ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 4, Winter 1980

SURGERY ON Λ-HOMOLOGY MANIFOLDS

BY

GERALD A. ANDERSON

1. Introduction

Let Λ be a subring of $\mathbf{Q}$ and K the set of primes invertible in Λ. This paper is devoted to the problem of finding a cobordism from a map $\phi: M \to X$, where M is a Λ-homology manifold, to a Λ-homotopy equivalence. Our main result is:

THEOREM 1.1. *Let $\phi: M \to X$ be a normal map of degree 1 between a Λ-homology manifold M of dimension $n \geq 5$ and a Λ-Poincare complex X, so that $\phi \mid \partial M: \partial M \to \partial X$ is a Λ-homotopy equivalence. Then there is an obstruction $\sigma(\phi) \in L_n^h(\Lambda[\pi_1(X)])$, depending only on the normal cobordism class of ϕ, so that $\sigma(\phi) = 0$ if and only if ϕ is normally cobordant to a Λ-homotopy equivalence.*

There are also simple and relative versions (cf. Section 5). By *normal map* we mean that there are Λ-homology cobordism bundles [4] ν over X and ξ over $M \times I$ so that ξ is a trivialization of $T_M \oplus \phi^*\nu$ (i.e. $\xi \mid M \times 0 = T_M \oplus \phi^*\nu$, $\xi \mid M \times 1 = M \times I^k$). A *$\Lambda$-Poincare complex* is a polyhedral pair $(X, \partial X)$ together with an element $[X, \partial X] \in H_n(X, \partial X)$ so that $\cap [X, \partial X]$: $H^i(X; \Lambda) \cong H_{n-i}(X, \partial X; \Lambda)$; degree 1 has the usual meaning. A *Λ-homotopy equivalence* is a map $f: A \to B$ so that $f_{\#}: \pi_i(A) \cong \pi_i(B)$ for $i \leq 1$ and $f_{\#}: \pi_i(A) \otimes \Lambda \cong \pi_i(B) \otimes \Lambda$ for $i \geq 2$. Normal cobordism is defined as usual, and $L_n^h(\Lambda[\pi_1(X)])$ denotes the Wall group of the group ring $\Lambda[\pi_1(X)]$ [24].

In Section 2, we show that the standard representability and stability properties hold for Λ-homology cobordism bundles. Our stability is not as strong as the result of Matumoto and Matsumoto [17] (for $\Lambda = \mathbf{Z}$), as there is no analogue of the Zeeman unknotting theorem, but suffices for our purposes.

Section 3 contains a straightforward generalization of the general position theorem for maps of Maunder [18] and a general position theorem for embeddings.

The simply connected case is considered in Section 4. It was claimed previously by Quinn [19], and the argument here, based on that of Matsui [15], is considerably simpler. This case is necessary for the argument of Section 5.

The main theorem is proven in Section 5. The results of Section 4 (surgery below the middle dimension) and Wall [24], Chapter 6, suffice to handle the odd dimensional case. If $n = 2k$, the argument goes as follows: By Section 4, we

Received November 27, 1978.

can assume $K_k(M; \Lambda) \cong \pi_{k+1}(\phi) \otimes \Lambda$ is the only non-zero homology kernel and is free over $\Lambda[\pi_1(X)]$. To define the surgery obstruction, we construct a codimension 0 submanifold $\hat{N} \subset M$, and a Λ-homotopy equivalence $\psi: N \to \hat{N}$ from a PL-manifold N, containing a basis for $\pi_{k+1}(\phi) \otimes \Lambda$ (reminiscent of the engulfing theorem of Jones [12]). There is a natural splitting of the surjection $K_k(N; \Lambda) \to K_k(M; \Lambda)$, and we define intersection and self-intersection forms on $K_k(M; \Lambda)$ geometrically in N.

The argument is reduced to that of Chapter 5 of [24] by showing that any element of $K_k(M; \Lambda)$ can be represented, up to multiplication by a unit in Λ, by an embedded k-sphere in some Λ-homology manifold h_K-cobordant to M. (W is an *h_K-cobordism* if $\partial_\pm W \to W$ are Λ-homotopy equivalences.)

2. Representability and stability of Λ-homology cobordism bundles

In this section, we show that Λ-homology cobordism bundles are representable over H_Λ-cell complexes and satisfy the same stability properties as PL-block bundles.

An H_Λ-cell complex is a cone complex in which each boundary is a Λ-homology sphere or disc. If X is an H_Λ-cell complex, then we let $k_n^\Lambda(X)$ denote the set of isomorphism classes of Λ-homology cobordism S^{n-1}-bundles over X (cf. [4]). If X' denotes a simplicial complex underlying X, then the amalgamation operation $\mathscr{A}$ of [14] defines a map $k_n^\Lambda(X') \to k_n^\Lambda(X)$. Furthermore, $k_n^\Lambda(X')$ is in bijective correspondence with $[X', BH_K(n)]$, where $H_K(n)$ is the Δ-set with i-simplexes Λ-homology cobordism S^{n-1}-bundles over $\Delta^i \times I$.

The first step in showing that $\mathscr{A}$ defines a bijection above is the following existence theorem for normal bundles.

THEOREM 2.1. *Let M^n, N^{n+k} be compact Λ-homology manifolds with M properly embedded as a full subcomplex. Then the simplicial neighborhood $N(M', N')$ in the first derived subdivision N' is a Λ-homology cobordism D^k-bundle over the dual cell decomposition M^* of M.*

Proof. We proceed by induction on n. Let A_n denote the statement of the theorem and B_n, C_n the following statements.

B_n. Let $\Sigma_1^{n-1} \subset \Sigma_2^{n+k-1}$ be Λ-homology spheres. Then $\partial N(\Sigma_1', \Sigma_2')$ is a trivial Λ-homology cobordism S^{k-1}-bundle over Σ_1^* and there exists a trivialization

$$G: \partial N(\Sigma_1', \Sigma_2') \cong \Sigma_1' \times S^{k-1}$$

that extends to an H_K-cobordism between $\Sigma_2' - N^\circ(\Sigma_1', \Sigma_2')$ and $c\Sigma_1' \times S^{k-1}$.

C_n. If Δ^n is a Λ-acyclic Λ-homology manifold and ξ is an orientable Λ-homology cobordism S^{k-1}-bundle over Δ^*, then ξ is trivial.

Clearly A_0 and C_0 hold.

$A_{n-1}, C_{n-1} \Rightarrow B_n$. By A_{n-1}, $\partial \overline{N(\Sigma'_1, \Sigma'_2)}$ defines an S^{k-1}-bundle ξ over Σ_1^*. Choose a vertex $v \in \Sigma_1$ and let $\Delta_i = \overline{\Sigma'_i - D(v, \Sigma_i)}$, $i = 1, 2$. By C_{n-1}, $\xi \mid \Delta_1^*$ is trivial. By the argument of Proposition 5.1 of [14], the trivialization G exists as stated.

$A_{n-1}, B_n \Rightarrow A_n$. The proof is identical to Corollary 5.2 of [14].

$A_n \Rightarrow C_n$. Let ξ be an orientable Λ-homology cobordism D^k-bundle over Δ^* with a zero-section $i: \Delta \to E(\xi)$. By the proof of Corollary 3.7 of [14], it suffices to show that ξ extends over $c\Delta^*$. Since $E(\xi) \searrow i(\Delta)$, A_n implies that

$$\overline{E(\xi) - N(i(\Delta^i), E(\xi)')}$$

defines an isomorphism between the sphere bundle $\hat{\xi}$ of ξ and $\hat{\xi}_0 = \partial N(i(\Delta'), E(\xi)')$.

Since both Δ and $E(\xi)$ are Λ-acyclic, both $c\Delta$ and $cE(\xi)$ are Λ-homology manifolds. Let η denote the space over $(c\Delta)^*$ defined by $N(c(i(\Delta))', cE(\xi)')$. Then $\eta \mid \Delta^* = \xi_0$ and, by A_n, $\eta \mid ((c\Delta)^*)^{(n)}$ is a Λ-homology cobordism D^k-bundle. By amalgamation, we may regard η as a space over $c\Delta^*$.

Let v be a vertex of Δ so that $D(v, \Delta)$ is an n-cone of Δ^*. As before, $\eta \mid c(\partial D(v, \Delta))$ is trivial, and a trivialization can be chosen to extend the trivialization of $\xi_0 \mid \partial D(v, \Delta)$ obtained from the structure of ξ_0 as a bundle over Δ^*. Therefore there is a trivialization G of $\eta \mid (D(v, \Delta) \cup c(\partial D(v, \Delta)))$. The total space $E(G)$ is an H_K-cobordism between

$$\eta(D(v, \Delta)) \cup \eta(c(\partial D(v, \Delta))) \quad \text{and} \quad (D(v, \Delta) \cup c(\partial D(v, \Delta))) \times D^k.$$

Extend G to a trivialization G' of $\eta \mid cD(v, \Delta)$ by setting

$$E(G') = c(\eta(cD(v, \Delta)) \cup E(G) \cup (cD(v, \Delta)) \times D^k);$$

clearly $E(G')$ defines a space G' over $cD(v, \Delta)$, extending G, and an H_K-cobordism between $\eta(cD(v, \Delta))$ and $(cD(v, \Delta)) \times D^k$. Therefore η is a Λ-homology cobordism D^k-bundle over $c\Delta^*$ and so ξ is trivial.

In particular, the tangent bundle T_M and stable normal bundle ν_M of a Λ-homology manifold M can be defined over M^* by the usual procedures. The following subdivision result shows that T_M, ν_M are defined over M'. Let X be an H_K-cell complex and Y a subcomplex whose cells are simplexes.

THEOREM 2.2. *Let ξ be a Λ-homology cobordism S^k-bundle over X. Then there is a Λ-homology cobordism S^k-bundle ξ' over X' so that $\xi' \mid Y = \xi \mid Y$ and $\mathscr{A}(\xi') \cong \xi$ by an isomorphism extending $(\xi \mid Y) \times I$ over Y.*

Proof. We first prove the result for X' a Λ-homology manifold. Let A_n denote the statement of the theorem if dim $X' = n$ and B_n the following statement: If Δ^n is a Δ-acyclic Λ-homology manifold and ξ is an orientable Λ-homology cobordism S^k-bundle over Δ, then ξ is trivial.

$A_n \Rightarrow B_n$. Let ξ/Δ^n be an orientable S^k-bundle. By A^n, there is a subdivision ξ' of ξ over the first derived Δ'. But ξ' amalgamates to an S^k-bundle over Δ^*, and so is trivial by the proof of Theorem 2.1.

$A_{n-1}, B_{n-1} \Rightarrow A_n$. The proof is the same as the completion of the proof of Theorem 4.5 of [14], since Q (as on page 107 of [14]) collapses onto $P - St^\circ(w, P)$, for some vertex w, which is a Λ-acyclic Λ-homology manifold (so that $D|(P - St^\circ(w, P))$ is trivial by B_{n-1}).

