B.$$

Proof. If we assume B is translation invariant then for a $t \in E$, $g_{-t} \in B$ if g does. Since ρ is the seminorm associated with B, we have

$$\rho(f_t) = \sup_{g \in B} \int_E |f_t \cdot g| \, d\mu = \sup_{g \in B} \int_E |f \cdot g_{-t}| \, d\mu = \rho(f).$$

For the converse we can assume without loss of generality that the weakly bounded $B \subset \Lambda^*$ is normal, convex and weakly closed in Λ^* and ρ is translation invariant. If

$$B^0 = \left\{ f \in \Lambda: \int |f \cdot g| \, d\mu \leq 1 \right\},$$

then the bipolar

$$(B^0)^0 = \left\{ g \in \Lambda^*: \int |f \cdot g| \, d\mu \leq 1 \text{ for all } f \in B^0 \right\}$$

coincides with B by [14].

If there existed a $t \in G$ and an $h \in B$ such that $h_t \notin B$, then because $B = (B^0)^0$, there would exist an $f \in B^0 \subset \Lambda$ satisfying $\sup_{g \in B} \int |f \cdot g| \, d\mu \leq 1$ but $\int |f \cdot h_t| \, d\mu > 1$. But ρ being translation invariant and $\rho(f) = \sup_{g \in B} \int |f \cdot g| \, d\mu \leq 1$ yields a contradiction since

$$\rho(f) = \rho(f_{-t}) = \sup_{g \in B} \int f_{-t} \cdot g| \, d\mu \geq \int |f_{-t} \cdot h| \, d\mu = \int |f \cdot h_t| \, d\mu > 1.$$

Thus the proposition is proved.

Remark 1. The space Ω and its Köthe dual Φ, the set of essentially bounded functions of compact support do not have translation invariant seminorms or ones that are translation bounded. The seminorms generating the strong topology on Λ are given by

$$\rho_n(f) = \int_{E_n} |f| \, d\mu, \quad n = 1, 2, \ldots.$$

If $E = R^1$ and $f(x) = \sum_{n=1}^{\infty} (x-n)^{1/n-1} \chi_{[n,n+1]} \in \Omega$ and $t_n = n$ $(n = 1, 2, \ldots)$, then $\rho_1(f_{t_n}) \geq \int_0^1 f_{t_n} \, d\mu = \int_n^{n+1} (x-n)^{1/n-1} \, dx \to \infty$ as $n \to \infty$.

Even if Λ is a translation invariant Banach space ρ may not be translation invariant or translation bounded. For this example let $E = R^1$ and set $g(x) = |x| + 1 \in \Omega$. Then $L_g^1 = \{f \in \Omega: \int f \cdot g \, d\mu < \infty\}$ can be easily shown to be a Banach function space with norm $\rho(f) = \int_{R^1} |f \cdot g| \, d\mu$. The function $g_t(x) = |x+t| + 1$ is a multiple of $g(x)$ and the bounded function $(|x + t| + 1)/(|x| + 1)$ showing that any f for which $\int_{R^1} |fg| \, d\mu < \infty$ will also have $\int_{R^1} |fg_t| \, d\mu < \infty$.

Therefore L_g^1 is not only a Banach space but translation invariant. If we consider the function $f(x) = x^{-3} \cdot \chi_{[1,\infty]} \in L_g^1$, then for each n,

$$\rho(f_{-n}) = \int f_{-n} g \, d\mu = \int f \cdot g_n \, d\mu = \int_1^{\infty} x^{-3}(x+n+1) = (n+3)/2.$$

This clearly shows that this Banach space does not have a translation bounded norm.

PROPOSITION 2. *If the set $\{g_{t_n}\}_{n=1}^{\infty}$ is weakly bounded in Λ^* for each $g \in \Lambda^*$ and sequence $\{t_n\}$ in E converging to 0, then for any seminorm ρ for $S(\Lambda; \Lambda^*)$ and any compact $K \subset E$ there exists a seminorm ρ' for $S(\Lambda, \Lambda^*)$ such that $\sup_{t \in K} \rho(f_t) \leq \rho'(f)$ for any $f \in \Lambda$.*

Proof. Let ρ and K be given. Let $B \subset \Lambda^*$ be the σ-closed bounded, normal subset associated with ρ; i.e. $\rho(f) = S_B(f)$. We show $D = \bigcup_{t \in K} B_t$ is weakly bounded.

