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ELEMENTARY AMENABLE GROUPS

BY

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1. In order to explain the Hausdorff-Banach-Tarski paradox, von Neumann [19] introduced the class of amenable groups in 1929. Since then the theory of amenable groups has advanced in many fronts, for a survey see Day [4] and Greenleaf [9]. But algebraically the only known amenable groups and non-amenable groups were provided by von Neumann. He showed that finite groups and abelian groups are amenable and that the class of amenable groups, AG, is closed under four standard processes of constructing new groups from given ones: (I) *subgroups*, (II) *factor groups*, (III) *group extensions* and (IV) *direct unions*. As in Day [3] let EG be the smallest class of groups which contains all finite groups and all abelian groups and is closed under processes (I)–(IV). Then EG is contained in AG and, in fact, algebraically they constitute the only *known* amenable groups. We will call groups in EG *elementary amenable* groups. Von Neumann also showed that if a group contains a free subgroup on two generators then it is not amenable. Therefore, NF, the class of groups without free subgroup on two generators, contains AG. The notations AG and NF were also introduced by Day [3].

Von Neumann [19] asked whether $AG = NF$ and Day [3] pointed out it is not known whether $EG = AG$ or even $EG = NF$. We are unable to provide any new examples of amenable or non-amenable groups. But in this paper a better description of the known amenable groups will be given. More precisely, we will show that the groups in EG can be constructed from abelian groups and finite groups by applying processes (III) and (IV) only. By combining this fact with the existence of non-locally finite periodic groups, cf. [5], [20], we are able to conclude that $EG \subsetneq NF$. Therefore either $EG \neq AG$ or $AG \neq NF$ or both.

A finitely generated group G with a finite generating set F is said to be exponentially bounded if $(\text{card } F^n)^{1/n} \to 1$ as $n \to \infty$ where

$$F^n = \{x_1 \cdots x_n : x_i \in F\}.$$

This property is independent of the choice of F. Milnor [17] and Wolf [22] showed that if a finitely generated solvable group is exponentially bounded then it contains a nilpotent subgroup of finite index. By applying our description of elementary amenable groups we are able to extend their result to finitely

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generated groups in EG. Note that it is easy to construct finitely generated groups in EG which are not solvable-by-finite.

A group G is said to have property (P) if given a finite subset F in G there exist a finite set $S \supset F$ in G and a set $X \subset G$ such that $Sx_1 \cap Sx_2 = \emptyset$ if $x_1, x_2 \in X$, $x_1 \neq x_2$ and $\bigcup\{Sx: x \in X\} = G$. We will show that groups in EG have property (P). This fact will be applied to study the almost convergent sets in such groups. Many groups not in EG also have property (P) e.g. free groups. We are unable to decide whether there exist groups which fail to have property (P).

2. Let EG_0 be the class of all finite groups and all abelian groups. Assume that $\alpha > 0$ is an ordinal and that we have defined EG_β for each ordinal $\beta < \alpha$. Then if α is a limit ordinal, set $EG_\alpha = \bigcup\{EG_\beta: \beta < \alpha\}$ and if α is not a limit ordinal, set EG_α is the class of groups which can be obtained from groups in $EG_{\alpha-1}$ by applying either process (III) or process (IV) once and only once.

PROPOSITION 2.1. *Each EG_α is closed under processes* (I) *and* (II).

Proof. Clearly if $G \in EG_0$ then a subgroup of G and a factor group of G also belong to EG_0. Assume that $\alpha > 0$ and that EG_β is closed under processes (I) and (II) if $\beta < \alpha$. Let $G \in EG_\alpha$ and let B be a subgroup of G and C be a homomorphic image of G. We have to show that $B, C \in EG_\alpha$. If α is a limit ordinal then, by definition of EG_α, $G \in EG_\beta$ for some $\beta < \alpha$ and therefore, by assumption B, C are contained in EG_β and hence in EG_α. If $\alpha - 1$ exists, then there are two ways to obtain G from $EG_{\alpha-1}$: (i) G is an extension of D by E with $D, E \in EG_{\alpha-1}$, i.e., there is an exact sequence $e \to D \to G \to E \to e$, and (ii) G is a direct union of $\{G_\tau\}$, $G_\tau \in EG_{\alpha-1}$. For (i) note that B is the extension of $D \cap B$ by a subgroup F of E and C is the extension of a homomorphic image D_1 of D by a homomorphic image E_1 of E. By assumption, $D \cap B$, F, D_1 and E_1 belong to $EG_{\alpha-1}$. Therefore $B, C \in EG_\alpha$. For (ii), note that B is the direct union of $\{G_\tau \cap B\}$ and C is the direct union of $\{\theta(G_\tau)\}$ where θ is the homomorphism which sends G to C. Since, by assumption, $G_\tau \cap A$ and $\theta(G_\tau)$ belong to $EG_{\alpha-1}$, we conclude that B and C belong to EG_α. By transfinite induction, the proof is completed.

