ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 3, Fall 1980

COMMUTATIVE POLYNOMIAL GROUP LAWS OVER VALUATION RINGS[1]

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Let A be a discrete valuation ring with fraction field K and residue field k. Let R be a finitely generated flat A-algebra, and suppose that $R \otimes K$ and $R \otimes k$ are polynomial rings. It is not known in general whether R must then be a polynomial ring. We show here that it is so when R is the ring of functions on a commutative group scheme.

The argument in this paper rests on Néron blow-ups of group schemes [4, Section 1], and I am grateful to Boris Weisfeiler for suggesting this problem as one where blow-ups might be useful. The results needed are summarized in the first section. We then require information on polynomial groups over the residue field: we show that any two primitive coordinate systems differ only by relatively simple variable changes. This allows us to make a specified subgroup occur as a coordinate hyperplane, and the argument from then on is essentially computational.

1. Review of Néron blow-ups

Let $G = \operatorname{Spec} A[G]$ be a flat affine group scheme of finite type over the discrete valuation ring A. Tensoring with the fraction field K, we can by flatness identify the Hopf algebra $A[G]$ with a subalgebra of $K[G] = A[G] \otimes_A K$. Let H be a closed subgroup of the special fiber G_k; it is defined by some ideal $J = (\pi, f_1, \ldots, f_n)$, where π is the uniformizer. Then $A[\pi^{-1}J] = A[G][\pi^{-1}f_1, \ldots, \pi^{-1}f_n]$ is another Hopf subalgebra of $K[G]$, and we say that the group scheme $G^H = \operatorname{Spec} A[\pi^{-1}J]$ is obtained by *blowing up H* in G. If G' is any other such flat group scheme, and $G' \to G$ is a homomorphism which on the special fiber sends G'_k into H, then it factors through a homomorphism $G' \to G^H$.

Suppose now that $G' \to G$ is an isomorphism over K. We can blow up the image of G'_k, getting a new group to which G' maps. The basic fact [4, 1.4] is that after finitely many repetitions of this process we obtain a group isomorphic to G'.

2. Primitive coordinate systems

In this section G will be an affine group scheme over a field k. We call G *polynomial* if $k[G]$ is a (finitely generated) polynomial ring. It is known that this

Received July 17, 1978.

[1] This work was partially supported by the National Science Foundation.

holds if and only if G is smooth, connected, unipotent, and k-solvable [1, p. 536]. Any quotient of G inherits these properties and hence is again polynomial.

Let $x_1, \ldots, x_n$ be coordinates on a polynomial group G. We call them *primitive* coordinates if for each index r we have

$$x_r(gh) = x_r(g) + x_r(h) + f_r(x_1(g), x_1(h), \ldots, x_{r-1}(g), x_{r-1}(h)).$$

It is known that G possesses primitive coordinate systems [3, p. 102]. We call a change of coordinates *permissible* if it arises by a sequence of changes each of which either multiplies some x_i by a constant or adds to x_i some polynomial in the other variables.

THEOREM 1. *Let G be a polynomial group over a field. Then any primitive coordinate system on G can be brought to agree with any other by a permissible change.*

Proof. We first construct a special primitive coordinate system. Inside G, consider all central subgroups which are isomorphic to G_a^r for some r. The product of two such is again one (quotient of the direct product), so there is a largest such subgroup G_1. It is nontrivial, since in fact $x_1 = \cdots = x_{n-1} = 0$ in any primitive coordinate system defines such a subgroup. As G/G_1 is unipotent, the group extension $1 \to G_1 \to G \to G/G_1 \to 1$ has a scheme-theoretic section [1, p. 535], and we can write G as $(G/G_1) \times G_1$ with $(h, x) \cdot (h', x') = (hh', x + x' + f(h, h'))$ for some cocycle f. Let $z_1, \ldots, z_r$ be additive coordinates on G_1, and $y_1, \ldots, y_m$ primitive coordinates on G/G_1; then $y_1, \ldots, y_m, z_1, \ldots, z_r$ are primitive coordinates on G.

The theorem is obvious when G has dimension one, and we proceed by induction on the dimension. Let $x_1, \ldots, x_n$ be any primitive coordinate system. It is enough to get the x_i by permissible changes from the y, z system above. Let N be defined by $x_1 = \cdots = x_{n-1} = 0$, so that $N \simeq G_a \subseteq G_1$. By another theorem of Rosenlicht [2, p. 688] we can make p-polynomial variable changes (which are certainly permissible) on $z_1, \ldots, z_r$ to get N defined inside G_1 by $z_1 = \cdots = z_{r-1} = 0$. Since each $z_i(gg') - z_i(g) - z_i(g')$ involves only y-terms, the coordinates constructed in this way are still primitive coordinates on G.