Thus the result holds for X' a Λ-homology manifold. Let p be a prime in K and M_p^r a compact PL-manifold of the homotopy type of the suspended Moore space $\Sigma^r(S^1 \cup_p D^2)$. Since $\tilde{H}_*(M_p^r; \Lambda) = 0$, the argument above implies that $[M_p^r, \mathrm{BSH}_K(k)] = 0$. By [1], $BSH_K(k)$ is Λ-local since $\pi_1(BSH_K(k)) = 0$. Therefore, by obstruction theory, if Δ is a Λ-acyclic simplicial complex and ξ is an orientable S^k-bundle over Δ, then ξ is trivial. The proof now follows exactly as the proof of Theorem 4.5 of [14].

COROLLARY 2.3. *$\mathscr{A} : k_n^\Lambda(X') \to k_n^\Lambda(X)$ is a bijection.*

By the proof of Theorem 2.2 we have the following result.

COROLLARY 2.4. *$BSH_K(n)$ is Λ-local.*

Theorem 2.2 allows us to define pull-backs. Let ξ be a Λ-homology cobordism D^k-bundle over a simplicial complex Y and f: $X \to Y$ a PL-map. Define $f^*\xi$ to be the amalgamation, over X, of $(\varepsilon^0 \times \xi)' | G_f$ where $X \times Y$ is subdivided so that the graph G_f of f is a subcomplex. In particular, we may define Whitney sums in the usual way.

We now consider stability properties of Λ-homology cobordism bundles. Our main result is the following.

THEOREM 2.5. *Suppose $n \geq \max\{k, 5 - k\}$. Then $\pi_k(PL^\sim, PL^\sim(n)) \otimes \Lambda \to \pi_k(H_K, H_K(n))$ is an isomorphism.*

The proof requires a number of lemmas. Let $\overline{PL}(n)$ be the Δ-set with k-simplexes PL $(n-1)$-sphere block bundles over $\Delta^k \times I$ which are a product over $\Delta^k \times \{0, 1\}$, and $PLH_K(n)$ the Δ-set with k-simplexes block-preserving PL H_K-cobordisms between $\Delta^k \times S^{n-1}$ and itself. (W is an H_K-cobordism if $H_*(W, \partial_\pm W; \Lambda) = 0$; the prefix PL indicates that W is a PL-manifold.) By [13], $PL^\sim(n)$ is homotopy equivalent to $\overline{PL}(n)$. Let ψ_n^K denote the group of PL H_K-cobordism classes of PL Λ-homology n-spheres of [3].

LEMMA 2.6. *Let $n \geq 2$. Then there is an exact sequence*

$$0 \to \psi_{n+k-1}^K \otimes \Lambda \to \pi_k(H_K(n), PLH_K(n)) \otimes \Lambda \to \psi_k^K \otimes \Lambda \to 0.$$

Proof. Let $x \in \pi_k(H_K(n), PLH_K(n))$. Then x is represented by a Λ-homology cobordism S^{n-1}-bundle over $\Delta^k \times I$, with total space W, which is a block-preserving PL H_K-cobordism over $\Delta^k \times I$ and the product bundle over $\partial\Delta^k \times \{0, 1\} \cup \Delta^{k-1} \times I$, where Δ^{k-1} is a fixed $(k-1)$-face of Δ^k.

Let $\mu_j(x) \in H_j(W; \psi^K_{n+k-j-1})$ be the first non-zero obstruction to finding a PL Λ-acyclic resolution of W rel (∂W) of [23]. If $j \geq n$, then $u\mu_{jj}(x) = 0$ for some $u \in \Lambda^{\cdot} \cap \mathbf{Z}$ and it follows by naturality that $\mu_j(ux) = 0$. Define

$$\alpha\colon \pi_k(H_K(n), PLH_K(n)) \otimes \Lambda \to \psi^K_k \otimes \Lambda$$

by

$$\alpha(x) = \frac{1}{u}\mu_{n-1}(ux)$$

where u is a unit in Λ so that the first (possibly) non-zero obstruction of ux is $\mu_{n-1}(ux)$. (We identify $\mu_{n-1}(ux)$ with its image under

$$H_{n-1}(W; \psi^K_k) \to H_{n-1}(W; \psi^K_k) \otimes \Lambda \cong \psi^K_k \otimes \Lambda.)$$

The map α is easily seen to be a homomorphism. To see that α is surjective, let Σ be a PL Λ-homology k-sphere, and regard $c\Sigma$ as a space over $\Delta^k \times I$ as in [13] Then $c\Sigma \times S^{n-1}$ represents an element $x \in \pi_k(H_K(n), PLH_K(n))$ with a single resolution obstruction $\mu_{n-1}(x) = [\Sigma]$.

Let $A = \ker(\alpha)$, and define $\beta\colon A \to \psi^K_{n+k-1} \otimes \Lambda$ by

$$\beta(x) = \frac{1}{u}\mu_0(ux)$$

where $u \in \Lambda^{\cdot}$ is chosen so that $\mu_0(ux)$ is the only resolution obstruction. If $\beta(x) = 0$, then there is a Λ-acyclic resolution, rel (∂W), $f\colon N \to W$ where N is a PL-manifold and W represents ux. Since N represents 0 in $\pi_k(H_K(n), PLH_K(n))$ and M_f defines a homotopy from N to W, $ux = 0$. Therefore β is injective.

Let $x \in A$ be represented by W as above where W is a PL-manifold except at vertices $v_1, \ldots, v_m$, and let $[\Sigma] \in \psi^K_{n+k-1}$. Define M to be the connected sum, along the boundary, of $St(v_i, W')$, $i = 1, \ldots, m$, which we can assume is a subcomplex of W. Then $[\partial M] = \beta(x)$. Let $\Sigma' = \Sigma \# (-\partial M)$ and $W' = W \# c\Sigma'$ along an $(n-k-1)$-disc lying over $(\partial\Delta^k - \Delta^{k-1}) \times I$. If $x' \in \pi_k(H_K(n), PLH_K(n))$ denotes the element defined by W', then $x' \in A$, since we introduced only one new singularity, and $\beta(x') = \beta(x) + [\Sigma'] = [\Sigma]$. Therefore β is surjective.

Let $i_\#\colon \pi_k(H_K(n), \overline{PL}(n)) \otimes \Lambda \to \pi_k(H_K(n), PLH_K(n)) \otimes \Lambda$ denote the map induced by inclusion.

LEMMA 2.7. $(\mathrm{Im}\, i_\#) \cap (\psi^K_{n+k-1} \otimes \Lambda) = 0$.

Proof. Let $x \in \pi_k(H_K(n), \overline{PL}(n))$ be represented by a total space W with only isolated singularities. Then $W|\partial(\Delta^k \times I)$ is a PL $(n-1)$-sphere block bundle, and so extends to a PL n-disc block bundle ξ. Let $\Sigma = W \cup E(\xi)$; then Σ is a Λ-homology $(n+k)$-sphere. Form $V = \Sigma - \mathrm{Int}\,(M)$ where M is constructed as in the proof of Lemma 2.6. Then $\beta(i_\#(x)) = [\partial M] = 0$ since $\partial M = \partial V$ and V is a Λ-acyclic PL-manifold.

LEMMA 2.8. *If* $n > \max\{k, 3-k\}$, *then* $\pi_k(PLH_K(n), \overline{PL}(n)) \otimes \Lambda \cong \psi^K_{n+k} \otimes \Lambda$.

Proof. Let $x \in \pi_k(PLH_K(n), \overline{PL}(n))$ be represented by a PL H_K-cobordism W between $\Delta^k \times S^{n-1}$ and itself, which is a PL block bundle over $\Delta^k \times I$, trivial over $\Delta^{k-1} \times I$. Extend $W|\partial(\Delta^k \times I)$ to a PL n-disc block bundle ξ and let $\Sigma = W \cup E(\xi)$ as in the proof of Lemma 2.7. Define γ: $\pi_k(PLH_K(n), \overline{PL}(n)) \to \psi^K_{n+k}$ by $\gamma(x) = [\Sigma]$.

The surjectivity of γ follows essentially as in Lemma 2.6. Let Σ be a PL Λ-homology $(n+k)$-sphere. Remove the interiors of two disjoint, trivially embedded copies of $\Delta^k \times D^n$ and represent the resulting manifold as a space over $\Delta^k \times I$. This defines an element x of $\pi_k(PLH_K(n), \overline{PL}(n))$ with $\gamma(x) = [\Sigma]$.

Suppose $\gamma(x) = 0$. We show that $ux = 0$ for some $u \in \Lambda^{\cdot}$. Let $[\Sigma] = \gamma(x)$. Then Σ bounds a Λ-acyclic PL-manifold N, which we may assume to be simply connected by Corollary 3.3 of [6]. Let i: $\partial(\Delta^k \times I) \to E(\xi)$ be the zero section. Since $\pi_1(N) = 0$ and $\tilde{H}_*(N; \Lambda) = 0$, i represents 0 in $\pi_k(N) \otimes \Lambda$ by the Hurewicz theorem, relative to the Serre class of abelian groups G with $G \otimes \Lambda = 0$. By replacing x by ux, where $u \in \Lambda^{\cdot}$ is so that $u[i] = 0$ in $\pi_k(N)$, we may assume that i is null-homotopic. Therefore if $k < n$, i extends to an embedding of $\Delta^k \times I$ by general position. (If $k = n-1$, there may be double points, which can be eliminated by the Whitney method since $n + k \geq 4$.)

Let η denote the normal block bundle of $\Delta^k \times I$ in N, and η° the associated open disc bundle. Represent $N - E(\eta^\circ)$ as a space over $\Delta^k \times I$. Then $N - E(\eta^\circ)$ defines a homotopy between ux and the element of $\pi_k(PLH_K(n), \overline{PL}(n))$ defined by the associated sphere bundle $S(\eta)$. By [21], $S(\eta)$ represents 0, and so $ux = 0$.