PROPOSITION 2.2. (a) $EG = \bigcup\{EG_\alpha$: *α an* ordinal$\}$.

(b) *EG is the smallest class of groups which contains all finite groups and all abelian groups and is closed under processes* (III) *and* (IV).

Proof. (a) Let $EG' = \bigcup_\alpha EG_\alpha$. Then $EG' \subset EG$. Clearly EG' is closed under (III) and by the above proposition it is also closed under (I) and (II). To see that EG' is closed under (IV), assume that G is a direct union of $\{G_\tau\}$ where each $G_\tau \in EG'$. Then for each τ there exists an ordinal α_τ such that $G_\tau \in EG_{\alpha_\tau}$. Let $\alpha = \sup_\tau \alpha_\tau$. Then $G \in EG_{\alpha+1} \subset EG'$. Therefore EG' is closed under (I)–(IV) and hence $EG = EG'$.

(b) is a direct consequence of (a).

Remark. Since the set of all finitely generated non-isomorphic groups has the cardinality of the continuum there exists a smallest ordinal α_0 such that if $\alpha \geq \alpha_0$ then EG_α and EG_{α_0} contain the same collection of finitely generated groups. Therefore $EG = EG_{\alpha_0+1}$.

Recall that a group G is periodic if each element of G is of finite order and G is locally finite if each finitely generated subgroup is finite. It is now well known that there exist periodic groups which are not locally finite, cf. Golod [5] and Novikov-Adjan [20].

THEOREM 2.3. *Every periodic group in EG is locally finite. Therefore* $EG \subsetneq NF$.

Proof. The second statement is a consequence of the first, because if G is a non-locally finite periodic group then $G \in NF \backslash EG$.

Clearly every periodic group in EG_0 is locally finite. Assume that $\alpha > 0$ and that each periodic group in EG_β is locally finite if $\beta < \alpha$. Let G be a periodic group in EG_α. If α is a limit ordinal then $G \in EG_\beta$ for some $\beta < \alpha$ and, by assumption, is locally finite. If α is not a limit ordinal then either G is a direct union of periodic and hence, by assumption, locally finite groups in $EG_{\alpha-1}$ or G is the middle term of an exact sequence $e \to B \to G \to C \to e$ where B, C are periodic (and hence locally finite) groups in $EG_{\alpha-1}$. Therefore G is locally finite by applying the well known and easy to prove facts: (i) A direct union of locally finite groups is locally finite. (ii) An extension of a locally finite group by another is locally finite. By transfinite induction and Proposition 2.2 we conclude that each periodic group in EG is locally finite.

Another consequence of Proposition 2.2 is the following.

COROLLARY 2.4. *A finitely generated simple group in EG is finite.*

Proof. Let G be a finitely generated simple group in EG. Let α be the smallest ordinal such that $G \in EG_\alpha$. Note that α is not a limit ordinal. If $\alpha > 0$ then since G is simple it has to be a direct union of groups $\{G_\tau\}$ in $G_{\alpha-1}$. Since G is finitely generated it equals G_τ for some τ. So $G \in EG_{\alpha-1}$, contradicting the minimality of α. So $\alpha = 0$ and hence G is finite.