If D is not bounded, then there exists an $f \in \Lambda$ for which

$$\sup_{t \in K} \sup_{g \in B} \int |fg_t| d\mu = \infty.$$

Thus we could extract a sequence $\{t_n\}_{n=1}^{\infty} \subset K$ and non-negative $\{g_n\}_{n=1}^{\infty} \subset B$ such that $\int |f \cdot (g_n)_{t_n}| d\mu \geq 2^n \cdot n$ for each n. Since $\{t_n\} \subset K$ is compact, there is a subsequence of $\{t_n\}$ and $t \in K$ which is the limit of the subsequence. Replacing the subsequence by $\{t_n\}$ and considering $\{t_n - t\}$ we can assume $t_n \to_n 0$. Since B is normal,

$$A = B^0 = \left\{ f \in \Lambda: \left| \int_E fg \, d\mu \right| \leq 1 \text{ for all } g \in B \right\}$$

is normal and

$$B = A^0 = \left\{ g \in \Lambda^*: \left| \int_E fg \, d\mu \right| \leq 1 \text{ for all } f \in A \right\} \qquad [14].$$

If $h_n = \sum_{i=1}^{n} g_i/2^i$ for $n = 1, 2, \ldots$, then for any $r(x) \in A$,

$$\left|\int r \cdot h_n \, d\mu\right| \leq \sum_{i=1}^{n} \frac{1}{2^i} \left|\int r \cdot g_i \, d\mu\right| \leq 1.$$

Thus $h_n \in B = A^0$ for each n. By [14] there is $h \in \Lambda^*$ which is $\lim h_n$. Since $r \cdot h \in L^1(E, \mu)$, we use the Lebesgue dominated convergence theorem to see that

$$\left|\int r \cdot h \, d\mu\right| = \lim_n \left|\int rh_n \, d\mu\right| \leq 1.$$

As this is true for all $r \in A$ it follows that $h \in B$. But

$$\int f \cdot h_{t_n} \, d\mu \geq \int f(h_n)_{t_n} \, d\mu \geq \frac{1}{2n} \int f(g_n)_{t_n} \, d\mu \geq n \to \infty.$$

This contradicts the hypothesis that $\{h_{t_n}\}_{n=1}^{\infty}$ is weakly bounded. Thus $D = \bigcup_{t \in K} B_t$ is weakly bounded. Replacing K by the compact $-K = \{-t \colon t \in K\}$ and denoting ρ' as the seminorm associated with D, we have for any $t \in K$ and $f \in \Lambda$,

$$\rho(f_t) = \sup_{g \in B} \int |f_t g| \, d\mu = \sup_{g \in B} \int |f \cdot g_{-t}| \, d\mu \leq \sup_{g \in D} \int |fg| \, d\mu = \rho'(f).$$

Hence $\sup_{t \in K} \rho(f_t) \leq \rho'(f)$.

For each $t \in E$ we define the function $T_t \colon \Lambda \to \Lambda$ by $T_t(f) = f_t$. We say $\{T_t\}_{t \in E}$ is a partition of the identity I on Λ if $\lim_{t \to 0} f_t = f$ for the strong topology on Λ for each $f \in \Lambda$.

COROLLARY 1. *If Ω is translation invariant and $\{T_t\}_t$ is a partition of I for the weak topology on Λ, then for seminorm ρ of $S(\Lambda, \Lambda^*)$ and K compact, there is a seminorm ρ' of $S(\Lambda, \Lambda^*)$ for which $\sup_{t \in K} \rho(f_t) \leq \rho'(t)$ for any $f \in \Lambda$.*

Proof. Let $t_n \to 0$ in E and let $f \in \Lambda$. By our hypothesis if $g \in \Lambda^*$, $\lim_n |\int_E (f_{-t_n} - f) g \, d\mu| = 0$. Thus there exists a positive number M for which

$$\left|\int f \cdot g_{t_n} \, d\mu\right| = \left|\int f_{-t_n} g \, d\mu\right| \leq \left|\int f \cdot g \, d\mu\right| + M \quad \text{for all } n.$$

Thus $\{g_{t_n}\}_n$ is weakly bounded in Λ^* and the corollary follows from Proposition 2.

COROLLARY 2. *If $B \subset \Lambda^*$ is σ-bounded, normal and there exists an open set V in E such that $\bigcup\{B_t \colon t \in V\}$ is σ-bounded, then $\bigcup\{B_t \colon t \in K\}$ is σ-bounded for any $K \subset E$ compact.*

Proof. Since V is open there exists a relatively compact 0-neighborhood $U \subset E$ and $y \in V$ such that $y + U \subset V$. Further

$$\int_E f(g)_{t+y} \, d\mu = \int f_{-y} g_t \, d\mu \quad \text{for } t \in U$$

implies $\bigcup\{B_t\colon t \in U\}$ is also σ-bounded by the hypothesis. Let K be any compact subset of E and $f \in \Lambda$. Then there exist n elements $\{t_1, t_2, \ldots, t_n\} \subset K$ for which $K \subset \bigcup_{i=1}^{n} (t_i + U)$.