Proof of Theorem 2.5. Consider the long exact homotopy sequence of triple $(H_K(n), PLH_K(n), \overline{PL}(n))$:

$$\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \pi_k(H_K(n), \overline{PL}(n)) \otimes \Lambda & & & & \\
 & & & & \downarrow{\scriptstyle i_\#} & & & & \\
0 & \longrightarrow & \psi^K_{n+k-1} \otimes \Lambda & \xrightarrow{\beta} & \pi_k(H_K(n), PLH_K(n)) \otimes \Lambda & \xrightarrow{\alpha} & \psi^K_k \otimes \Lambda & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle \partial_\#} & & & & \\
0 & \longrightarrow & \psi^K_{n+k-1} \otimes \Lambda & \xrightarrow{\gamma^{-1}} & \pi_{k-1}(PLH_K(n), \overline{PL}(n)) \otimes \Lambda & \longrightarrow & 0 & & \\
 & & & & \downarrow & & & & \\
 & & & & \vdots & & & &
\end{array}$$

The rows in this diagram are exact by the lemmas and the square on the left commutes by construction. By a diagram chase, $\alpha \circ i_{\#}\colon \pi_k(H_K(n), \overline{PL}(n)) \otimes \Lambda \to \psi_k^K \otimes \Lambda$ is an isomorphism for $k < n$, $n \geq 3$, $n + k \geq 5$. It follows from [3] that

$$\pi_k(H_K(n), \overline{PL}(n)) \otimes \Lambda \cong \pi_k(H_K, \overline{PL}) \otimes \Lambda.$$

The result for $k < n$ now follows from the exact ladder

$$\begin{array}{ccccccc} \cdots \to \pi_k(B\overline{PL}(n)) \otimes \Lambda & \to & \pi_k(BH_K(n)) \otimes \Lambda & \to & \pi_{k-1}(H_K(n), \overline{PL}(n)) \otimes \Lambda & \to \cdots \\ \downarrow & & \downarrow & & \downarrow & \\ \cdots \to \pi_k(B\overline{PL}) \otimes \Lambda & \to & \pi_k(BH_K) \otimes \Lambda & \to & \pi_{k-1}(H_K, \overline{PL}) \otimes \Lambda & \to \cdots \end{array}$$

by Corollary 2.4, since $BSH_K(n)$ is the universal cover of $BH_K(n)$ and

$$\pi_1(BH_K(n)) \cong \pi_1(BH_K) \cong \mathbf{Z}/2$$

by [4].

For $k = n$, the argument above shows that $\pi_n(PL, PL(n)) \otimes \Lambda \to \pi_n(H_K, H_K(n))$ is surjective, and so an isomorphism if $n = 3$, 7, or n is odd and $2 \in K$ (cf. [21]). If n is odd, $\neq 3$, 7, and $2 \notin K$, then

$$\pi_n(PL^{\sim}, PL^{\sim}(n)) = \ker\,(\pi_n(BPL^{\sim}(n)) \to \pi_n(BPL^{\sim})) \cong \mathbf{Z}/2$$

is generated by T_{S^n}, which is not fiber homotopically trivial. Since $n \geq 3$, the map $BPL^{\sim}(n) \to BG(n) \to BG_K(n)$ factors through $BH_K(n)$ by Theorem 2.7 of [4] so that T_{S^n} is non-zero in $\pi_n(BH_K(n))$ (since $\pi_n(BG_K(n)) \cong \pi_n(BG(n)) \otimes \Lambda$).

If n is even, then $\pi_n(PL^{\sim}, PL^{\sim}(n)) \cong \mathbf{Z}$ is again generated by T_{S^n} and the result follows as above since $\pi_n(G, G(n)) \cong \mathbf{Z} \oplus$ torsion.

3. General position

In this section, we develop general position theorems for maps of a polyhedron to a Λ-homology manifold. The arguments given are due to Maunder [18], handling the case $\Lambda = \mathbf{Z}$. All spaces are assumed to be compact polyhedral and all maps will be PL.

Let $f\colon K \to L$. Recall that the *singularity set* of f is the subpolyhedron of K defined by

$$S(f) = \overline{\{x \in K\colon |f^{-1}f(x)| > 1\}}$$

and that f is *non-degenerate* if each point inverse $f^{-1}(x)$ is finite.

THEOREM 3.1. *Suppose $f\colon K^n \to M^m$ where M is a Λ-homology manifold. Let P be a subpolyhedron of K so that $f\,|\,P$ is an embedding and* $\dim\,(K - P) = p$. *Then there exists a Λ-acyclic resolution $g\colon \bar{K}^n \to K$ and a non-degenerate map $h\colon \bar{K} \to M$ so that*

(1) $g\colon g^{-1}(P) \cong P$, $h \circ g^{-1}\,|\,P = f\,|\,P$,
(2) $\dim S(h) \leq n + p - m$.

The proof requires a number of lemmas. (Compare [18]).

LEMMA 3.2. *Suppose L is a full subcomplex of K. Then there is a simplicial map $f : K' \to v*L'$ $(v \notin K)$ so that if x is in the interior of the simplex $vb_{\sigma_0} \ldots b_{\sigma_n}$, then $f^{-1}(x) \cong \partial D(\sigma_n, K) - \partial D(\sigma_n, L)$.*

The map f is defined by

$$f(b_\sigma) = \begin{cases} b_\sigma & \text{if } \sigma \in L \\ v & \text{if } \sigma \notin L. \end{cases}$$

LEMMA 3.3 *Suppose $f : K \to L$, where $\dim K = n$ and $\tilde{H}_i(L; \Lambda) = 0$ for $i \leq n$. Then f factors, up to homotopy, through a Λ-acyclic polyhedron.*

The proof is identical to the proof of Proposition 2.2 of [18]. Lemma 3.3 immediately implies the following.

LEMMA 3.4. *If $f : K \to L$ factors through a Λ-acyclic polyhedron M, then we may assume* $\dim M \leq \dim K + 1$.

PROPOSITION 3.5. *Suppose $f : K^n \to K^m$ where M is a Λ-homology manifold. Let P be a subpolyhedron of K so that $f \mid P$ is non-degenerate. If $n \leq m$, then there exists a Λ-acyclic resolution $g : \bar{K}^n \to K$ and a non-degenerate map $h : \bar{K} \to M$ so that $g : g^{-1}(P) \cong P$ and $h \circ g^{-1} \mid P = f \mid P$.*

Proof. The proof is by induction on m. Let A_m denote the statement of the theorem for Λ-homology manifolds M of dimension $\leq m$, and consider the following statement:

B_m. Let K^n be a cone complex, L a cone subcomplex and M^k a H_Λ-cell complex with $n \leq k \leq m$. Let $f : K \to M$ be a conewise map so that

(a) f is injective on simplexes of K with no vertex in L, and
(b) if v, w are vertexes of p-, q-cones and $f(v) = w$, then $q \geq k - n + p$.

Then there exists a Λ-acyclic resolution $g : \bar{K}^n \to K$, and a non-degenerate map $h : \bar{K} \to M$ so that

(i) if C, D are cones with $f(C) \subset D$, then $h(g^{-1}(C)) \subset D$, and
(ii) if σ is a simplex of K with no vertex in L and $x \in \sigma$, then $|g^{-1}(x)| = 1$ and $h(g^{-1}(x)) = f(x)$.

Both A_0 and B_0 are obvious.

$B_m \Rightarrow A_m$. Let $f : K \to M$ satisfy the hypothesis of the statement A_m. Assume f is simplicial and P is a subcomplex of K. Then the induced map of the dual-cell decompositions satisfies the hypothesis of B_m, with L taken to be the subcomplex of cones not meeting P, and A_m follows.

$(A_{m-1}, B_{m-1}) \Rightarrow B_m$. Let $f: K^n \to M^m$ and L satisfy the hypothesis of B_m, and let $P = f^{-1}(M^{(m-1)})$. Let $g: \bar{P} \to P$, $h: \bar{P} \to M^{(m-1)}$ be the maps obtained from $f \mid P: P \to M^{(m-1)}$ using B_{m-1}. Since, $n \leq m$, g extends to a Λ-acyclic resolution $\bar{P}' \to K$ by adding vertexes to $\bar{P}$ corresponding to vertexes of K not in P, and h and f define a non-degenerate map $\bar{P}' \to M$. Thus we may assume that $f \mid P$ is non-degenerate.

We first extend the construction over $P \cup L^{(n-1)}$. Assume $f: Q \to M^{(m-1)}$ is non-degenerate, where $P \cup L^{(s-1)} \subset Q \subset P \cup L^{(s)}$, $s \leq n-1$, and let C be an s-cone of L, not in Q, whose vertex is sent to the vertex of an m-cone D of M. Let $R = f^{-1}(D) \cap Q$. Clearly $\partial C \subset R$ and $f \mid R: R \to \partial D$ is non-degenerate.

By Lemma 3.3, $f \mid \partial C: \partial C \to \partial D$ extends to a map $f': \bar{C} \to \partial D$, where $\bar{C}$ is a Λ-acyclic polyhedron of dimension s containing ∂C as a subcomplex. Let $g': \bar{C} \to C$ be the map constructed in Lemma 3.2; g' is a Λ-acyclic resolution rel (∂C), and so defines a Λ-acyclic resolution $g'': R \cup \bar{C} \to R \cup C$ rel (R). The maps f and f' define a map $f'': R \cup \bar{C} \to \partial D$, non-degenerate on R. Since ∂D is a Λ-homology manifold, by A_{m-1}, there is a Λ-acyclic resolution $g: \bar{R} \to R \cup C$ rel (R) and a non-degenerate map $\hat{h}: \bar{R} \to \partial D$ so that $h \circ g^{-1} \mid R = f'' \mid R$. Continuing in this manner, there exists a Λ-acyclic resolution $g_0: \bar{K}_0 \to P \cup L^{(n-1)}$ and a non-degenerate map $h_0: \bar{K}_0 \to M^{(m-1)}$ satisfying conditions (i), (ii) above.

Finally, we join up the remaining vertexes of K as before, getting a Λ-acyclic resolution $g: \bar{K} \to K$, and define $h: \bar{K} \to M$ by the maps h_0 and f. It is now easily checked that g, h satisfy the conclusion of B_m.

Proof of Theorem 3.1. Again we proceed by induction on m. Let A_m denote the statement of the theorem, assuming f is non-degenerate and $\dim M \leq m$, and B_m the following statement:

Let K^n be a cone complex, L^p a cone subcomplex and M^k an H_Λ-cell complex, $k \leq m$. Let $f: K \to M$ be a conewise map so that

(a) f is injective on verticies of cones not in L,
(b) if f identifies the vertices of C and D, then $C = D$ or $C \cap D = \emptyset$,
(c) if v, w are vertices of q-, v-cones and $f(v) = w$, then $r \geq k - n + q$, and $r \geq k - p + q$ if $v \in L$.

Then there exists a Λ-acyclic resolution $g: \bar{K}^n \to K$ and a non-degenerate map $h: \bar{K} \to M$ so that

(i) if C, D are cones with $f(C) \subset D$, then $h(g^{-1}(C)) \subset D$,
(ii) if σ is a simplex of K with no vertex in L and $x \in \sigma$, then $|g^{-1}(x)| = 1$ and $h(g^{-1}(x)) = f(x)$,
(iii) $\dim S(h) \leq n + p - k$.

Both A_0 and B_0 are obvious, and B_m implies A_m as in the proof of Proposition 3.5.