The first finitely generated infinite simple group was constructed by Higman [10]. Let $H = \langle a, b, c, d; a^{-1}ba = b^2, b^{-1}cb = c^2, c^{-1}dc = d^2, d^{-1}ad = a^2 \rangle$ and N be a maximal ($\neq H$) normal subgroup of H. Higman showed that H/N is a (4-generated) simple non-periodic group and hence it doesn't belong to EG. We do not know whether it is possible to choose N so that H/N contains no free subgroups on two generators. If it could be chosen then we would have an example of a group in $NF \backslash EG$ which doen't depend on the existence of non-locally finite periodic groups. In fact a stronger conclusion could be made.

Let PA be the smallest class of groups which contains all abelian groups and all periodic groups and is closed under processes (I)–(IV). Let PA_0 be the class of all abelian groups and all periodic groups. Construct PA_α inductively by applying processes (III) and (IV) as in the case of EG. Then $PA = \bigcup_\alpha PA_\alpha$. In this setting, Proposition 2.4 can be stated as follows: A finitely generated simple group in PA belongs to PA_0. Since Higman's simple group $H/N \notin PA_0$ it doesn't belong to PA either. Note that by applying Novikov-Adjan [20] and Kostriken [11], B. H. Neumann [17, p. 71] has constructed an infinite simple finitely generated group with exponent p, p a prime larger than 665.

If it is possible to find N such that Higman's simple group belongs to NF then we may further ask whether it is possible to choose N so that $H/N \in AG$. If possible, then we would have found a group in $AG \backslash EG$. No matter whether it is possible to choose N so that $H/N \in NF$ we speculate that $PA \subsetneqq NF$.

3. Let G be a group with a finite generating set F. G is said to have polynomial growth if there exist positive integers c and k such that $|F^n| \leq cn^k$ for each positive integer n where for a set A, $|A|$ is the number of elements in A. Milnor [16] showed that $\lim |F^n|^{1/n} = v$ always exists. If $v > 1$ then G is said to have exponential growth and if $v = 1$ then G is said to be exponentially bounded. The conditions mentioned above do not depend on the choice of the finite generating set F, cf. Wolf [22]. Clearly if G has polynomial growth then it is exponentially bounded. However it is not known whether the converse holds. Milnor [17] and Wolf [22] proved that a finitely generated solvable group G is exponentially bounded if and only if it has polynomial growth and if and only if it is almost nilpotent, i.e., G contains a nilpotent subgroup of finite index. Wolf [22] suggested that their theorem ought to hold for all finitely generated groups. By applying the description of EG in Section 2 we are able to prove that Wolf's conjecture holds for groups in EG.

First note that a finite-by-nilpotent group G is almost nilpotent. For if $e \to F \to G \to N \to e$ is an exact sequence of groups where F is finite and N is nilpotent then for $s \in F$ set $K_s = \{x \in G\colon xs = sx\}$, i.e., the centralizer of s in G. Using the fact that F is finite and normal one sees that $[G\colon K_s] < \infty$ and therefore $[G\colon K] < \infty$ where $K = \bigcap\{K_s\colon s \in F\}$ is the centralizer of F in G. Consider the exact sequence

$$e \to F \cap K \to K \to K/F \cap K \to e.$$

Note that $K/F \cap K$ is nilpotent since it can be embedded into the nilpotent group N and that $F \cap K$ is contained in the center of K. Therefore K is nilpotent and hence G is almost nilpotent as claimed.

LEMMA 3.1. *If $e \to A \to B \to C \to e$ is an exact sequence of groups where A and C are almost nilpotent and B is finitely generated then B is either almost nilpotent or it has exponential growth.*

Proof. Assume that B is exponentially bounded. Since B is finitely generated so is its homomorphic image C. Let C_1 be a normal nilpotent subgroup of C with $[C: C_1] < \infty$. Then C_1 is also finitely generated and hence is finitely presented. Now C is finitely presented since it is the extension of the finitely presented group C_1 by the finite group C/C_1. By applying Lemmas 1 and 2 of Milnor [17] we conclude that A is finitely generated. (Milnor's Lemmas 1 and 2 are stated for abelian A. The fact that A is abelian is only used on line 10 of p. 448. This can be avoided because what is needed is to express α_m as a word in $\alpha_1, \ldots, \alpha_{m-1}$:

$$\alpha_m^{j_m - i_m} = (\alpha_1^{j_m} \cdots \alpha_{m-1}^{j_{m-1}})^{-1} \cdot \alpha_1^{i_1} \cdots \alpha_{m-1}^{i_{m-1}}.$$