As $\bigcup\{B_t\colon t \in U\}$ is σ-bounded and $\{f_{-t_i}\}_{i=1}^{n}$ is a finite set of functions in Λ we have

$$\sup_{i \leq n} \sup_{g_t \in \cup_U B_t} \left\{\int |f_{-t_i} \cdot g_t|\, d\mu \colon\right\} < \infty.$$

Because every $s \in K$ is of form $s = t_i + t$ for some $t \in K$ and some $i \leq n$ we have

$$\sup_{s \in K} \sup_{g \in B} \int |f \cdot g_s|\, d\mu = \sup_{i \leq n} \sup_{t \in U} \sup_{g \in B} \int |f_{-t_i} \cdot g_t|\, d\mu < \infty.$$

This holds for any $f \in \Lambda$; so the corollary is proved.

Notice that if Λ is a Banach space with norm ρ and satisfies the conditions of Proposition 2 or its corollaries, since each E_n: $n = 1, 2, \ldots$ is compact there exist numbers $\{M_n\colon n = 1, 2, \ldots\}$ satisfying

$$\sup_{t \in E_n} \rho(f_t) \leq M_n \cdot \rho(f) \quad \text{for each } n \text{ and } f \in \Lambda.$$

As shown in Remark 1 we cannot conjecture any further that there exists a constant M such that $\sup_{t \in E_n} \rho(f_t) \leq M\rho(f)$ for all n.

5. In this section we show that the net $\{T_t\}$ being a partition of I is equivalent to the Köthe dual being the same as the topological dual. The assumption of Proposition 2 in the previous section is vital to the discussion; we give an example showing the assumption is not true even in a restrictive case.

Example. We construct Köthe space Λ which is the intersection of Köthe spaces associated with the translations of a single function of compact support and demonstrate that $\{g_{t_n}\}_n$ is not weakly bounded in Λ^* for a $g \in \Lambda^*$ and a sequence $t_n \to 0$ in E.

Let $E = R^1$ with μ Lebesgue measure and let

$$g(x) = \sum_{n=2}^{\infty} \frac{1}{n} \left(x - \frac{1}{n}\right)^{-1/2 + 1/n4} \chi_{[1/n, 1/(n-1)]}.$$

Then $g(x) \in L^1$ since

$$\int_{R1} g(x)\, dx = \sum_{n=2}^{\infty} \frac{1}{n} \int_{1/n}^{1/(n-1)} \left(x - \frac{1}{n}\right)^{-1/2 + 1/n4} \cdot dx$$

$$= \sum_{2}^{\infty} \left(\frac{\frac{1}{n} \left[\frac{1}{(n-1)n}\right]}{\frac{1}{2} + \frac{1}{n^4}} \right)^{1/2 + 1/n4} < \sum_{2}^{\infty} \frac{2}{n2} < \infty.$$

The sets $L^1_{g_t} = \{f \in \Omega\colon \int_{R^1} |f \cdot g_t|\, d\mu < \infty\}$ are Köthe spaces for each $t \in R^1$ as is the space $\Lambda = \bigcap_t L^1_{g_t}$ which is translation invariant.

Let $f = x^{-1/2} \cdot \chi_{[0,1]}$. To see that $f \in \Lambda$ we must show $f_t \cdot g$ is integrable for every $t \in R^1$. Clearly the only t we need check are $t = 0, -1/n$ $(n = 1, 2, \ldots)$.

For $t = 0$,

$$\begin{aligned}\int f \cdot g\, d\mu &= \sum \frac{1}{n} \int_{1/n}^{1/(n-1)} x^{-1/2} \left(x - \frac{1}{n}\right)^{-1/2 + 1/n4} dx \\ &\leq \sum \frac{1}{n} \int_{1/n}^{1/(n-1)} \sqrt{n} \left(x - \frac{1}{n}\right)^{-1/2 + 1/n4} \\ &\leq 2 \sum \frac{1}{n^{3/2}} < \infty.\end{aligned}$$

For any fixed integer $m = 1, 2, \ldots$ set $t_m = -1/m$ and we have

$$\begin{aligned}\int f_{-1/m} \cdot g\, d\mu &= \int_{1/m}^{1/(m-1)} \frac{1}{m} \left(x - \frac{1}{m}\right)^{-1/2} \left(x - \frac{1}{m}\right)^{-1/2 + 1/m4} dx \\ &\quad + \sum_{n=1}^{m-1} \frac{1}{n} \int_{1/n}^{1/(n-1)} \left(x - \frac{1}{m}\right)^{-1/2} \cdot \left(x - \frac{1}{n}\right)^{-1/2} dx \\ &= \frac{1}{m} \left(\frac{1}{m(m-1)}\right)^{1/m4} m^4 + \sum_{n=1}^{m-1} \frac{1}{n} \int (\quad)(\quad)\, dx\end{aligned}$$

is clearly finite. But

$$\int f \cdot g_{1/m}\, d\mu = \int f_{-1/m}\, g\, d\mu \geq m^3 \quad \text{for each } m = 1, 2, \ldots.$$

Thus $\{g_{1/m}\}_m$ is not weakly bounded in Λ^*.