Now $x_1, \ldots, x_{n-1}$ and $y_1, \ldots, y_m, z_1, \ldots, z_{r-1}$ are both primitive coordinates on G/N. By induction we can make permissible changes to get them to agree. We have then

$$k[x_1, \ldots, x_{n-1}][x_n] = k[y_1, \ldots, z_{r-1}][z_r] = k[x_1, \ldots, x_{n-1}][z_r].$$

This forces z_r to equal $cx_n + w$ with c constant and w in $k[x_1, \ldots, x_{n-1}]$; for base change to $k(x_1, \ldots, x_{n-1})$ shows that z_r must have degree one in x_n, and base change to residue fields shows that c must be invertible in $k[x_1, \ldots, x_{n-1}]$. Thus finally we can make a permissible change from z_r to x_n. ∎

3. Blow-ups of polynomial groups

THEOREM 2. *Let A be a discrete valuation ring, G a commutative affine group scheme over A. Assume that $A[G]$ is a polynomial ring $A[X_1, \ldots, X_n]$ with the X_i reducing to primitive coordinates on the special fiber G_k. Let H be a polynomial subgroup of G_k. Then $A[G^H]$ is a polynomial ring $A[W_1, \ldots, W_n]$ with the W_i reducing to primitive coordinates on the special fiber.*

Proof. The argument used for G_1 at the start of Theorem 1 shows that there is some primitive coordinate system $y_1, \ldots, y_n$ on G_k such that H is defined by $y_1 = \cdots = y_r = 0$. Theorem 1 shows that we can obtain the y_i by permissible changes from the images of the X_i. The crucial fact now is that permissible changes obviously all lift to A. Hence we can change the X_i and assume that H is defined by $X_1 \equiv \cdots \equiv X_r \equiv 0$. Changing by constants, we may also assume $\varepsilon(X_i) = 0$. The ring $A[G^H]$ is obtained from $A[G]$ by adjoining the elements $Z_1, \ldots, Z_r$ with $Z_i = \pi^{-1} X_i$. Thus $A[G^H]$ is a polynomial ring $A[X_{r+1}, \ldots, X_n, Z_1, \ldots, Z_r]$.

It remains to see that these coordinates (in this order) give a primitive system on the special fiber. The comultiplication sends X_i to $X_i \otimes 1 + 1 \otimes X_i$ plus various other terms $a_{\alpha\beta} X^\alpha \otimes X^\beta$ such that:

(i) no other terms involve $1 = X^0$; and

(ii) if $a_{\alpha\beta}$ is not divisible by π, then the term involves only the X_j with $j < i$.

We now substitute πZ_j for X_j whenever $j \leq r$. Consider first the image of an X_i with $i > r$. A term $X^\alpha \otimes X^\beta$ involving any X_j with $j \leq r$ becomes divisible by π when we rewrite it using Z_j, so over k we have only terms involving the reductions of $X_{r+1}, \ldots, X_{i-1}$.

For $i \leq r$ the image of Z_i is π^{-1} times the image of X_i. In the image of X_i, a term $a_{\alpha\beta} X^\alpha \otimes X^\beta$ with π not dividing $a_{\alpha\beta}$ can involve only X_j with $j < i \leq r$; each factor has degree at least one, so when we rewrite using the Z_j we get at least a factor π^2. The terms with π dividing $a_{\alpha\beta}$ also become divisible by π^2 if they involve any X_j with $j \leq r$. Thus for the image of Z_i we get over k only terms involving the reductions of $X_{r+1}, \ldots, X_n$. ∎

4. Characterization of commutative polynomial groups

THEOREM 3. *Let G be a smooth commutative affine group scheme of finite type over the discrete valuation ring A. Assume $K[G]$ and $k[G]$ are polynomial rings. Then $A[G]$ is a polynomial ring.*

Proof. Choose primitive coordinates $Y_1, \ldots, Y_n$ on G_k, scaling them so that $Y_i \in A[G]$. Set $X_1 = Y_1$. Write $X_2 = \pi^{m(2)} Y_2$. The comultiplication Δ sends Y_2 to

$$Y_2 \otimes 1 + 1 \otimes Y_2 + f_2(Y_1 \otimes 1, 1 \otimes Y_1),$$

so by choosing $m(2)$ large enough we force ΔX_2 to have coefficients in A and be congruent to $X_2 \otimes 1 + 1 \otimes X_2$ modulo π. If we set $X_3 = \pi^{m(3)} Y_3$, we similarly then get these properties holding when $m(3)$ is large. In this way we get a Hopf algebra

$$A[F] = A[X_1, \dots, X_n]$$

inside $A[G]$, and the reductions of the X_i are primitive coordinates on F_k (which is $\simeq G_a^n$). The Hopf algebra inclusion $A[F] \subseteq A[G]$ corresponds to a homomorphism $G \to F$ which is an isomorphism over K. The image H of G_k in F_k is a quotient of G_k and hence is polynomial. Theorem 2 shows then that the blow-up F^H again has polynomial coordinates primitive on the special fiber. Repeating this inductively, we eventually reach a group isomorphic to G, and thus $A[G]$ is a polynomial ring. ▮

COROLLARY 4. *Let A be a discrete valuation rine with both K and k perfect. Let G be a commutative affine group scheme over A. Then $A[G]$ is a polynomial ring if and only if G is smooth with unipotent connected fibers.*

Proof. The nontrivial implication follows from the theorem, since k-solvability is automatic for smooth unipotent groups over perfect fields [1, p. 495]. ▮

Commutativity actually enters the arguments only in Theorem 2—if H is to be defined by $y_1 = \cdots = y_r = 0$ in primitive coordinates, it must be normal. This is no restriction in dimension 2, so in that case the the results are also valid for noncommutative groups.

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