Let A_1 be a nilpotent subgroup of finite index in A. Since A is finitely generated, by a theorem of M. Hall (see [12, p. 56]) it has only finitely many subgroups with index $[A: A_1]$, say, $A_1, A_2, \ldots, A_n$. Let $A_0 = \bigcap_{i=1}^n A_i$. Then A_0 is a nilpotent characteristic subgroup of A with finite index. Therefore A_0 is normal in B. Consider the exact sequence

$$e \to A/A_0 \to B/A_0 \xrightarrow{\theta} C \to e.$$

As before, let C_1 be a nilpotent subgroup of finite index in C. Since the sequence

$$e \to A/A_0 \to \theta^{-1}(C_1) \to C_1 \to e$$

is exact, C_1 is nilpotent and A/A_0 is finite, by the remark given before this lemma, $\theta^{-1}(C_1)$ is almost nilpotent. Since $[B/A_0: \theta^{-1}(C_1)] < \infty$, B/A_0 is also almost nilpotent.

Now consider the exact sequence

$$e \to A_0 \to B \xrightarrow{\pi} B/A_0 \to e.$$

Let K be a nilpotent normal subgroup of B/A_0, $[B/A_0: K] < \infty$. Then we have the exact sequence

$$e \to A_0 \to \pi^{-1}(K) \to K \to e.$$

Since A and B/A_0 are finitely generated A_0 and K are also finitely generated. Therefore $\pi^{-1}(K)$ is a finitely generated solvable group and by assumption is exponentially bounded. Hence, by the theorem of Milnor-Wolf, $\pi^{-1}(K)$ is almost nilpotent. Again since $B/\pi^{-1}(K) \approx B/A_0/K$ is finite we finally conclude that B is almost nilpotent.

THEOREM 3.2. *Let G be a finitely generated group in EG. Then G is either almost nilpotent or it has exponential growth.*

Proof. By Proposition 2.2, it suffices to prove that this theorem holds for groups in each EG_α. As before, transfinite induction will be applied. If $G \in EG_0$ then G is of course almost nilpotent. Assume that $\alpha > 0$ and that we have proved the theorem for finitely generated groups in EG_β, $\beta < \alpha$. Let G be a finitely generated exponentially bounded group in EG_α. We may assume that

$G \notin EG_\beta$ if $\beta < \alpha$. Then α is not a limit ordinal, i.e., $\alpha - 1$ exists. Since G is finitely generated, the only way to obtain G from $EG_{\alpha-1}$ is by group extension, say, we have the exact sequence $e \to A \to G \to C \to e$ where $A, C \in EG_{\alpha-1}$. Since G is exponentially bounded, so are A and C. By inductive assumption A and C are both almost nilpotent. Therefore, by Lemma 3.1, G is almost nilpotent.

In [21], Rosenblatt modified the proofs of Milnor and Wolf to obtain the following: If G is a finitely generated solvable group then G is either almost nilpotent or it contains a free subsemigroup on two generators. Clearly, if a group contains a free subsemigroup on two generators then it has exponential growth. Therefore, Rosenblatt's theorem is somewhat stronger than Milnor-Wolf's result. As in the proof of Lemma 3.1, we may apply Lemmas 4.8 and 4.9 in [21] together with the understanding that A doesn't have to be abelian there to obtain the following: If $e \to A \to B \to C \to e$ is exact, A, C are almost nilpotent, B is finitely generated and contains no free subsemigroup on two generators then B is almost nilpotent. As a consequence, we may state Theorem 3.2 as follows:

THEOREM 3.2′. *A finitely generated group in EG is either almost nilpotent or it contains a free subsemigroup on two generators.*

The fact that a finitely generated exponentially bounded group is amenable is first given in Milnor [16, Lemma 5], see also Corollary 3.5 of Rosenblatt [21] for a stronger conclusion. For the convenience of the reader we like to include a short proof here: Let F be a finite generating set of an exponentially bounded group G. Since $|F^n|^{1/n} \to 1$ there exists a subsequence n_k such that $|F^{n_k+1}|/|F^{n_k}| \to 1$, as $k \to \infty$. Let ϕ be a w^*-limit point of (ϕ_k) where $\phi_k \in l^\infty(G)^*$ is defined by

$$\phi_k(f) = (1/|F^{n_k}|) \sum \{f(x): x \in F^{n_k}\}, \quad f \in l^\infty(G).$$

Then ϕ is a left invariant mean on $l^\infty(G)$; so G is amenable. Therefore if $EG = AG$ then we would have a positive answer to Wolf's conjecture.