In view of the preceding remarks we make the following definition.

A translation invariant Köthe space Λ is Translation Bounded if $\{g_{t_n}\}_n$ is weakly bounded in Λ^* for any $g \in \Lambda^*$ and any sequence $\{t_n\}$ in E convergent to 0. Because of Proposition 2 this is equivalent to having $\bigcup_{t \in K} B_t = \bigcup_{t \in K} \{f_t\colon f \in B\}$ weakly bounded in Λ^* for any $B \subset \Lambda^*$ that is weakly bounded and $K \subset E$ that is compact.

With our consideration of the functions $\{f_t\}$ we have to make use of the functions $T_U f$ for $f \in \Lambda$ and relatively compact 0-neighborhood $U \subset E$ defined by

$$T_U f(x) = \frac{1}{\mu(U)} \int f(x + t)\, d\mu(t) \quad \text{for } x \in E.$$

From now on we use dt or dx in place of $d\mu(t)$ or $d\mu(x)$. With the obvious definition $\{U\colon U \in \mathscr{U}\}$ is a directed set and $\{T_U\}_{U \in \mathscr{U}}$ is a net.

The following theorem will be useful.

THEOREM 1. *If Λ is translation invariant and translation bounded, then we have the following*:

(i) *If $f \in \Lambda$, then $T_U f \in \Lambda$ for $f \in \Lambda$ and $U \in \mathscr{U}$.*
(ii) *$T_U f$ is continuous for $f \in \Omega$ and $U \in \mathscr{U}$.*
(iii) *If $f_t \to f$ as $t \to 0$, then $T_U f \to f$ as $U \to 0$ for $S(\Lambda, \Lambda^*)$ and $f \in \Lambda$.*

Proof. (i) Let $f \in \Lambda$, $g \in \Lambda^*$ and $U \in \mathscr{U}$. Since U is relatively compact and Λ is translation bounded the set $\{\int_E f_t g \, d\mu : t \in U\}$ is bounded by some constant M. We then have

$$\begin{aligned}\int_E T_U f \cdot g \, d\mu &= \int_E \frac{1}{\mu(U)} \int_U (f(x+t) \, dt) g(x) \, dx \\ &= \frac{1}{\mu(U)} \int_U \left(\int_E f(x+t) g(x) \, dx \right) dt \\ &= \frac{1}{\mu(U)} \int_U \left(\int_E f_t \cdot g \, d\mu \right) dt < M.\end{aligned}$$

Thus $T_U f \in \Lambda$.

(ii) The proof of (ii) runs similarly to the proof of Lemma 2 in [13].

Let $f \in \Omega$, $U \in \mathscr{U}$ and $x \in E$. Let E be a positive number. There exists an integer n such that $x \in E_n$, and an integer $m > n$ such that the compact $E_n + U + U + U \subset E_m$. Since f is integrable on E_m, there exist a $\delta > 0$ for which $A \subset E_m$ and $\mu(A) < \delta$ implies $\int_A |f| \, d\mu < \mu(U)(\varepsilon/2)$.

Let V be a symmetric open 0-neighborhood in E such that $\mu(V + U \mid U) < \delta$ and $V \subset U$. If $y \in x + V$, then for $t \in V + U$ we have $t + y \in E_m$ and $t + x \in E_m$ implies the following

$$\begin{aligned}\mu(U) |T_U f(x) - T_U f(y)| &= \left| \int_U f(t+x) \, dt - \int_U f(t+y) \, dt \right| \\ &= \left| \int_{U+x} f(t) \, dt - \int_{U+y} f(t) \, dt \right| \\ &= \left| \int_{U+x/U+y} f(t) \, dt \right| + \left| \int_{U+y/U+x} f(t) \, dt \right| \\ &= \left| \int_{x-y+U/U} f(t+y) \, dt \right| + \left| \int_{y-x+U/U} f(t+x) \, dt \right| \\ &\leq \int_{V+U/U} |f(t+y)| \, dt + \int_{V+U/U} |f(t+x)| \, dt \\ &< \mu(U)(\varepsilon/2) + \mu(U)(\varepsilon/2) \\ &= \mu(U)\varepsilon.\end{aligned}$$

Thus $T_U f$ is continuous at x.