$(A_{m-1}, B_{m-1}) \Rightarrow B_m$. Let K^n, L^p, M^m and f be as in the hypothesis of B_m, and define $P = f^{-1}(M^{(m-1)})$. As in the proof of Proposition 3.5, we may assume $f \mid P$ is non-degenerate and $\dim S(f \mid P) \leq (n-1) + (p-1) - (m-1) = n + p - m - 1$. Again mimicking Proposition 3.5, there is a Λ-acyclic resolution $g_0 \colon \bar{K}_0 \to P \to L^{(p-1)}$ and a non-degenerate map $h_0 \colon \bar{K}_0 \to M^{(m-1)}$ satisfying (i), (ii) above with $\dim S(h_0) \leq n + p - m - 1$; together with the map f, these maps define a Λ-acyclic resolution $g \colon \bar{K} \to K$ and a non-degenerate map $h \colon \bar{K} \to M$ satisfying (i) and (ii).

Case 1. $n + p - m \geq 0$. In this case, $\dim S(h) \leq n + p - m$. Suppose $h(v) = h(w)$ is the vertex of an m-cone of M, and $v * \sigma$, $w * \tau$ are q-simplexes of K so that $h(v * \sigma) = h(w * \tau)$. Since h is non-degenerate, σ and τ are in $P \cup L^{(p-1)}$, and it follows easily that $q - 1 \leq n + p - m - 1$.

Case 2. $p < n$ or $n + p - m \leq -2$. In this case the construction above can be extended to get a Λ-acyclic resolution $g'_0 \colon \bar{K}'_0 \to P \cup L$ and an embedding $h'_0 \colon \bar{K}_0 \to M^{(m-1)}$. Proceeding as before, we get maps $g' \colon \bar{K}' \to K$, $h \colon \bar{K}' \to M$ satisfying properties (i), (ii), (iii).

Case 3. $p = n$ and $n + p - m = -1$. We must modify h once again. Let D be a $(2n+1)$-cone whose vertex is the image under h of vertices of n-cones $C_1, \ldots, C_k$ of L and at most one vertex of a cone E not in L. Then $h \colon h^{-1}(\partial D) \to \partial D$ is an embedding and $h \colon \partial C_i \to \partial D$ extends to a Λ-acyclic polyhedron $\bar{C}_i^n$. By A_{m-1} we may assume

$$\dim S(h \mid (h^{-1}(\partial D) \cup \bar{C}_1 \cup \cdots \cup \bar{C}_k)) = 0.$$

The singularities of $h \mid (h^{-1}(\partial D) \cup \bar{C}_1 \cup \cdots \cup \bar{C}_k)$ are of two types:

(a) $h(x) = h(y)$, $x \in \operatorname{Int}(\bar{C}_i)$, $y \in \bar{C}_j$,
(b) $h(x) = h(y)$, $x \in \operatorname{Int}(\bar{C}_i)$, $y \in \partial E$.

To eliminate points of type (a), let U be a small closed neighborhood of x missing both $\partial \bar{C}_i$ and $S(h) - \{x\}$. Using a collared neighborhood of ∂ in D, deform h, relative to $\bar{K}$-Int (U), to a map sending Int (U) to Int (D). A similar argument works if more than two points are identified.

For points of type (b), we first apply the argument above to assume that $y \in \operatorname{Int}(E)$. Let F be a $(2n+1)$-cone neighborhood of $h(x)$ in Int (D) so that, letting $C'_i = h^{-1}(F) \cap \bar{C}_i$ and $E' = h^{-1}(F) \cap E$, $h \mid \partial C_i$ and $h \mid \partial E'$ are disjoint embeddings.

Since $\partial D'$ is a Λ-homology sphere or disc of dimension $2n$ and $\dim h(\partial E') \leq n - 1$,

$$\tilde{H}_k(\partial D' - h(\partial E'); \Lambda) = 0 \quad \text{for } k \leq n - 1$$

by Alexander duality. By Lemma 3.3, $h \mid \partial C'_i \colon \partial C'_i \to (\partial D' - h(\partial E'))$ extends to a Λ-acyclic polyhedron $\bar{C}$. Modifying h as before, we exchange our singularity for a new set of isolated double points away from $\partial E'$, which are of type (a).

Therefore we may assume h is an embedding of $h^{-1}(D)$ in D. Doing this for every $(2n + 1)$-cone, we get $S(h) = \emptyset$.

Remark. By subdividing K, we may assume h is arbitrarily close to $f \circ g$, and so $h \simeq f \circ g$ rel $g^{-1}(P)$.

COROLLARY 3.6. *Let $P_0 \subset P^p, Q^q$ be subpolyhedra of a Λ-homology manifold M^m with $p \leq q$, $p + q < m$. Then there exists a Λ-acyclic resolution $g\colon \bar{P}^p \to P$ and an embedding $h\colon \bar{P} \to M$ so that*

(1) $g\colon g^{-1}(P_0) \cong P_0$, $h|g^{-1}(P_0) = g|g^{-1}(P_0)$,
(2) $h(\bar{P} - g^{-1}(P_0)) \cap Q = \emptyset$.

Proof. Let $K = P \cup_{P_0 \cap Q} Q$ and $f : K \to M$ induced by inclusion. By Theorem 3.1, there exists a Λ-acyclic resolution $g : \bar{K} \to K$ and a non-degenerate map $h : \bar{K} \to M$ so that $\hat{g} : \hat{g}^{-1}(Q) \cong Q$, $\hat{h}|\hat{g}^{-1}(Q) = f \circ \hat{g}|\hat{g}^{-1}(Q)$ and $\dim S(\hat{h}) \leq q + p - m < 0$. Letting $\bar{P} = \hat{g}^{-1}(P)$, $g = \hat{g}\,|\,\bar{P}$, $h = \hat{h}\,|\,\bar{P}$, we get the result.

4. The simply connected case

In this section we prove the main theorem for simply connected Λ-Poincare complexes. Let $\phi\colon M \to X$ be a normal map, where M is a Λ-homology manifold of dimension $n \geq 5$ and X is a finite polyhedron. We say that ϕ is (k, Λ)-*connected* if $\phi_{\#}\colon \pi_i(M) \cong \pi_i(X)$ for $i \leq 1$ and $\pi_i(\phi) \otimes \Lambda = 0$ for $i \leq k$.

Our first result shows that ϕ can be made highly Λ-connected.

THEOREM 4.1. *ϕ is normally cobordant to a $([n/2], \Lambda)$-connected map.*

The proof depends on the following embeddability result. (Compare Matsui [15].)

PROPOSITION 4.2. *Let $\alpha \in \pi_{k+1}(M, x_0)$, $k < n/2$. Then there exists a Λ-homology manifold M' of dimension n and a Λ-acyclic resolution $p : M \to M'$ so that $p_{\#}(\alpha) \in \pi_k(M', p(x_0))$ is represented by an embedding.*

Proof. Represent α by a simplicial map $f\colon (S^k, s_0) \to (M, x_0)$ and let Δ_1, Δ_2 be disjoint closed k-simplexes of S^k so that $s_0 \in \partial\Delta_1$; let

$$N = \overline{S^k - (\Delta_1 \cup \Delta_2)}.$$

Choose an n-simplex σ of M not meeting $f(S^k)$ and an embedding $i\colon S^k \to \operatorname{Int}(\sigma)$. Let q be a path in M from x_0 to $i(s_0)$ so that

$$q(I) \cap i(S^k) = i(s_0).$$

Define $g_j\colon \Delta_j \to S^k$, $j = 1, 2$, by $\Delta_j \to \Delta_j/\partial\Delta_j \cong S^k$, sending $\partial\Delta_j$ to s_0, and $h\colon N \to I$ by

$$N \cong S^{k-1} \times I \to I,$$

sending $\partial\Delta_1$ to 0 and $\partial\Delta_2$ to 1. Finally, define $f_0: S^k \to M$ by

$$f_0 \mid \Delta_1 = f \circ g_1, \quad f_0 \mid \Delta_2 = i \circ g_2, \quad f_0 \mid N = q \circ h.$$

Clearly $f_0 \simeq f$ rel (s_0).

By Theorem 3.1, there is a Λ-acyclic resolution $\mu: \Sigma^k \to S^k$ and an embedding $\bar{f}: \Sigma \to M$ so that $\mu^{-1}(\Delta'_2 \cup \{s_0\}) \cong \Delta'_2 \cup \{s_0\}$ for some k-simplex $\Delta'_2 \subset \text{Int}(\Delta_2)$,

$$\bar{f} \mid \mu^{-1}(\Delta'_2 \cup \{s_0\}) = f_0 \circ \mu \mid \mu^{-1}(\Delta'_2 \cup \{s_0\})$$

and

$$\bar{f} \simeq f_0 \circ \mu \text{ rel } \mu^{-1}(\Delta'_2 \cup \{s_0\}).$$

Let R be a relative regular neighborhood of $(i \circ g_2(\Delta'_2), i \circ g_2(\partial\Delta'_2))$ and Q a regular neighborhood of $\bar{f}(\Sigma - \text{Int}(\Delta'_2))$ in $M - \text{Int}(R)$. Clearly Q is Λ-acyclic. Define $M' = M/Q \cong (M - \text{Int}(Q)) \cup c(\partial Q)$, $p: M \to M'$ the collapsing map, and let $f': S^k \to M'$ be the composition

$$S^k \cong \Delta'_2/\partial\Delta'_2 \xrightarrow{p \circ \bar{f}} M'.$$

We have that $p_{\#}(\alpha) = [p \circ f_0] = [f']$ (obtained by collapsing a homotopy between $\bar{f}$ and $f_0 \circ \mu$ as above) and f' is clearly an embedding.

COROLLARY 4.3. *Let $\alpha \in \pi_{k+1}(\phi)$, $k < n/2$. Then there exists a Λ-acyclic resolution $p: M \to M'$ and a normal map $\phi': M' \to X$ so that ϕ and ϕ' are normally cobordant, $\phi' \circ p \simeq \phi$ and $p_{\#}(\alpha)$ is represented by an embedding.*

Proof. Construct p as in the proposition and define $W = M \times I \cup c(Q \times 1)$. Since $\bar{f} \simeq f_0 \circ \mu$, $\phi \mid Q$ is null-homotopic and so $\phi \circ \pi_1: M \times I \to X$ extends to $\Phi: W \to X$. We have $\partial W \cong M \cup M'$, and let $\phi' = \Phi \mid M'$. Let $\hat{\xi}$ be the trivialization of $T_{M \times I} \oplus (\phi \circ \pi_1) \uparrow \nu$ determined by ξ; $\hat{\xi}$ is a homology cobordism bundle over $M \times I \times I$, trivial over $M \times I \times 1$. Let $i: M \times I \times I \to E(\hat{\xi})$ be the zero-section and

$$\gamma: (E(\hat{\xi} \mid M \times I \times 1), i(M \times I \times 1)) \to (M \times I \times D^r, M \times I \times 0)$$

a PL-homeomorphism. Define a Λ-homology cobordism bundle ξ' over $M \times I \times I/Q \times 1 \times I$ by collapsing $i(Q \times 1 \times I)$ to a point and identifying $\gamma^{-1}(x, 1, t)$ with $\gamma^{-1}(y, 1, t)$ for $x, y \in Q$. Note that ξ' is trivial over $M \times I \times 1/M \times 0 \times 1$. Let q denote the contractible resolution $W \times I \cong (M \times I/Q \times 1) \times I \to M \times I \times I/Q \times 1 \times I$; then $q \uparrow \xi'$ is the desired trivialization.