Adjan [1] stated that each Burnside group $B(m, n)$ with $m > 1$, n odd and ≥ 665, has exponential growth. Rosenblatt has kindly communicated to us that in many cases the Golod-Shafarevitch's p-groups have exponential growth. This fact can be readily seen from Golod's construction. Let R be the polynomial ring of non-commuting variables $x_1, \ldots, x_d$ over the field of residue classes modulo p, $d \geq 2$. Let I be the ideal of R generated by homogeneous elements f_j of degree ≥ 2 in which for every $i \geq 2$ the number of f_j's of degree i is finite and equal to r_i. Write $A = R/I = A_0 + A_1 + \cdots$ and $\dim A_n = b_n$. Golod-Shafarevitch [6] showed that if $1/(1 - dt + \sum_{i=2}^\infty r_i t^i) \geq 0$ as a formal power series then $\sum_{n=0}^\infty b_n t^n \geq 1/(1 - dt + \sum_{i=2}^\infty r_i t^i)$. In [5], Golod showed that if

$$r_i \leq \varepsilon^2 (d - 2\varepsilon)^{i-2} \quad (0 < \varepsilon < \tfrac{1}{2}) \tag{1}$$

then $1/(1 - dt + \sum_{i=2}^{\infty} r_i t_i) \geq 0$ and hence (by the remark prior to [6, Lemma 4])

$$b_n \geq (n+1)(d-\varepsilon)^n - n(d-\varepsilon)^{n-1}(d-2\varepsilon) \geq (d-\varepsilon)^n. \tag{2}$$

Furthermore, he showed that it is possible to choose f_j such that (1) is satisfied and that the multiplicative semigroup G generated by $1 + \bar{x}_1, \ldots, 1 + \bar{x}_d$ in A is a p-group. ($\bar{x}_i$ stands for the image of x_i in A.) In this case, let

$$F = \{1 + \bar{x}_1, \ldots, 1 + \bar{x}_d\}.$$

Then $|F^n| \geq b_n$ and hence by (2) $\lim |F^n|^{1/n} \geq \liminf b_n^{1/n} \geq d - \varepsilon > 1$. So G has exponential growth.

We like to conclude this section with two examples of groups in EG.

Example 1. For each pair of integers $m, n, m < n$, consider a symbol e_{mn}. Let M be the set of all expressions of the form $1 + \sum a_{mn} e_{mn}$ where $a_{mn} \in \mathbf{Z}$ and only finitely many of the a_{mn}'s are different from zero. If $x = 1 + \sum a_{mn} e_{mn}$ and $y = 1 + \sum b_{kl} e_{kl}$ belong to M then define

$$x \cdot y = 1 + \sum a_{mn} e_{mn} + \sum b_{kl} e_{kl} + \sum a_{mn} b_{kl} e_{mn} e_{kl}$$

where $e_{mn} e_{kl} = e_{ml}$ if $n = k$; $= 0$ otherwise. With this multiplication M becomes a locally nilpotent group with generators $1 + e_{mn}$, $m < n$. Of course M is just a slight modification of the group considered by McLain [15]. Note that $M \notin EG_n$ for $n = 1, 2, \ldots$. Let $G = \mathbf{Z} \times M$ be the semi-direct product of $\mathbf{Z}$ and M with $\mathbf{Z}$ acting on M: $k \in \mathbf{Z}$ sends $1 + \sum a_{mn} e_{mn}$ to $1 + \sum a_{mn} e_{m+k,n+k}$. A simple calculation shows that G is generated by $(1, 1)$ and $(0, 1 + e_{12})$. Therefore G is a finitely generated torsion free group in EG but $G \notin \bigcup_{n=1}^{\infty} EG_n = EG_\omega$ where ω is the first infinite ordinal. Since G is not almost solvable, by applying Theorem 3.2′, we know that G contains a free subsemigroup on two generators.