(iii) Let $B \subset \Lambda^*$ be normal and weakly bounded. If $\varepsilon > 0$ is given, then $f_t \to f$ as $t \to 0$ implies there is a $U \in \mathscr{U}$ such that $S_B(f_t - f) < \varepsilon$ if $t \in U$. Consequently, if $V \subset U$ we have

$$\begin{aligned} S_B(T_v f - f) &= \sup_{g \in B} \int_E (T_v f - f) g \, d\mu \\ &= \sup_{g \in B} \int_E \left[\frac{1}{\mu(V)} \int_V (f(x+t) - f(x)) \, dt \right] g(x) \, dx \\ &= \sup_{g \in B} \frac{1}{\mu(V)} \int_V \left[\int_E (f(x+t) - f(x)) g(x) \, dx \right] dt \\ &\leq \frac{1}{\mu(V)} \int_V \left[\sup_{g \in B} \int_E (f_t - f) g \, d\mu \right] d\mu(t) \\ &= \frac{1}{\mu(V)} \int_V S_B(f_t - f) \, d\mu(t) \\ &< \varepsilon. \end{aligned}$$

Thus $T_U f \to f$ as $U \to 0$ for $S(\Lambda, \Lambda^*)$.

THEOREM 2. *If Λ is a translation invariant Köthe space given the strong topology $S(\Lambda, \Lambda^*)$, the following two statements are equivalent:*

(1) *Λ is translation bounded, Λ^* can be identified with the topological dual Λ' of Λ^*.*

(2) *$\{T_t\}_{t \in E}$ is a partition of the identity in Λ.*

Proof. We first show (1) implies (2). We begin by showing $\lim_{t \to 0} \chi_{A+t} = \chi_A$ and $\lim \chi_{A+V} = \chi_A$ as $V \to 0$ ($V \in \mathscr{U}$) a.e. and for $S(\Lambda, \Lambda^*)$ whenever $A \subset E$ with $\mu(A) < \infty$. Since

$$\mu\{x \colon |\chi_{V+A} - \chi_A| \neq 0\} = \mu(V + A/A) \to 0 \quad \text{as } V \to 0 \ (V \in \mathscr{U})$$

because of the regularity of μ, we have $\lim \chi_{A+V} = \chi_A$ a.e. Further if $\varepsilon > 0$ is given there is a $V \in \mathscr{U}$ for which $\mu(V + A/A) < \varepsilon/2$. Consequently for $t \in V$ symmetric we have

$$\begin{aligned} \mu\{x \colon |\chi_{A+t}(x) - \chi_A(x)| \neq 0\} &= \mu(A + t/A) + \mu(A/A + t) \\ &= \mu(A + t/A) + \mu(A - t/A) \\ &\leq \mu(V + A/A) + \mu(V + A/A) \\ &< \varepsilon. \end{aligned}$$

Thus $\lim_{t\to 0} \chi_{A+t} = \chi_A$ a.e. For convergence in $S(\Lambda, \Lambda^*)$ let us consider the case where A is compact and let $B \subset \Lambda^*$ be normal weakly bounded and weakly closed (without loss of generality). Given $\varepsilon > 0$, for each V a compact symmetric neighborhood of 0 in E consider the sets

$$A_V = \left\{ g \in \Lambda^* \colon \int g \cdot \chi_{A+V} \, d\mu < \varepsilon \right\}.$$

The sets are open for the weak topology $\sigma(\Lambda^*, \Lambda)$ in Λ^*. Since each $g \in \Lambda^*$ is also in Ω and $\mu(V + A/A) \to 0$ as $V \to 0$, we can always find a V for which $\int_{A+V/A} g < \varepsilon$ and therefore $\{A_V \colon V \in U\}$ covers $B \subset \Lambda^*$. Because of [15, Theorem 2] B is compact for $\sigma(\Lambda^*, \Lambda)$. Hence there are a finite number of symmetric 0-neighborhoods $V_1, V_2, \ldots, V_n$ in E such that $B \subset \bigcup_{i=1}^n A_{V_i}$. Letting $V_0 = \bigcap_{i=1}^n V_i$, we have

$$\int \chi_{A+V} \, g \, d\mu \le \int \chi_{A+V_0} \, g \, d\mu < \varepsilon \quad \text{for all } V \subset V_0 \text{ and } g \in B.$$

Thus

$$S_B(\chi_{A+V} - \chi_A) = S_B(\chi_{A+V/A}) = \sup_{g \in B} \int \chi_{A+V} \, g \, d\mu < \varepsilon \quad \text{for } V \subset V_0.$$