LEMMA 4.4. *ϕ is normally cobordant to a 2-connected map.*

Proof. Clearly we may assume that ϕ is 1-connected. Let $\alpha \in \pi_2(\phi)$. By Theorem 3.1, α is represented by an embedding $f: S^1 \to M$ missing the dual 2-skeleton of M. Then the normal bundle of f is a trivial PL-block bundle by Theorem 2.5 and the proof follows as in [24].

LEMMA 4.5. *Suppose ϕ is (k, Λ)-connected, $1 < k < n/2$, and $\alpha \in \pi_{k+1}(\phi)$. If p, ϕ' denote the maps constructed in Corollary 4.3, then ϕ' is (k, Λ)-connected and*

$$p_{\#}: \pi_{k+1}(\phi) \otimes \Lambda \cong \pi_{k+1}(\phi') \otimes \Lambda.$$

Proof. We have a commutative diagram

$$\begin{array}{ccc} \pi_1(M) & \xrightarrow{\phi_{\#}} & \pi_1(X) \\ {\scriptstyle p_{\#}}\downarrow & \nearrow_{\phi'_{\#}} & \\ \pi_1(M') & & \end{array}$$

where $\phi_{\#}$ is an isomorphism, and $p_{\#}$ is surjective by the Van Kampen theorem. Therefore $\phi'_{\#}$ is an isomorphism. The result now follows from the Hurewicz theorem, since

$$p_*: K_i(M; \Lambda) \cong K_i(M', \Lambda) \text{ and } K_i(M; \Lambda) \cong \pi_{i+1}(\phi) \otimes \Lambda \quad \text{for } i \leq k.$$

Proof of Theorem 4.1. Assume ϕ is (k, Λ)-connected, $k < n/2$, and let $\alpha \in \pi_{k+1}(\phi)$ be represented by an embedding $f: S^k \to M$ with stably trivial normal bundle ν_f. By Theorem 2.5, ν_f is trivial; let η be a homotopy between ν_f and $S^k \times D^{n-k}$. We may assume that η possesses a zero-section $i: S^k \times I \to E(\eta)$ which is a homotopy equivalence. Since $\phi \circ f$ is null-homotopic and $i \mid S^k \times 0$ is a homotopy equivalence, there is a cobordism

$$\Phi: W \to X, \quad W = M \times I \cup E(\eta),$$

between ϕ and $\phi': M' \to X$ so that ϕ' is (k, Λ)-connected,

$$\pi_{k+1}(\phi) \otimes \Lambda \cong \pi_{k+1}(\phi') \otimes \Lambda$$

(by the Hurewicz theorem since $k > 1$ and $E(\eta)$ is an H_K-cobordism of pairs), and the element corresponding to α is represented by an embedding of $S^k \times D^{n-k}$.

Using the same methods, the following can be proven as Theorem 1.4 of [24].

COROLLARY 4.6. *Let $\phi: (M, \partial M) \to (Y, X)$ be a normal map where M is a Λ-homology manifold of dimension $n \geq 6$. Then ϕ is normally cobordant to a map $\phi': (M', \partial M') \to (Y, X)$ so that ϕ' is homologically $([(n+1)/2], \Lambda)$-connected, $\phi \mid M'$ is $([n/2], \Lambda)$-connected and $\phi \mid \partial M$ is $([(n-1)/2], \Lambda)$-connected.*

To complete the simply connected case we must extend Corollary 4.3 and Lemma 4.5 to include $k = n/2$. (See [15] for the case $\Lambda = \mathbf{Z}$.)

PROPOSITION 4.6. *Suppose $\phi: M^{2k} \to X$ is (k, Λ)-connected, $k \geq 3$, $H_1(X; \Lambda) = 0$, and $\alpha \in \pi_{k+1}(\phi)$. Then there exists a Λ-acyclic resolution $p: M \to M'$ and a normal map $\phi: M' \to X$ so that ϕ and ϕ' are normally cobordant, ϕ' is (k, Λ)-connected, $p_{\#}: \pi_{k+1}(\phi) \otimes \Lambda \cong \pi_{k+1}(\phi') \cong \Lambda$, and $p_{\#}(\alpha)$ is represented by an embedding.*

Proof. By the proofs of Corollary 4.3 and Lemma 4.5, it suffices to prove the analogue of Proposition 4.2. Let f represent α and construct $\mu: \Sigma \to S^k$, $\bar{f}: \Sigma \to M$ as in the proof of Proposition 4.2; $\bar{f}$ is an embedding except for isolated singularities.

Let $x_1, x_2 \in S(\bar{f})$ with $\bar{f}(x_1) = \bar{f}(x_2)$ and choose an embedded path q in Σ-Int (Δ_2') from x_1 to x_2, missing $S(\bar{f}) - \{x_1, x_2\}$. Since $H_1(M; \Lambda) \cong H_1(X; \Lambda) = 0$, by Lemma 3.3, there is a Λ-acyclic 2-complex Δ_0 containing S^1 and a map $h_0: \Delta_0 \to M$ extending $\bar{f} \circ q$. By Theorem 3.1, there is a Λ-acyclic resolution $r: \Delta \to \Delta_0$ rel (S^1) and an embedding $h: \Delta \to M$ so that $h | S^1 = \bar{f} \circ q$ and $h \simeq h_0 \circ r$ rel (S^1). Construct such a map $h_{\bar{x}}: \Delta_{\bar{x}} \to M$ for every $\bar{x} \in R$, where R is a minimal set of generators of the equivalence relation $x \sim y$ iff $\bar{f}(x) = \bar{f}(y)$ on $S(\bar{f})$. By Corollary 3.6, we may assume that

$$h_{\bar{x}}(\Delta_{\bar{x}} - S^1) \cap \bar{f}(\Sigma) = \phi \text{ and } h_{\bar{x}}(\Delta_{\bar{x}} - S^1) \cap h_{\bar{y}}(\Delta_{\bar{y}} - S^1) = \phi \quad \text{if } \bar{x} \neq \bar{y}.$$

Let $\Sigma' = \bar{f}(\Sigma) \cup \bigcup_{\bar{x} \in R} h_{\bar{x}}(\Delta_{\bar{x}})$, which is clearly a Λ-homology sphere. The proof now follows exactly as in Proposition 4.2, letting Q be a regular neighborhood of Σ'-Int $(i \circ g_2(\Delta_2'))$.

We can now prove the main theorem of this section.

THEOREM 4.7. *Let $\phi: M \to X$ be a normal map of degree 1 between a Λ-homology manifold M of dimension $n \geq 5$ and a Λ-Poincare complex X so that $\phi | \partial M$ is a Λ-homotopy equivalence and $\pi_1(X) = 0$ (respectfully, $\phi | \partial M$ is a Λ-homology equivalence and $H_1(X; \Lambda) = 0$). Then ϕ is normally cobordant to a Λ-homotopy equivalance (Λ-homology equivalence) if and only if an obstruction $\sigma(\phi) \in L_n(\Lambda[1])$ vanishes.*

Proof. Recall that $L_n(\Lambda[1])$ is 0 if n is odd, $\mathbf{Z}/2 \otimes \Lambda$ if $n \equiv 2 \bmod (4)$ and $\bar{W}(\Lambda)$, the Witt group of even quadratic forms over Λ, if $n \equiv 0 \bmod (4)$. If $n = 4k + 2$ and $1/2 \notin \Lambda$, then we may define a quadratic form ψ on $K^{2k+1}(X, \partial X; Z/2)$ as in [7, III.4.5], and we let $\sigma(\phi)$ be the Arf invariant of ψ. If $n = 4k$, then Corollary III.3.4 of [7] implies that the cup product pairing on $K^{2k}(X, \partial X; \Delta)$/tor is even if $1/2 \notin \Lambda$ (which is automatic if $1/2 \in \Lambda$) and we define $\sigma(\phi)$ to be the Witt class of this pairing. The proof now follows exactly as in Chapter IV of [7] by Theorems 2.5, 4.1 and Proposition 4.6.

5. The general case

In this section, we prove the main theorem of the paper, as stated in the introduction. It is implicit in the theorem that X has a Stiefel-Whitney class $\omega: \pi_1(X) \to \{\pm 1\}$ and that the involution on $\Lambda[\pi_1(X)]$ is twisted by this homomorphism. The simple (i.e. zero torsion) and relative cases will be treated at the end of the section.

If n is odd, then the proof of Theorem 1.1 follows exactly as in [8, Section 1.2], or [24, Chapter 6], using Theorem 4.1. Therefore assume $n = 2k$. We first construct the obstruction $\sigma(\phi)$.

By Theorem 4.1, we may assume ϕ is (k, Λ)-connected, and that $K_k(M; \Lambda)$ is a free $\Lambda[\pi_1(X)]$-module, adding trivial handles if necessary, by [2], Theorem 2.1. The Hurewicz theorem implies that $\pi_{k+1}(\phi) \otimes \Lambda \cong K_k(M; \Lambda)$, and we let g: $\bigvee_{i=1}^r S_i^k \to M$ be a simplicial map representing a basis for $K_k(M; \Lambda)$. Let $\hat{N}$ be a regular neighborhood of the image of g in M. The following operations show that we may assume that $\pi_1(\partial\hat{N}) \cong \pi_1(\hat{N}) \cong \pi_1(M)$.

1. By Poincaré duality and Theorem 3.1, every element of $g_1(\hat{N}, \partial\hat{N})$ is represented by an embedded path $\alpha: (I, \dot{I}) \to (\hat{N} - \mathrm{Im}\,(g), \partial\hat{N})$, and we change $\hat{N}$ to $\hat{N} - \mathrm{Int}\,(R)$, where R is a regular neighborhood of $\alpha(I)$, as many times as necessary to assume $\pi_1(\partial\hat{N}) \to \pi_1(\hat{N})$ is surjective.

2. By Theorem 3.1, every element of $\pi_2(\hat{N}, \partial\hat{N})$ is represented by a map $f: D^2 \to \hat{N}$ with $f \mid S^1: S^1 \to \partial\hat{N}$ an embedding. Again, there is a Λ-acyclic 2-complex $\Delta \supset S^1$ and a map $\hat{f}: \Delta \to \hat{N}$, factoring through f and extending $f \mid S^1$. Let $\hat{\Delta}$ be a regular neighborhood of Δ and change $\hat{N}$ to $\hat{N}/\hat{\Delta}$ a finite number of times to assume $\pi_1(\partial\hat{N}) \cong \pi_1(\hat{N})$.