Example 2. Let E be the group of finite even permutations on $\mathbf{Z}$. Then E is a locally finite simple group, $E \in EG_1$. Let $G = \mathbf{Z} \times E$ be the semi-direct product of $\mathbf{Z}$ and E with $\mathbf{Z}$ acting on E: If $n \in \mathbf{Z}$ and $\pi \in E$, $n \cdot \pi \in E$ is defined by $(n \cdot \pi)(k + n) = \pi(k) + n$, $k \in \mathbf{Z}$. A simple calculation shows that G is generated by $(1, e)$ and $(0, (1, 2, 3))$ where e stands for the identity permutation on $\mathbf{Z}$ and $(1, 2, 3)$ is the 3-cycle which sends 1 to 2, 2 to 3, and 3 to 1. Therefore G is a finitely generated group in EG_2 and G is not almost solvable. By Theorem 3.2′, G contains a free subsemigroup on two generators.

4. In this section we will apply Proposition 2.2 to study almost convergent sets in an elementary amenable group. First of all let us introduce a certain packing property concerning groups which may prove to be interesting in its own right.

DEFINITION. A group G is said to have property (P) if given a finite set F in G there exist a finite set $S \supset F$ and a set X in G such that the mapping from

$S \times X$ to G which sends (s, x) to $s \cdot x$, $s \in S$, $x \in X$, is one-one and onto. For convenience, we will say that the pair (S, X) forms a packing of G.

Clearly every finite group has property (P). The additive group of integers $\mathbf{Z}$ also has property (P): if F is a finite subset of $\mathbf{Z}$ choose n large enough so that $F \subset \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n\} = S$ then the pair S and $X = \{k \cdot (2n+1) : k \in \mathbf{Z}\}$ forms a packing of $\mathbf{Z}$. A similar construction shows that every finitely generated abelian group has property (P).

LEMMA 4.1. (a) *If G is a direct union of $\{G_\tau\}$ and each G_τ has property (P) then G also has property (P).*
(b)[2] *If*

$$e \to A \to G \xrightarrow{\theta} B \to e$$

is an exact sequence of groups and A, B have property (P) then G also has property (P).

Proof. (a) If F is a finite set in G then there exists τ such that $F \subset G_\tau$. By assumption there exist a finite set S in G_τ and a set Y in G_τ such that (S, Y) forms a packing of G_τ. Let $Z \subset G$ be a set of representatives of the right cosets $\{G_\tau a : a \in G\}$. Then (S, YZ) forms a packing of G.

(b) Let F be a given finite subset of G. Let F_1 be a subset of F such that the mapping $\theta: F_1 \to \theta(F)$ is one-one and onto. Let E be the set of $a \in A$ such that a can be written as $q^{-1}p$ for some $p \in F$ and $q \in F_1$. Then E is a finite subset of A and $F \subset F_1 E$. Choose a finite set $T \supset E$ in A and a set $Y \subset A$ such that

(1) (T, Y) forms a packing of A.
Let U_1 be a finite subset of B, $U_1 \supset \theta(F)$, and $X_1 \subset B$ such that
(2) (U_1, X_1) forms a packing of B.

Choose subsets U and X of G such that $U \xrightarrow{\theta} U_1$, $X \xrightarrow{\theta} X_1$ are one-one and onto and $U \supset F_1$. Note that $U \cdot T \supset F_1 \cdot E \supset F$ is a finite subset of G. We claim that $(U \cdot T, Y \cdot X)$ forms a packing of G.

Let $r \in G$. Then, by (2), $\theta(r) = u_1 x_1$, $u_1 \in U_1$, $x_1 \in X_1$. So there exist $u \in U$ and $x \in X$ such that $u_1 = \theta(u)$, $x_1 = \theta(x)$. So $u^{-1}rx^{-1} \in A$. By (1), $u^{-1}rx^{-1} = ty$, $t \in T$, $y \in Y$, i.e., $r = utyx \in (U \cdot T) \cdot (Y \cdot X)$. By a similar method one sees that the decomposition of r as $(ut) \cdot (yx)$ is unique. Therefore $(U \cdot T, Y \cdot X)$ forms a packing of G as claimed.