We have now $\lim \chi_{A+V} = \chi_A$ for $S(\Lambda, \Lambda^*)$. Now let A be a set with $\mu(A) < \infty$ there is a sequence of compact sets $\{A_n\}_{n=1}^\infty$ such that $A_n \subset A_{n+1}$, $n = 1, 2, \ldots$ and $\bigcup_n A_n = A$. Since

$$\chi_A - \chi_{A_n} = \chi_{A/A_n} \downarrow n_0 \text{ a.e.} \quad \text{and} \quad \chi_{A+V} - \chi_{A_n+V} = \chi_{A+V/A_n+V} \downarrow n_0 \text{ a.e.},$$

the fact that $\Lambda^* = \Lambda'$ implies (by [15])

$$S_B(\chi_{A+V} - \chi_{A_n+V}) \downarrow n_0 \quad \text{and} \quad S_B(\chi_A - \chi_{A_n}) \downarrow 0.$$

If $\varepsilon > 0$ is given and fixing a V_0 there is an n for which

$$S_B(\chi_{A+V_0} - \chi_{A_n+V_0}) < \varepsilon/3 \quad \text{and} \quad S_B(\chi_A - \chi_{A_n}) < \varepsilon/3.$$

For this fixed n there is a $V \subset V_0$ such that $S_B(\chi_{V+A_n} - \chi_{A_n}) < \varepsilon/3$. Thus for $V \subset V_0$,

$$S_B(\chi_{V+A} - \chi_A) \le S_B(\chi_{V+A} - \chi_{V+A_n}) + S_B(\chi_{V+A_n} - \chi_{A_n}) + S_B(\chi_{A_n} - \chi_A) < \varepsilon.$$

Therefore, $\lim \chi_{A+V} = \chi_A$ for any A of finite measure. We use this information to show $\chi_{A+t} \to \chi_A$ as $t \to 0$. To do this fix a $V_0 \in \mathscr{U}$ and let $D = \bigcup_{t \in V_0} B_t$ which is also weakly bounded in Λ^* because of our hypothesis and the fact that V_0 is relatively compact. Since $\lim \chi_{A+V} = \chi_A$ for any $\varepsilon > 0$ there is a $V \subset V_0$ such that $S_D(\chi_{A+V} - \chi_A) < \varepsilon/2$. Then for $t \in V$

$$\begin{aligned}
S_B(\chi_{A+t} - \chi_A) &= S_B(\chi_{A+t/A} + \chi_{A/A+t}) \\
&\le S_B(\chi_{A+t/A}) + S_B(\chi_{A/A+t}) \\
&= \sup_{g \in B} \int \chi_{A+t/A}\, g \, d\mu + \sup_{g \in B} \int \chi_{A/A+t}\, g \, d\mu \\
&= \sup_{g \in B} \int \chi_{A+t/A}\, g \, d\mu + \sup_{g \in B} \int \chi_{A-t/A}\, g_{-t} \, d\mu \\
&\le \sup_{g \in B} \int \chi_{V+A/A}\, g \, d\mu + \sup_{g \in B} \int \chi_{V+A/A}\, g_{-t} \, d\mu \\
&\le \sup_{g \in D} \int \chi_{V+A/A}\, g \, d\mu + \sup_{g \in D} \int \chi_{V+A/A}\, g \, d\mu \\
&= S_D(\chi_{A+V} - \chi_A) + S_D(\chi_{V+A} - \chi_A) \\
&< \varepsilon.
\end{aligned}$$

Now that we have shown $\chi_{A+t} \to \chi_A$ as $t \to 0$ for the strong topology if $\mu(A) < \infty$ it follows that $S_t \to S$ as $t \to 0$ for $S(\Lambda, \Lambda^*)$ for any simple function S whose support is of finite measure.

We are now ready to show that $\{T_t\}$ forms a partition of the identity. For this purpose let $f \in \Lambda$ and we will assume without loss of generality that $f \ge 0$ a.e.; let $B \subset \Lambda^*$ be normal and weakly bounded, and suppose $\varepsilon > 0$ is given. If U is a relatively compact symmetric open set containing 0 in E; then $D = \bigcup \{B_t \colon t \in U\}$ is also weakly bounded in Λ^* because Λ is translation bounded. Let $\{S_n\}_{n=1}^{\infty}$ be an increasing sequence of simple functions each of whose support is of finite measure such that $S_n \uparrow f$ a.e. Because $\Lambda^* = \Lambda'$ implies (by [15] $(f - S_n) \downarrow 0$ for $S(\Lambda, \Lambda^*)$, there exists an n_0 such that for $n \ge n_0$, $S_D(f - S_n) < \varepsilon/3$ for $n \ge n_0$. Choose an $n > n_0$; since S_n is a simple function the fact that $S_B((S_n)_t - S_n) \to 0$ as $t \to 0$ implies there is a symmetric 0-neighborhood $V \subset U$ such that $S_B((S_n)_t - S_n) < \varepsilon/3$ for all $t \in V$. Consequently for $t \in V$ we have