3. Since M is a PL-manifold in a neighborhood of its dual 3-skeleton, every element of $\pi_1(M, \hat{N})$ is represented by an embedding path $\alpha: (I, I) \to (M, \hat{N})$ intersecting only n- and $(n-1)$-simplexes. Trading $\hat{N}$ for $\hat{N} \cup R$, where R is a regular neighborhood of $\alpha(I)$, we may assume $\pi_1(\hat{N}) \to \pi_1(M)$ is surjective; $\pi_1(\partial\hat{N})$ remains isomorphic to $\pi_1(\hat{N})$ by the Van Kampen theorem.

4. Do Step 2 to $\pi_2(M, N)$ to get $\pi_1(\partial\hat{N}) \cong \pi_1(\hat{N}) \cong \pi_1(M)$.

The proof of Lemma 4.5 shows that the operations above can be accomplished through a normal cobordism of ϕ to a map that is still (k, Λ)-connected. Clearly the image of g is unaffected and so $\hat{N}$ can be chosen as stated.

Since $\phi \mid \hat{N}$ is null-homotopic, the stable trivialization of $T_M \oplus \phi^*\nu$ induces one of $T_{\hat{N}}$, and so the Spivak fibration $\nu_{\hat{N}}$ of $\hat{N}$ has a PL-reduction. This reduction induces a normal map $\psi: N \to \hat{N}$, N a PL-manifold, of degree $d \in \Lambda^{\cdot}$, which we may take to be a Λ-homotopy equivalence by [8], Theorem 3.1, since $\pi_1(\partial\hat{N}) \cong \pi_1(\hat{N})$ and $n \geq 6$ (replacing the fundamental class $[\hat{N}, \partial\hat{N}]$ by $d[\hat{N}, \partial\hat{N}]$ to make ψ have degree 1). Since

$$\psi_\#: \pi_k(N) \otimes \Lambda \cong \pi_k(\hat{N}) \otimes \Lambda,$$

there exist $d_1, \ldots, d_r \in \Delta$ so that, if $\mu: \bigvee_{i=1}^r S_i^k \to \bigvee_{i=1}^r S_i^k$ is of degree d_i on S_i^k, then $g \circ \mu$ lifts to N.

Since N is a PL-manifold, the constructions of [8, Section I.1], define intersection and self-intersection forms λ_0, μ_0 on

$$K_k(N; \Lambda) = \ker\left(H_k(N; \Lambda) \xrightarrow{(\phi\varphi)_*} H_k(X; \Lambda)\right).$$

The lift of $g \circ \mu$ to N defines a splitting of the surjection $\psi_*: K_k(N; \Lambda) \to K_k(M; \Lambda)$, and so λ_0, μ_0 restrict to a $(-1)^k$-Hermitian form $(\lambda, \mu, K_k(M; \Lambda))$. Define $\sigma(\phi) \in L_n^h(\Lambda[\pi_1(X)])$ to be the class of this form.

LEMMA 5.1. *$\sigma(\phi)$ is well defined.*

Proof. Clearly $\sigma(\phi)$ does not depend on the units $d_1, \ldots, d_r$ and the number of trivial handles added to make $K_k(M; \Lambda)$ free. Suppose $g, g' : \bigvee_{i=1}^{r} S_i^r \to M$ represent bases for $K_k(M; \Lambda)$ and let $\hat{N}$, $\hat{N}'$ be regular neighborhoods of the images of g, g'. Extend g, g' to maps, $\bar{g}, \bar{g}' : \bigvee_{i=1}^{2r} S_i^k \to \hat{N}, \hat{N}'$ so that $\bar{g}|S_{i+r}^k \simeq g|S_i^k$ and $\bar{g}'|S_{i+r}^k \simeq g|S_i^k$. Then there exists a homotopy G: $(\bigvee_{i=1}^{2r} S_i^k) \times I \to M \times I$ from $\bar{g}$ to $\bar{g}'$; let $\hat{W}$ be a regular neighborhood of the image of G. As before, we can assume that there is PL-manifold W and a Λ-homotopy equivalence Ψ: $(W; N, N') \to (\hat{W}; \hat{N}, \hat{N}')$.

We have a commutative diagram

$$\begin{array}{ccccccc} & K_{k+1}(W, N \cup N'; \Lambda) & \to & K_k(N \cup N'; \Lambda) & \to & K_k(W; \Lambda) & \\ & \Psi\downarrow & & \downarrow & & \downarrow & \\ 0 \to & K_{k+1}(M \times I, M \times \{0, 1\}; \Lambda) & \to & K_k(M \times \{0, 1\}; \Lambda) & \to & K_k(M \times I; \Lambda) & \to 0 \end{array}$$

with exact rows. The elements $G\,|\,S_i^k \times I$, $i = 1, \ldots, r$ represent a basis for

$$K_{k+1}(M \times I, M \times \{0, 1\}; \Lambda)$$

and, modulo units in Λ, have canonical lifts to $(W, N \cup N')$. This defines a compatible splitting of the surjection Ψ_*. By the proof of Lemma 5.7 of [24], intersections and self-intersections vanish on $K_{k+1}(W, N \cup N'; \Lambda)$, and so

$$K_{k+1}(M \times I, M \times \{0, 1\}; \Lambda)$$

is a subkernel of

$$K_k(M \times \{0, 1\}; \Lambda),$$

implying the result.

LEMMA 5.2. *$\sigma(\phi)$ is a normal cobordism invariant.*

Proof. Let Φ: $V \to X \times I$ be a normal cobordism, rel (∂M), between ϕ and ϕ': $M' \to X$. We may assume that ϕ, ϕ' and Φ are (k, Λ)-connected and that $K_k(V, M \cup M'; \Lambda) = 0$ by Corollary 4.6. The long exact homology sequence of the pair $(V, M \cup M')$ then reduces to

$$0 \to K_{k+1}(V, M \cup M'; \Lambda) \to K_k(M \cup M'; \Lambda) \to K_k(V; \Lambda) \to 0,$$

and we may assume all modules to be free.

Choose simplicial maps

$$g_i : (S^k \times I, S^k \times \{0, 1\}) \to (V, M \cup M'), \quad i = 1, \ldots, r,$$

$$h_i : S^k \to M \cup M', \qquad i = 1, \ldots, s,$$

representing bases for $K_{k+1}(V, M \cup M'; \Lambda)$, $K_k(V; \Lambda)$, that extend to a basis of $K_k(M \cup M'; \Lambda)$. Let $\hat{W}$ denote a regular neighborhood of the union of the images of $g_1, \ldots, g_r, h_1, \ldots, h_s$, modified to have the correct fundamental group as before. Letting $\hat{N} = \hat{W} \cap M$, $\hat{N}' = \hat{W} \cap M'$, we may assume that there is a

PL-manifold W, with disjoint submanifolds $N, N' \subset \partial W$, and a Λ-homotopy equivalence $\Psi\colon (W; N, N') \to (\hat{W}; \hat{N}\ \hat{N}')$.

The maps g_i, h_i define splittings of the exterior vertical maps in the diagram

$$\begin{array}{ccccc} K_{k+1}(W, N \cup N'; \Lambda) & \to & K_k(N \cup N'; \Lambda) & \to & K_k(W; \Lambda) \\ \downarrow & & \downarrow & & \downarrow \\ 0 \to K_{k+1}(V, \partial V; \Lambda) & \to & K_k(\partial V; \Lambda) & \to & K_k(V; \Lambda) \to 0 \end{array}$$

and so induce a splitting of the interior vertical map. By Lemma 5.1, $K_k(\partial V; \Lambda)$, with λ and μ induced from $K_k(N \cup N'; \Lambda)$, represents $\sigma(\phi) - \sigma(\phi')$. Again by [24, Lemma 5.7], intersections and self-intersections vanish on $K_{k+1}(W, N \cup N'; \Lambda)$, and so $K_{k+1}(V, \partial V; \Lambda)$ is a subkernel of $K_k(\partial V; \Lambda)$.

The crucial step in the proof of Theorem 1.1 is the following generalization of Proposition 4.6. (Compare Matumoto [16].)

PROPOSITION 5.3. *Let $\alpha \in K_k(M; \Lambda)$. Then there is a normal h_K-cobordism $\Phi\colon W \to X$ from ϕ to $\hat{\phi}\colon \hat{M} \to X$ and an embedding $i\colon S^k \to \hat{M}$ so that $\hat{j}_* i_*[S^k] = j_*(u\alpha)$ for some $u \in \Lambda^{\cdot}$, where $j, \hat{j}$ denote the inclusions of M, $\hat{M}$ into W.*

Let $f_0\colon S^k \to \hat{N}$ be a PL-map representing $v\alpha$ for some $v \in \Lambda^{\cdot}$. By Theorem 3.1, there is a Λ-acyclic resolution $g\colon \Sigma^k \to S^k$ and a map $h\colon \Sigma \to \hat{N}$, with isolated singularities, close enough to f_0 so that $f_0(S^k)$ is contained in a regular neighborhood N_0 of $h(\Sigma)$ and $h \simeq f_0 \circ g$. It follows from the Mayer-Vietoris sequence that $H_*(N_0; \Lambda) \cong H_*(U; \Lambda)$, where U is a regular neighborhood of $S^k \wedge \bigvee_{i=1}^m S^1$, for some m, in $\mathbf{R}^{2k}$.

LEMMA 5.4. *There exists a Λ-homology equivalence $\psi_0\colon N_0 \to U$ so that $P = \psi_0^{-1}(S^k)$ is a Λ-homology submanifold of N_0 and the surgery obstruction $\sigma(\psi_0 | P) \in L_k(\Lambda[1])$ is 0.*

Proof. Since N_0 is a regular neighborhood of a k-dimensional complex, it is $(k+1)$-coconnected. Therefore, by the Hopf theorem, [11], $\pi^k(N_0) \cong H^k(N_0)$, which has rank 1. Let $p\colon N_0 \to S^k$ denote an infinite cyclic generator. We also have $\pi^1(N_0) \cong H^1(N_0)$ is of rank m, and let $q_1, \ldots, q_m\colon N_0 \to S^1$ denote generators.

Since $S^1 \vee S^1 \simeq S^1 \times S^1 - *$, we can inductively construct, from $q_1, \ldots, q_m$, a map

$$q : N_0 \to \bigvee_{i=1}^m S^1$$

inducing an isomorphism on $H_1(\ ; \Lambda)$. By obstruction theory, there is a lift ψ_0 of

$$p \times q\colon N_0 \to S^k \times \left(\bigvee_{i=1}^m S^1\right) \quad \text{to} \quad S^k \wedge \bigvee_{i=1}^m S^1 \subset U,$$

which is clearly a Λ-homology equivalence.

By [10], ψ_0 may be changed by a homotopy to be transverse to S^k, and we get a normal map $\psi_0 | P$: $P \to S^k$ with stable trivialization induced from ξ, and degree a unit in Λ. If k is odd or $1/2 \in \Lambda$, $k \equiv 2 \bmod (4)$, then $\sigma(\psi_0 | P)$ vanishes trivially; otherwise we are divided into 3 cases.