By Lemma 4.1, the remarks before Lemma 4.1, Proposition 2.2 and transfinite induction we get the following.

PROPOSITION 4.2. *Every group in EG has property (P).*

[2] The proof of this lemma was communicated to us by Professor S. Yuan. We wish to thank him for giving us permission to include it here.

We will apply the above proposition to obtain a result concerning almost convergent sets. Let G be an amenable group and let $LIM(G)$ be the set of all left invariant means on $l^\infty(G)$. Each $\mu \in LIM(G)$ can be identified with a left translation invariant finitely additive probability measure on the family of all subsets of G. For each subset $A \subset G$, the upper density of A is

$$\bar{d}(A) = \sup \{\mu(A) \colon \mu \in LIM(G)\}$$

and the lower density of A is

$$\underline{d}(A) = \inf \{\mu(A) \colon \mu \in LIM(G)\}.$$

The reason for calling $\bar{d}(A)$ ($\underline{d}(A)$) the upper (lower) density of A can be best explained by quoting (a special case of) Proposition 3 in Granirer [8]:

$$\bar{d}(A) = \inf \left\{(1/|F|) \sup_{x \in G} |A \cap F \cdot x| \colon F \text{ a finite subset of } G\right\}$$

$$\underline{d}(A) = \sup \left\{(1/|F|) \inf_{x \in G} |A \cap F \cdot x| \colon F \text{ a finite subset of } G\right\}.$$

If $\bar{d}(A) = \underline{d}(A)$ then we say that A is almost convergent and denote the common value by $d(A)$. (The adjective "almost convergent" was introduced by Lorentz [13]). By Granirer's result quoted above one sees that a set is almost convergent if it is "evenly distributed" in G. The family of all almost convergent subsets of G will be denoted by $\mathfrak{U}(G)$. For example, if (S, X) forms a packing of an amenable group G then $X \in \mathfrak{U}(G)$ and $d(X) = 1/|S|$.

The following proposition is perhaps true for every amenable group. But we can only prove it for elementary amenable groups.

PROPOSITION 4.3. *If $G \in EG$ anf $A \subset G$ with $\underline{d}(A) > 0$ then there exists $B \subset A$ such that $B \in \mathfrak{U}(G)$ and $d(B) > 0$.*

Proof. Since $\underline{d}(A) > 0$, by [2, Lemma 4.1 (1)] or [8, Proposition 3], there exists a finite set F in G such that $F \cdot x \cap A \neq \emptyset$ for each $x \in G$. Since G has property (P), there exists a finite set $S \supset F$ and a set X such that (S, X) forms a packing of G. For each $x \in X$ pick $t(x) \in F \cdot x \cap A$ and let $B = \{t(x) \colon x \in X\}$. Then, by Lemma 4.1 (3) in [2], $B \in \mathfrak{U}(G)$, $d(B) = d(X) = 1/|S| > 0$ and, from the construction, $B \subset A$.

Combining the above proposition with Theorem 4.2 and Remark (1) on p. 692 in [2], we get the following.

PROPOSITION 4.4. *Let G be an infinite group in EG. If $A \in \mathfrak{U}(G)$ and $0 < d(A) < 1$ then there exists $B \in \mathfrak{U}(G)$ such that $A \cap B \notin \mathfrak{U}(G)$.*

In [7], Granirer proved that if G is an amenable group such that either (i) G contains an element of infinite order or (ii) G contains an infinite locally finite subgroup then for each $r \in [0, 1]$ there exists $A \in \mathfrak{U}(G)$ such that $d(A) = r$. Since, by Theorem 2.3, each infinite group in EG satisfies either (i) or (ii), a consequence of his theorem can be stated as follows:

PROPOSITION 4.5. *If G is an infinite group in EG then for each $r \in [0, 1]$ there exists an almost convergent set A such that $d(A) = r$.*

The class of groups with property (P) is much bigger than EG as the following lemma shows.

LEMMA 4.6. (a) *Let G be a group such that for each finite set $F \subset G$ there exists a normal subgroup K such that G/K has property (P) and $xK \neq yK$ if $x, y \in F$, $x \neq y$. Then G has property (P).*

(b) *Let G be a group with a family of normal subgroups $\{G_\gamma\}_{\gamma \leq \alpha}$ indexed by ordinals $\leq \alpha$, $G_0 = G$, $G_\alpha = \{e\}$, $G_{\gamma_1} \subset G_{\gamma_2}$ if $\gamma_1 \geq \gamma_2$, for $\gamma \leq \alpha$, $G_{\gamma-1}/G_\gamma$ has property (P) if $\gamma - 1$ exists and $G_\lambda = \bigcap \{G_\beta : \beta < \gamma\}$ if γ is a limit ordinal. Then G has property (P).*

Proof. (a) Let $F \subset G$ be a finite set. Choose a normal subgroup K of G such that the restriction of the natural homomorphism $\theta \colon G \to G/K$ to F is one-one and that G/K has property (P). Pick a finite subset $S_1 \supset \theta(F)$ in G/K and a set $X_1 \subset G/K$ such that (S_1, X_1) forms a packing of G/K. Choose a finite set $S \supset F$ such that $\theta \colon S \to S_1$ is one-one and onto. Choose $X \subset G$ such that $\theta \colon X \to X_1$ is one-one and onto. Then $(S, X \cdot K)$ forms a packing of G.

(b) Clearly $G/G_0 = \{e\}$ has property (P). Suppose that G/G_β has property (P) if $\beta < \gamma$. We claim that G/G_γ has property (P). If $\gamma - 1$ exists, then $e \to G_{\gamma-1}/G_\gamma \to G/G_\gamma \to G/G_{\gamma-1} \to e$ is an exact sequence, $G_{\gamma-1}/G_\gamma$ and $G/G_{\gamma-1}$ have property (P). Therefore by Lemma 4.1, G/G_γ has property (P). If γ is a limit ordinal, given a finite set $F = \{x_1, \ldots, x_n\} \subset G/G_\gamma$, then there exists an ordinal $\beta < \gamma$ such that $\bar{x}_i \neq \bar{x}_j$ if $i \neq j$ where $\bar{x}_i$ stands for the image of x_i under the natural homomorphism $G/G_\gamma \to (G/G_\gamma)/(G_\beta/G_\gamma)$. Now

$$(G/G_\gamma)/(G_\beta/G_\gamma) \approx G/G_\beta$$

has property (P) and hence, by (a), G/G_γ also has property (P). So, by induction, $G = G/G_\alpha$ has property (P).

COROLLARY 4.7. *If G is residually in EG, i.e., for each $x \neq e$ in G there exists a normal subgroup K of G such that $x \notin K$ and $G/K \in EG$, then G has property (P).*

Proof. Let $x_1 \neq e$, $x_2 \neq e$ be in G. Then there exist normal subgroups K_1 and K_2 such that $x_1 \notin K_1$, $x_2 \notin K_2$ and G/K_1, G/K_2 belong to EG. Denote the natural homomorphism $G \to G/K_2$ by θ. Then $K_1/K_1 \cap K_2 \approx \theta(K_1) \subset G/K_2$. So $K_1/K_1 \cap K_2 \in EG$. Since $e \to K_1/K_1 \cap K_2 \to G/K_1 \cap K_2 \to G/K_1 \to e$ is exact, $K_1/K_1 \cap K_2 \in EG$ and $G/K_1 \in EG$, so $G/K_1 \cap K_2 \in EG$. This corollary follows easily from the above observation, Proposition 4.2 and Lemma 4.5(a).

As a consequence of the above corollary, we see that each free group has property (P) since it is residually finite. Therefore, the free product of two non-trivial groups also has property (P) by applying problems 13, 23, 24 of Section 4.1 in [14]. A Golod-Shafarevitch's infinite p-group G has property (P) since $\bigcap G^{(n)} = (e)$ where $G^{(n)}$ stands for the nth derived group of G, see [5]. We are unable to decide whether every group has property (P). In particular, we do not know whether each amenable group has property (P).

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