$$\begin{aligned}
S_B(f_t - f) &\le S_B(f_t - (S_n)_t) + S_B((S_n)_t - S_n) + S_B(S_n - f) \\
&< \sup_{g \in B} \int_E (f_t - (S_n)_t) g \, d\mu + \varepsilon/3 + \sup_{g \in B} \int_E (S_n - f) g \, d\mu \\
&= \sup_{g \in B} \int_E (f - S_n) \cdot g_{-t} \, d\mu + \varepsilon/3 + \sup_{g \in B} \int_E (S_n - f) g \, d\mu \\
&\le \sup_{g \in D} \int_E (f - S_n) g \, d\mu + \varepsilon/3 + \sup_{g \in D} \int_E (S_n - f) \cdot g \, d\mu \\
&= 2 S_D(f - S_n) + \varepsilon/3 \\
&< \varepsilon.
\end{aligned}$$

This completes the proof that (1) implies (2).

Now we show (2) implies (1). First, if $\{T_t\}_{t \in E}$ is a partition of the identity for the strong topology, then it is a portion of I for the weak topology. Because of this Corollary 1 implies Λ is translation bounded. We show $\Lambda^* = \Lambda'$ by showing an equivalent property—Λ contains a countable dense subset [15]. Fix an E_m and let $\{V_n\}_{n=1}^{\infty}$ be a sequence of relatively compact, symmetric, open 0-neighborhoods such that $V_{n+1} \subset V_n$, $n = 1, 2, \ldots$ and $V_n \downarrow \{0\}$. Since E_m is compact, for each n there exist a finite number k_n and points $= \{x_j^n : j = 1, \ldots, k_n\}$ such that the sets

$$\{V_j^n = V_n + x_j^n : j = 1, \ldots, k_n\}$$

cover E_m. For each n we consider the disjoint sets

$$\left\{V_j^n \Big/ \bigcup_{j \neq i} V_i^n : j = 1, \ldots, k_n\right\}, \qquad \left\{V_j^n \cap V_i^n \Big/ \bigcup_{p \neq i,j} V_p^n : i, j = 1, \ldots, k_n\right\}.$$

The sets $S(n, m)$ of simple functions with a single rational value on each of these disjoint sets is countable for each fixed $n, m = 1, 2, \ldots$, and the collection of all such simple functions over all n, m is a countable set. Suppose ϕ is a continuous function whose support is contained in E_m. Given $\varepsilon \to 0$, since ϕ is uniformly continuous there exists an n such that $|\phi(x) - \phi(y)| < \varepsilon$ if $x - y \in V_n$. Choose a set of rational numbers $\{\Gamma_j : j = 1, \ldots, k_n\}$ such that $|\phi(x_j) - \Gamma_j| < \varepsilon/2$, $j = 1, \ldots, k_n$, and let $S(x)$ be the simple function contained in $S(n, m)$ of previous construction with value Γ_j on $V_j^n / \bigcup_{j \neq i} V_i^n$ ($j = 1, \ldots, k_n$) and value Γ_j or Γ_i on $V_j^n \cap V_i^n / \bigcup_{p \neq i,j} V_p^n$ ($j, i = 1, \ldots, k_n$). If $x \in E_m$, there is an index j for which $x \in V_j^n = V_n + x_j^n$ and Γ_j is the value of $S(x)$. Since $x - x_j^n \in V_n$ we have $|\phi(x) - \phi(x_j)| < \varepsilon/2$ and therefore

$$|\phi(x) - S(x)| = |\phi(x) - \Gamma_j| \leq |\phi(x) - \phi(x_j)| + |\phi(x_j) - \Gamma_j| < \varepsilon.$$

This shows us that for any continuous function of compact support and any $\varepsilon > 0$ there is an $S \in \bigcup_{n,m} S(n, m)$ (a countable collection) such that $\|\phi - S\|_\infty < \varepsilon$.

Now let $f \in \Lambda$ and $B \subset \Lambda^*$ be normal and weakly bounded. Since it is known that $S_B(f \cdot (1 - \chi_{E_m}) \downarrow 0$ (see [14]), there is an m such that $S_B(f \cdot (1 - \chi_{E_m})) < \varepsilon/3$. Since $\{T_t\}_{t \in E}$ is a partition of I, Theorem 1 (iii) implies $T_U f \to f$ as $U \to 0$. Thus there exist a compact 0-neighborhood $U \subset E$ such that $S_B(T_U(f \cdot \chi_{E_m}) - f \cdot \chi_{E_m}) < \varepsilon/3$; and since Theorem 1 implies that $T_U(f \cdot \chi_{E_m}) \in \Lambda$ is a continuous function of compact support contained in $E_m + U$, there is an $S \in \bigcup_{n,m} S(n, m)$ such that

$$\|T_U(f \cdot \chi_{E_m}) - S\|_\infty < \frac{\varepsilon}{3S_B(\chi_{E_m} + U)}.$$

Therefore

$$\begin{aligned} S_B(f - S) &\leq S_B(f \cdot (1 - \chi_{E_m})) + S_B(f \cdot \chi_{E_m} - T_U(f \cdot \chi_{E_m})) \\ &\quad + S_B(T_U(f \cdot \chi_{E_m}) - S) \\ &< \varepsilon/3 + \varepsilon/3 + \|T_U(f \cdot \chi_{E_m}) - S\|_\infty S_B(\chi_{E_m} + U) \\ &< \varepsilon. \end{aligned}$$

This completes the proof of (2) implies (1) and the proof of Theorem 2.