(1) $1/2 \notin \Lambda$, $k \equiv 0 \bmod (4)$. Let β_p: $L_0(\Lambda[1]) \to W(\mathbf{F}_p)$ denote the second residue homomorphisms. By [4], Sign $(\sigma(\psi_0 | P)) =$ Sign (P), $\beta_p(\sigma(\psi_0 | P)) = \beta_p(P)$, and it suffices to show that both vanish. Since $1/2 \notin \Lambda$, there are homomorphisms

$$\hat{\sigma}_\infty: \pi_k(SG(K)/SH(K)) \to \Lambda, \quad \hat{\sigma}_p: \pi_k(SG(K)/SH(K)) \to W(\mathbf{F}_p)$$

so that if γ: $S^k \to SG(K)/SH(K)$ is the classifying map for $\psi_0 | P$, then

$$\hat{\sigma}_\infty(\gamma) = (a_K \deg(\psi_0 | P))^{-1} \operatorname{Sign}(P), \qquad \hat{\sigma}_p(\gamma) = (\deg(\psi_0 | P))^{-1} \beta_p(P).$$

Since γ is trivial by construction, Sign $(P) = 0$, $\beta_p(P) = 0$, and therefore $\sigma(\psi_0 | P) = 0$.

(2) $1/2 \notin \Lambda$, $k \equiv 2 \bmod (4)$. The same argument holds, using the Kervaire class of [4].

(3) $1/2 \in \Lambda$, $k \equiv 0 \bmod (4)$. It follows as in case 1 that Sign $(P) = 0$, but $\beta_p(P)$ need not vanish. Let $W_1 = N_0 \times I \cup E \times I$, where E is a regular neighborhood of P in N_0,

$$V_1 = U \times I \cup S^k \times D^k \times I, \quad V_2 = U \times I \cup S^k \times D^k \times [0, 1/2],$$

and define $\hat{\psi}_0$: $N_0 \to U$ by

$$N_0 \cong (N_0 - \operatorname{Int}(E)) \times 1 \cup S(E) \times I \cup E \times 1 \subset W_1 \xrightarrow{\psi_0} V_1 \xrightarrow{1 \times \mu} V_2$$
$$\supset (U - S^k \times D^{\circ k}) \times 1 \cup (S^k \times S^k \times I) \cup S^k \times D^k \times 1 \cong U,$$

where μ: $[0, 1] \to [0, 1/2]$ is the folding map

$$\mu(t) = \begin{cases} t & 0 \leq t \leq \frac{1}{2} \\ 1 - t & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Since $1/2 \in \Lambda$, $\hat{\psi}_0$ remains a Λ-homology equivalence, and $\hat{\psi}^{-1}(S^k) = P \cup P$. Repeating this process, we get Sign $(\psi_0^{\wedge\wedge -1}(S^k)) = 4$ Sign $(P) = 0$ and $\beta_p(\psi_0^{\wedge\wedge -1}(S^k)) = 4\beta_p(P) = 0$ since $4W(\mathbf{F}) = 0$.

Let i_0: $P \to M$ be the inclusion; clearly $(i_0)_*[P] = v'\alpha$ for some $v' \in \Lambda^{\cdot}$, and we may assume P is 1-connected by adding handles inside of M.

LEMMA 5.5 *There is an h_K-cobordism (W_0, V_0) from (M, P) to (M', P') so that P' is an s-parallelizable PL-manifold. Furthermore, ϕ extends to W_0.*

Proof. We consider 3 cases.

(1) $k \leq 3$. In this case P is already a PL-manifold, and we let $P' = P$.

(2) $k \geq 5$. By Theorem 4.7 and Lemma 5.4, $\psi_0 | P$ is normally cobordant to a Λ-homotopy equivalance ψ'_0: $\Sigma \to S^k$. Let c: $\Sigma \to S^k$ denote the collapse of the exterior of a k-simplex and d: $S^k \to S^k$ a map of the same degree as ψ'_0. By the

Hopf theorem and Theorem 1.7 of [5], $\pi^k(\Sigma) \cong H^k(\Sigma) \cong H_0(\Sigma) = \mathbf{Z}$, and so elements of $\pi^k(\Sigma)$ are classified by their degree. Since deg $(c) = 1$, $\psi'_0 \simeq d \circ c$.

Let $F: \Sigma \times I \to S^k \times I$ be a homotopy between ψ'_0 and $d \circ c$; F induces a normal cobordism $F' : \Sigma \times I \cup M_c \to S^k \times I$ between ψ'_0 and d. Combining this with the normal cobordism from ψ_0 to ψ'_0, we get a cobordism Q from P to S^k together with a stable framing of T_Q. By Theorem 2.3 of [19], there is a PL-manifold Q' and a degree 1 normal map (in the sense of [19]), $h: (Q'; P', S^k) \to (Q; P, S^k)$ obtained from the lift of ν_Q from $BH(K)$ to BPL defined by the stable framings of T_Q and $T_Q \oplus \nu_Q$. Since $k \geq 5$, we may choose h to be a Λ-homotopy equivalence by Theorem 2.1 of [19].

By construction, there is a normal cobordism $H: V_0 \to P \times I$ from $h|P'$ to 1_p. Let $x \in L_{k+1}(\Lambda[1])$ be the surgery obstruction of H and choose a normal map $\phi_x: M_x \to S^{k+1}$ with $\sigma(\phi_x) = -x$. (M_x may be chosen to be a PL-manifold, with one singularity if $x \notin L_{k+1}(1)$; see [4].) Replacing V_0 with $V_0 \# M_x$ and H with $H \cup \phi_x$, we get $\sigma(H) = 0$, and so may assume that H is a Λ-homotopy equivalence by Theorem 4.7.

Let $\pi_1 : P \times I \to P$ be the projection. Then $W_0 = M \times I \cup E(H^*\pi_1^*\nu_P)$ is the desired h_K-cobordism, where ν_P is the normal Λ-homology cobordism disc bundle of P in M. Finally, ϕ extends over V_0 since $\phi \circ i_0$ is null-homotopic, and ϕ extends to W_0 since $V_0 \simeq E(H^*\pi_1^*\nu_p)$.

(3) $k = 4$. In this case, P is a PL-manifold except at isolated singularities, $x_1, \ldots, x_p$. By preliminary surgeries, we can assume that $\psi_0^{-1}\psi_0(x_i) = \{x_i\}$. Form P_0, S_0^4 from P, S^4 by removing small open neighborhoods of x_i, $\psi_0(x_i)$, so that ψ_0 maps $(P_0, \partial P_0)$ to $(S_0^4, \partial S_0^4)$. By Lemma 5.4 and the proof of Theorem 16.6 of [24], $\psi_0|P_0$ is normally cobordant to a Λ-homotopy equivalence, relative to ∂P_0. Filling in the neighborhoods we removed, we get that $\psi_0|P$ is normally cobordant to a Λ-homotopy equivalence $\psi'_0: \Sigma \to S_1^4$. The proof is now the same as case 2, with the exception of the construction of the Λ-homotopy equivalence $h|P'$. We may still construct a normal map $h: (Q'; P', S^4) \to (Q; P, S^4)$ and it follows from [4] that the surgery obstruction of $h|P'$ vanishes since both P and P' are cobordant to S^4. Arguing as before, $h|P'$ is normally cobordant to a Λ-homotopy equivalence.

Remark. If $K = \emptyset$, then it follows from [9] that P resolves to a PL-manifold, and so is H-cobordant to one. By [22], Section 5, this H-cobordism is actually an h-cobordism, and so the argument above can be avoided.

Proof of Proposition 5.3. Construct a codimension 0 submanifold $\hat{N}'$ engulfing a basis of $K_k(M'; \Lambda)$ and a Λ-homotopy equivalence $\psi': N' \to \hat{N}'$ as above. We may clearly assume $P' \subset \hat{N}'$, and, since $T_{P'}$ is stably framed and P' has codimension $k \geq 3$, we may choose N' and ψ' so that there is an embedding l of P' in N' and $\psi'|l(P'): l(P') \to P'$ is a PL-homeomorphism. (See [24, Chapter 3].)

The construction of P' yields a framed cobordism between P' and S^k. (If $k \leq 3$, $\psi_0 | P'$ is in fact normally cobordant to S^k). Let $Q = P' \times I \cup H_1 \cup \cdots \cup H_r$ be a handle decomposition of such a cobordism with no handles of index 0, 1, k or $k + 1$. By Theorem 1.3 of [24] and general position, the embedding l extends to an embedding $L: Q \to N'$.

Let

$$Q_s = L(P' \times I \cup H_1 \cup \cdots \cup H_s)$$

and write $\partial Q_s = l(P') \cup P_s$. We claim that there is an h_K-cobordism W_s from M' to M'_s, rel (P'), an extension ψ' to $\psi'_s: N' \to M'_s$ and a framed cobordism $\hat{Q}_s \subset \hat{M}_s$, from P' to P'_s, so that ψ'_s: $(Q_s, \partial Q_s) \to (\hat{Q}_s, \partial \hat{Q}_s)$ is a Λ-homotopy equivalence.

Let $W'_0 = M' \times I$, and construct M'_{s+1} from M' by the following 3 steps.

(1) Let $c_{s+1}: (D^{i+1}, S^i) \to (N', P_s)$ be the embedding of the core of H_{s+1}. We may collapse a Λ-acyclic codimension 0 submanifold of M_s to a point, as in Corollary 4.3, to assume that $\psi \circ c_{s+1} | S^i$ is an embedding.

(2) Let μ: $(\Delta, S^i) \to (M'_s, P'_s)$ be a map close to $\psi_s \circ c_{s+1} | D^{i+1}$, rel (S^i), in general position as in Theorem 3.1. Since $i + 1 < k$, μ is an embedding and we may again collapse a regular neighborhood of the exterior of an $(i + 1)$-simplex of Δ, with an i-face on S^i, to a point to assume that $\psi'_s \circ c_{s+1}$: $(D^{i+1}, S^i) \to (M'_s, P'_s)$ is an embedding.

(3) The normal bundle of D^{i+1} in M'_s is trivial and we may change M'_s by an h-cobordism, as in Lemma 4.4, to assume that

$$\psi'_s \circ (c_{s+1} \times 1_{D^{k-i}}): (D^{i+1} \times D^{k-i}, S^i \times D^{k-i}) \to (M'_s, P'_s)$$

is an embedding.

Define W'_{s+1} to be the union of W'_s, the mapping cylinders of the Λ-acyclic resolutions constructed in steps 1 and 2, and the h-cobordism of step 3. Clearly, ψ'_s extends to $\psi'_{s+1}: N' \to M'_{s+1}$, and, as in Section 4, $\phi': M' \to X$ extends to M'_{s+1}. Define $\hat{Q}_{s+1}$ by adding the handle of step 3 to $\hat{Q}_s$.