COROLLARY. *If Λ is translation invariant and $\{T_t\}_{t \in E}$ is a partition of the identity for the strong topology, then $C_0(E)$ the continuous functions of compact support is dense in Λ for the strong topology.*

Proof. This follows obviously from the last part of the proof of Theorem 2.

THEOREM 3. *If Λ is translation bounded and $C_0(E)$ is strongly dense in Λ, then $\{T_t\}$ is a partition of the identity in Λ for the strong topology.*

Proof. We first show that for any $\phi \in C_0(E)$ a continuous function of compact support $\phi_t \to \phi$ as $t \to 0$. Suppose ϕ has its support in K compact. Let $B \subset \Lambda^*$ be weakly bounded and normal and let $\varepsilon > 0$ be given. Fix V a relatively compact open 0-neighborhood in E. If $S_B(\chi_{V+K}) = 0$, then because ϕ, ϕ_t for $t \in V$ will have its support in $K + V$ we can easily see

$$S_B(\phi_t - \phi) = \sup_{g \in B} \int (\phi_t - \phi) \cdot \chi_{V+K} \cdot g \, d\mu = 0$$

and the result $\phi_t \to \phi$. If $S_B(\chi_{V+K}) \neq 0$, then the uniform continuity of ϕ implies there exists a 0-neighborhood $U \subset V$ for which

$$|\phi(x) - \phi(y)| < \frac{\varepsilon}{S_B(\chi_{V+K})} \quad \text{whenever } x - y \in U.$$

Then for $t \in U$, since ϕ, ϕ_t have support in $K + V$,

$$\begin{aligned} S_B(\phi_t - \phi) &= \sup_{g \in B} \int (\phi(t + x) - \phi(x)) \cdot \chi_{K+V} \cdot g(x) \, dx \\ &< \sup_{g \in B} \int \frac{\varepsilon}{S_B(\chi_{V+K})} \chi_{V+K} \cdot g \, d\mu \\ &= \frac{\varepsilon}{S_B(\chi_{V+K})} S_B(\chi_{V+K}) \\ &= \varepsilon. \end{aligned}$$

Thus $\{T_t\}$ is a partition of I in $C_0(E)$ strongly.

Now let $f \in \Lambda$, $B \subset \Lambda^*$ weakly bounded and $\varepsilon > 0$ be given. As Λ is translation bounded we can fix a symmetric relatively compact open 0-neighborhood V in E such that $D = \bigcup_{t \in V} B_t$ is weakly bounded. Since $C_0(E)$ is strongly dense in Λ, there exists a $\phi \in C_0(E)$ satisfying $S_D(f - \phi) < \varepsilon/3$. By virtue of $\{T_t\}$ partitioning I in $C_0(E)$ there is a 0-neighborhood $U \subset V$ such that $t \in U$ implies $S_B(\phi_t - \phi) < \varepsilon/3$. For such t we have

$$\begin{aligned} S_B(f_t - f) &\leq S_B(f - \phi) + S_B(\phi - \phi_t) + S_B(f_t - \phi_t) \\ &< \varepsilon/3 + \varepsilon/3 + \sup_{g \in B} \int (f_t - \phi_t) g \, d\mu \\ &= \varepsilon/3 + \varepsilon/3 + \sup_{g \in B} \int (f - \phi) g_{-t} \, d\mu \\ &\leq 2\varepsilon/3 + S_D(f - \phi) \\ &< \varepsilon. \end{aligned}$$

Thus $\{T_t\}$ is a partition of I in Λ.

COROLLARY. *If Λ is translation bounded and $C_0(E)$ is strongly dense in Λ, then $\Lambda^* = \Lambda'$.*

Proof. This is an obvious consequence of Theorem 3 and Theorem 2, (2) implies (1).

Remark. Even though all of our results had the strong topology on Λ, they could have been generalized to any of the Köthe topologies on Λ which are generated by subsets of weakly bounded sets in Λ^* provided we stipulated that there always exists a compact 0-neighborhood $U \subset E$ for which $\bigcup_{t \in U} B_t$ is a member of that set of subsets whenever B is a member.

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