By induction, $P'_r \subset M'_r$ is a Λ-homotopy sphere. Collapsing a regular neighborhood of the exterior of a k-simplex in P'_r to a point, we have that M' is h_K-cobordant to a Λ-homology manifold M so that $[P'] \in K_k(M'; \Lambda)$ is represented by an embedding j: $S^k \to \hat{M}$. This implies the result as in Section 4.

Proof of Theorem 1.1. By Lemma 5.2, we need only verify sufficiency. Suppose $\sigma(\phi) = 0$, so that $K_k(M; \Lambda)$ is a kernel with basis $e_1, f_1, \ldots, e_m, f_m$. Let Φ: $W \to X$ be a normal h_K-cobordism from ϕ to ϕ': $M' \to X$ so that $(j'_*)^{-1} j_*(e_m)$ is represented by an embedding i: $S^k \to M'$. Let v_i be the normal bundle of i. By the proof of Proposition 5.3, we may assume that the basis of $K_k(M'; \Lambda)$ determined by $e_1, f_1, \ldots, e_m, f_m$ is engulfed in a patch ψ: $N' \to M'$ so that $\psi^{-1}(S^k) \cong S^k$ with normal bundle equivalent (as Λ-homology cobordism bundles) to $(i^{-1} \circ \psi | \psi^{-1}(S^k))^* v_i$.

Let $e'_m = i_*[S^k] \in K_k(M'; \Lambda)$, $e''_m = (\psi^{-1} \circ i)_*[S^k] \in K_k(N; \Lambda)$. Since W is an h_K-cobordism, it follows that $\lambda'(e'_m, e'_m) = 0$, $\mu'(e'_m) = 0$. Letting λ_0, μ_0 denote the intersection and self-intersection functions on $K_k(N; \Lambda)$, we have $\lambda_0(e''_m, e''_m) = 0$, $\mu_0(e''_m) = 0$ by construction. By [24], Theorem 5.2(iii), the Euler class of the normal bundle of $S^k \subset N$ vanishes, and so v_i is zero in $\pi_k(BH(K)_k)$ by Theorem 2.5. Arguing as in the proof of Theorem 4.1, we may change M' by an h_K-cobordism to represent e'_m by an embedding of $S^k \times D^k$. The proof now follows from the proof of Theorem 5.6 of [24].

Theorem 1.1 can be extended to cover the simple case. Define $Wh(\pi; \Lambda) = K_1(\Lambda[\pi]/G$, where G is the subgroup generated by π and $\Lambda^{\cdot}$. Λ-Poincare complexes, Λ-homotopy equivalences and h_K-cobordisms determine well-defined torsions in $Wh(\pi; \Lambda)$, defined as in [2], [24], and the prefix simple is applied to those objects above with vanishing torsion. By the proof of Theorem 2.1 of [24], a Λ-homology manifold is a simple Λ-Poincare complex.

THEOREM 5.6. *Under the conditions of Theorem* 1.1, *if X and $\phi|\partial M$ are simple, then the obstruction $\sigma(\phi)$ lies in $L^s_n(\Lambda[\pi_1(X)])$, and vanishes if and only if ϕ is normally cobordant to a simple Λ-homotopy equivalence.*

The proof is the same as that of Theorem 1, noting the following two facts.

(1) Theorem 3.1 of [8] extends to the simple case when $\mathscr{F}$ is the map $\mathbf{Z}[\pi] \to \Lambda[\pi]$ and torsion is computed in $Wh(\pi; \Lambda)$. (Note that our Whitehead group differs from the one defined in [8].)

(2) In Proposition 5.3, $(V; M, M)$ is a simple h_K-cobordism, since V is found from $M \times I$ by adjoining simply-connected h-cobordisms and mapping cylinders of maps of the form $M \to (M - \text{Int}(\Delta)) \cup c(\partial\Delta)$, where $\tilde{H}_*(\Delta; \Lambda) = 0$ and $\pi_1(\Delta) \to \pi_1(M)$ is 0, which clearly have zero torsion.

The relative versions of Theorem 1.1 may also be considered. The crucial geometric result is the following "π-π Theorem".

THEOREM 5.7. *Let ϕ: $(M; \partial_+, M, \partial_- M) \to (X; \partial_+ X, \partial_- X)$ be a normal map of degree* 1 *between a Λ-homology manifold triad M of dimension $n \geq 6$ and a* (*simple*) *Λ-Poincare triad X, so that $\phi|\partial_- M$ is a* (*simple*) *Λ-homotopy equivalence and $\pi_1(\partial_+ X) \cong \pi_1(X)$. Then ϕ is normally cobordant*, rel $(\partial_- M)$, *to a* (*simple*) *Λ-homotopy equivalence.*

Proof. Case 1. $n = 2k + 1$. This case follows exactly as the proof of Theorem 3.1 of [8].

Case 2. $n = 2k$. By Corollary 4.6, we may assume that $\phi|\partial_+ M$ is $(k - 1, \Lambda)$-connected and $\phi|M$ is (k, Λ)-connected; as before, we may take $K_k(M, \partial_+ M; \Lambda)$ to be free, with a basis represented by f_i: $(D^k, S^{k-1}) \to (M, \partial_+ M)$, $i = 1, \ldots, r$. The following lemma reduces the proof to the construction in [24, Chapter 4], or [8, Theorem 3.1].

LEMMA 5.8. *There is a Λ-acyclic resolution $p\colon M \to M'$,* rel $(\partial_- M)$, *and a map $\phi'\colon M' \to X$ so that ϕ and ϕ' are normally cobordant, $\phi' \circ p \simeq \phi$, ϕ', $\phi' | \partial_+ M$ are (k, Λ)-, $(k-1, \Lambda)$-connected, respectfully and the maps $p \circ f_i$, $i = 1, \ldots, r$, are homotopic to disjoint embeddings.*

Proof. By Proposition 4.2, there is a Λ-acyclic resolution $q\colon M \to M_0$ and maps

$$g_i\colon (D^k, S^{k-1}) \to (M_0, \partial_+ M_0),$$

homotopic to $q \circ f_i$, so that $g_i | S^{k-1}$, $i = 1, \ldots, r$, are disjoint embeddings. Applying Theorem 3.1, there exist Λ-acyclic resolutions $\mu_i\colon \Delta_i \to D^k$ rel (S^{k-1}) and maps $h_i\colon \Delta_i \to M_0$ in mutual general position so that $g_i \circ \mu_i \simeq h_i$ rel (S^{k-1}).

For each double point x of $\bigcup h_i\colon \bigcup \Delta_i \to M_0$, choose embedded paths α, β along different branches from x to $\partial_+ M_0$, missing all other singularities, and let γ be a path in $\partial_+ M_0$ from $\alpha(1)$ to $\beta(1)$. Since $\pi_1(\partial_+ M_0) \cong \pi_1(M_0)$ (see the proof of Lemma 4.5), the loop $\alpha^* \gamma^* \beta^{-1}$ is null-homotopic and so extends to $F\colon D^2 \to M_0$. By general position, $\alpha^* \gamma^* \beta^{-1}$ can be engulfed in a Λ-acyclic 2-complex. The remainder of the proof follows from Proposition 4.6, Corollary 4.3 and Lemma 4.5.

Theorem 5.7 may be used, exactly as in Chapter 9 of [24], to set up a geometric obstruction theory for surgery on Λ-homology manifold m-ads. Details are left to the reader.

REFERENCES

1. D. W. ANDERSON, *Localizing C.W. complexes*, Illinois J. Math., vol. 16 (1972), pp. 519–525.
2. G. A. ANDERSON, *Surgery with coefficients*, Lecture Notes in Mathematics #591, Springer-Verlag, New York, 1977.
3. ———, *Groups of PL Λ-homology spheres*, Trans. Amer. Math. Soc., vol. 241 (1978), pp. 55–67.
4. ———, *Λ-homology cobordism bundles*, to appear.
5. ———, *Resolutions of generalized polyhedral manifolds*, Tohoku Math. J., vol. 31 (1979), pp. 495–517.
6. J. BARGE, J. LANNES, F. LATOUR and P. VOGEL, *Λ-spheres*, Ann. Sci. Ecole Norm. Sup., vol. 4 (1974), pp. 463–506.
7. W. BROWDER, *Surgery on simply connected manifolds*, Ergebnisse der Math., Band 65, Springer-Verlag, New York, 1972.
8. S. E. CAPPEL and J. L. SHANESON, *The codimension two placement problem and homology equivalent manifolds*, Ann. of Math., vol. 99 (1974), pp. 277–348.
9. A. L. EDMONDS and R. J. STERN, *Resolutions of homology manifolds: a classification theorem*, J. London Math. Soc., vol. 11 (1975), pp. 474–480.
10. D. E. GALEWSKI and R. J. STERN, *Geometric transversality and bordism theory*, preprint.
11. S. T. HU, *Homotopy theory*, Academic Press, New York, 1959.
12. L. JONES, *Patch spaces*, Ann. of Math., vol. 97 (1973), pp. 306–343.
13. N. MARTIN, *On the difference between homology and piecewise-linear bundles*, J. London Math. Soc., vol. 6 (1973), pp. 197–204.
14. N. MARTIN and C. R. F. MAUNDER, *Homology cobordism bundles*, Topology, vol. 10 (1971), pp. 93–110.
15. A. MATSUI, *Surgery on 1-connected homology manifolds*, Tohoku Math. J., vol. 26 (1974), pp. 169–172.

16. T. MATUMOTO, *Varietes simpliciales dehomologie et varietes topologiques metrisables*, Ph.D. Dissertation, Orsay, 1976.
17. T. MATUMOTO and Y. MATSUMOTO, *The unstable difference between homology cobordism and piecewise linear block bundles*, Tohoku Math. J., vol. 27 (1975), pp. 57–68.
18. C. R. F. MAUNDER, *General position theorems for homology manifolds*, J. London Math. Soc., vol. 4 (1972), pp. 137–155.
19. F. QUINN, "Semifree group actions and surgery on *PL* homology manifolds" in *Geometric topology*, Lecture Notes in Mathematics #438, Springer-Verlag, New York, 1975.
20. C. P. ROURKE and B. J. SANDERSON, *Block bundles I*, Ann. of Math., vol. 87 (1968), pp. 1–28.
21. ——, *Block bundles III*, Ann. of Math., vol. 87 (1968), pp. 431–483.
22. H. SATO, *Constructing manifolds by homotopy equivalences I*, Ann. Inst. Fourier (Grenoble), vol. 22 (1972), pp. 271–286.
23. D. P. SULLIVAN, "Singularities in space", in *Proc. Liverpool singularities*, Lecture Notes in Mathematics #209, Springer-Verlag, New York, 1971.
24. C. T. C. WALL, *Surgery on compact manifolds*, Academic Press, New York, 1973.

THE PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA