ILLINOIS JOURNAL OF MATHEMATICS
Volume 24, Number 4, Winter 1980

ON THE ISOMORPHISM PROBLEM FOR INCIDENCE RINGS

BY

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Incidence rings were first defined by G. C. Rota in [10]. Given a locally finite partially ordered set X (see Section 1) and a ring Δ, the incidence ring $I(X, \Delta)$ is, loosely speaking, the ring of X by X matrices that have arbitrary elements of Δ in the (x, y) position if $x \leq y$, and 0 in the (x, y) position otherwise. If X is finite, then $I(X, \Delta)$ is just a tic-tac-toe ring in the sense of B. Mitchell [5, p. 229]. R. P. Stanley proved in [3] (also a sketch of proof appeared in [11]) that if Δ is a field and X and Y are locally finite partially ordered sets such that $I(X, \Delta) \cong I(Y, \Delta)$, then X is order isomorphic to Y. P. Ribenboim [9] has recently generalized Stanley's result to commutative noetherian rings in a graph theoretic setting. W. Belding [2] and N. A. Nachev [7] extended the definition of incidence rings for locally finite pre-ordered sets and proved that if Δ is simple artinian, then $I(X, \Delta) \cong I(Y, \Delta)$ implies that X is order isomorphic to Y as pre-ordered sets. Such theorems are partial answers to what Belding [2] called the *isomorphism problem for incidence rings*. In this paper, the methods of non-commutative ring theory allow us to modify Stanley's proof to obtain the more general results:

(1) *If X and Y are locally finite partially ordered sets and Δ is a ring that is indecomposable modulo the radical, then $I(X, \Delta) \cong I(Y, \Delta)$ implies that X and Y are order isomorphic.*

(2) *If X and Y are locally finite pre-ordered sets and Δ is a ring that is simple artinian modulo the radical, then $I(X, \Delta) \cong I(Y, \Delta)$ implies that X and Y are order isomorphic.*

Moreover we are able to generalize the Stanley-Belding-Nachev results in quite another direction by extending J. Hashimoto's results [4] on partially ordered sets in order to prove:

(3) *If X and Y are locally finite pre-ordered sets and Δ is an indecomposable semiperfect* (e.g., *indecomposable artinian*) *ring, then $I(X, \Delta) \cong I(Y, \Delta)$ implies that X and Y are order isomorphic.*

Received November 13, 1978.

1. Preliminaries

A set X is said to be *partially ordered* provided X has a relation $\leq$ that is reflexive, transitive, and antisymmetric (that is, $x \leq y$ and $y \leq x \Rightarrow x = y$). If $\leq$ is only reflexive and transitive, then X is said to be *pre-ordered*. In either case, X is *locally finite* provided each interval $\{z \in X \mid x \leq z \leq y\}$ has a finite number of elements. If X is a pre-ordered set, the relation $\sim$, given by $x \sim y$ if and only if $x \leq y$ and $y \leq x$, is an equivalence relation on X. The equivalence classes $[x] \in \hat{X} = X/\sim$ form a partially ordered set with $[x] \leq [y]$ if and only if $x \leq y$. We shall denote elements of $\hat{X}$ by Greek letters α, β, γ, etc.

Let X be a locally finite pre-ordered set and let Δ be a ring (with identity). The *incidence ring* $I(X, \Delta)$ of X over Δ consists of all functions f: $X \times X \to \Delta$ such that $f(x, y)$ is zero whenever $x \not\leq y$. If $f, g \in I(X, \Delta)$ then $f + g$ and fg are defined by

$$(f+g)(x, y) = f(x, y) + g(x, y), \qquad (fg)(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y).$$

We will say that a ring Δ *respects pre-order* if, given locally finite pre-ordered sets X and Y, $I(X, \Delta) \cong I(Y, \Delta)$ implies that X is order isomorphic to Y. If we only require that X and Y be partially ordered, then we say that Δ *respects partial order*. Unless otherwise specified, X, Y, and Z will always denote locally finite pre-ordered sets in this paper.

Although our purpose is to prove that certain classes of rings respect pre-order (or partial order), we first give a simple construction that provides a large class of rings that do not respect even partial order. Let Z be any locally finite pre-ordered set and let z_0 be an arbitrary element of Z. Define $Z^{(\omega)}$ to be the set of all sequences of elements from Z that eventually have constant value z_0. For $\sigma, \tau \in Z^{(\omega)}$, letting $\sigma \leq \tau$ if and only if $\sigma_i \leq \tau_i$ for $i = 1, 2, \ldots$ gives a locally finite pre-ordering of $Z^{(\omega)}$. The *product* of two pre-ordered sets X and Y is defined to be the Cartesian product $X \times Y$ with the ordering $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. If X and Y are locally finite, then so is $X \times Y$ and it is evident that

$$I(X, I(Y, \Delta)) \cong I(X \times Y, \Delta),$$

for any ring Δ, via $f \mapsto g$ where

$$g((x, y), (x', y')) = [f(x, x')](y, y').$$

These remarks allow us to prove:

PROPOSITION 1. *Let Z be a locally finite pre-ordered set. Then there exists a ring Γ such that $\Gamma \cong I(Z, \Gamma)$.*

Proof. It is easily seen that $Z \times Z^{(\omega)}$ is order isomorphic to $Z^{(\omega)}$ via $(z, (z_1, z_2, \ldots)) \mapsto (z, z_1, z_2, \ldots)$. Let Δ be any ring and let $\Gamma = I(Z^{(\omega)}, \Delta)$. Then

$$\Gamma = I(Z^{(\omega)}, \Delta) \cong I(Z \times Z^{(\omega)}, \Delta) \cong I(Z, I(Z^{(\omega)}, \Delta)) = I(Z, \Gamma). \qquad \blacksquare$$

Notice that if $\hat{Z}$ consists of more than one equivalence class and if $\Gamma \cong I(Z, \Gamma)$ then Γ does not respect partial order. Indeed if we take $Z = \{1, 2, \ldots, n\}$ with the usual partial ordering, Proposition 1 provides us with a ring Γ that not only does not respect partial order but is also isomorphic to the ring of $n \times n$ upper triangular matrices over itself.

In order to obtain the results promised above we shall need some facts about how certain idempotents of an incidence ring relate to its Jacobson radical. Let X be a locally finite pre-ordered set, let Δ be a ring, and as before, let $\hat{X} = X/\sim$. For each $x \in X$ we define the idempotent $e_x \in I(X, \Delta)$ by $e_x(u, v) = 1$ if $u = v = x$ and $e_x(u, v) = 0$ otherwise. For each $\alpha \in \hat{X}$ we define the idempotent $e_\alpha \in I(X, \Delta)$ by $e_\alpha(u, v) = 1$ if $u = v \in \alpha$ and $e_\alpha(u, v) = 0$ otherwise. Notice that e_α is just the sum of the e_x for $x \in \alpha$. It is not difficult to check that the orderings of X and $\hat{X}$ are intimately related to these idempotents as follows:

$$x \leq y \text{ in } X \text{ if and only if } e_x I(X, \Delta) e_y \neq 0$$

and

$$\alpha \leq \beta \text{ in } \hat{X} \text{ if and only if } e_\alpha I(X, \Delta) e_\beta \neq 0.$$

Intuitively, if $f \in I(X, \Delta)$, $e_\alpha f e_\beta$ corresponds to the (α, β) block of f when viewed as an $X \times X$ block matrix. In fact,

$$e_\alpha f e_\beta(x, y) = \begin{cases} f(x, y) & \text{if } x \in \alpha \text{ and } y \in \beta \\ 0 & \text{otherwise.} \end{cases}$$

In this context the following lemma says that we may multiply elements of the incidence ring by block matrix multiplication.

LEMMA 2. *Let $\alpha, \beta \in \hat{X}$ with $\alpha \leq \beta$ and let $f, g \in I(X, \Delta)$. Then*

$$e_\alpha(fg)e_\beta = \sum_{\alpha \leq \gamma \leq \beta} (e_\alpha f e_\gamma)(e_\gamma g e_\beta).$$

Proof. Let $x, y \in X$. Unless $x \in \alpha$ and $y \in \beta$, both sides of the above equation are zero when evaluated at (x, y). Hence we may assume $x \in \alpha$ and $y \in \beta$. Then $e_\alpha h e_\beta(x, y) = h(x, y)$ for each $h \in I(X, \Delta)$. Thus we have

$$\begin{aligned}
[e_\alpha(fg)e_\beta](x, y) &= (fg)(x, y) \\
&= \sum_{x \leq z \leq y} f(x, z) g(z, y) \\
&= \sum_{\alpha \leq \gamma \leq \beta} \sum_{z \in \gamma} f(x, z) g(z, y) \\
&= \sum_{\alpha \leq \gamma \leq \beta} \left[\sum_{x \leq z \leq y} (e_\alpha f e_\gamma)(x, z) \cdot (e_\gamma g e_\beta)(z, y) \right] \\
&= \left[\sum_{\alpha \leq \gamma \leq \beta} (e_\alpha f e_\gamma)(e_\gamma g e_\beta) \right] (x, y). \qquad \blacksquare
\end{aligned}$$

We devote the remainder of this section to calculating the Jacobson radical of the incidence ring $I(X, \Delta)$. Since we will use the quasi-regular characterization of the radical, we will first need to determine the invertible elements of $I(X, \Delta)$. Rota [8, Proposition 1] showed that if Δ is a field and $\zeta \in I(X, \Delta)$ where $\zeta(x, y) = 1$ if $x \leq y$ and $\zeta(x, y) = 0$ otherwise, then ζ has an inverse μ which he called the *Möbius function* for X. Extending Rota's method of direct calculation to block matrices, we are able to determine the invertible elements of $I(X, \Delta)$ for any ring Δ (cf. [2, Theorem 1.16] and [9, Proposition 1]).

PROPOSITION 3. *Let X be a locally finite pre-ordered set and let Δ be a ring. Then an element f of $R = I(X, \Delta)$ is left (right) invertible if and only if $e_\alpha f e_\alpha$ is left (right) invertible in the ring $e_\alpha R e_\alpha$ for each $\alpha \in \hat{X}$.*

Proof. ($\Rightarrow$) Assume $gf = 1_R$ for some $g \in R$. If $\alpha \in \hat{X}$ then by Lemma 2 we have

$$(e_\alpha g e_\alpha)(e_\alpha f e_\alpha) = e_\alpha(gf)e_\alpha = e_\alpha.$$

Since e_α is the identity of $e_\alpha R e_\alpha$, $e_\alpha g e_\alpha$ is a left inverse of $e_\alpha f e_\alpha$.

($\Leftarrow$) For each $\alpha, \beta \in \hat{X}$ with $\alpha \leq \beta$ we define $r_{\alpha\beta} \in e_\alpha R e_\beta$ as follows: for each α, let $r_{\alpha\alpha}$ be a left inverse of $e_\alpha f e_\alpha$ in $e_\alpha R e_\alpha$. Now fix α, let $\beta > \alpha$, and assume that we have defined $r_{\alpha\gamma}$ for each $\alpha \leq \gamma < \beta$. Then let

$$r_{\alpha\beta} = -\left[\sum_{\alpha \leq \gamma < \beta} r_{\alpha\gamma}(e_\gamma f e_\beta)\right] r_{\beta\beta}.$$

Now define $g \in R$ by

$$g(x, y) = \begin{cases} r_{\alpha\beta}(x, y) & \text{if } x \leq y \text{ and } x \in \alpha,\ y \in \beta \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, this makes a matrix out of the blocks $r_{\alpha\beta}$, that is, $e_\alpha g e_\beta = r_{\alpha\beta}$ for $\alpha \leq \beta$. Let $x, y \in X$ and let $\alpha, \beta \in \hat{X}$ such that $x \in \alpha$ and $y \in \beta$. If $\alpha = \beta$, then by Lemma 2,

$$\begin{aligned} gf(x, y) &= e_\alpha(gf)e_\alpha(x, y) = (e_\alpha g e_\alpha)(e_\alpha f e_\alpha)(x, y) \\ &= r_{\alpha\alpha}(e_\alpha f e_\alpha)(x, y) = e_\alpha(x, y) = 1_R(x, y). \end{aligned}$$

If $\alpha < \beta$ then again by Lemma 2,

$$\begin{aligned} gf(x, y) = e_\alpha(gf)e_\beta(x, y) &= \left[\sum_{\alpha \leq \gamma \leq \beta} (e_\alpha g e_\gamma)(e_\gamma f e_\beta)\right](x, y) \\ &= \left[\sum_{\alpha \leq \gamma < \beta} (r_{\alpha\gamma} e_\gamma f e_\beta) + r_{\alpha\beta} e_\beta f e_\beta\right](x, y) = 0, \end{aligned}$$

after inserting the definition of $r_{\alpha\beta}$. Hence $gf = 1_R$. For right invertible, the proof is analogous. ∎

For partial order, the Jacobson radical of $I(X, \Delta)$ has been calculated in [3, Proposition 3.3] when Δ is a field. Nachev [8, Theorem 1] determined the radical of $I(X, \Delta)$ when X is a pre-ordered set and Δ is a semisimple ring. For any ring Δ we have:

PROPOSITION 4. *Let X be a locally finite pre-ordered set and let Δ be a ring. Let R be the incidence ring $I(X, \Delta)$. If N is the Jacobson radical $J(\Delta)$ of Δ, then the Jacobson radical of R is $J(R) = \{f \in R \mid f(x, y) \in N \text{ if } x \sim y\}$.*

Proof. Let I denote the above set, which is a (two-sided) ideal of R. If $\alpha \in \hat{X}$, then $e_\alpha R e_\alpha$ is isomorphic to the ring $M_n(\Delta)$ of $n \times n$ matrices over Δ, where $n = \operatorname{card} \alpha$, via $e_\alpha f e_\alpha \mapsto (A_{ij})$ where $A_{ij} = f(i, j)$ for $i, j \in \alpha$. Since this isomorphism takes $e_\alpha I e_\alpha$ onto $M_n(N) = J(M_n(\Delta))$ we have $J(e_\alpha R e_\alpha) = e_\alpha I e_\alpha$. Hence for $f \in I$, each $e_\alpha f e_\alpha$ is quasi-regular in $e_\alpha R e_\alpha$, that is, $e_\alpha(1 - f)e_\alpha$ is invertible in $e_\alpha R e_\alpha$. Then $1 - f$ is invertible by Proposition 3. Thus I is a quasi-regular ideal, so $I \subseteq J(R)$. Using quasi-regularity and Proposition 3, it is also immediate that $e_\alpha J(R) e_\alpha \subseteq J(e_\alpha R e_\alpha) = e_\alpha I e_\alpha$ for each $\alpha \in \hat{X}$. It then follows easily from the definition of I that if $f \in J(R)$ then $f \in I$. Hence $J(R) = I$. ∎

Notice that a useful consequence of Proposition 4 is that if $f \in R$ and $e_\alpha f e_\alpha \in J(R)$ when $\alpha \in \hat{X}$, then $f \in J(R)$. Roughly speaking, a matrix belongs to the radical of $I(X, \Delta)$ if and only if the blocks on the main diagonal are matrices over $N = J(\Delta)$.

2. Incidence rings over a ring that is indecomposable modulo the radical

Stanley [11] proved that fields respect partial order. In this section, we shall prove (Theorem 1) that all rings which are indecomposable modulo the radical (e.g., the integers or a primitive ring) respect partial order. As a consequence, we prove (Theorem 2) that rings which are simple artinian modulo the radical (e.g., local rings) respect pre-order. This generalizes the result of Belding [2] and Nachev [7] that simple artinian rings respect pre-order.

THEOREM 1. *Let Δ be a ring that is indecomposable modulo $J(\Delta)$. Let X and Y be locally finite pre-ordered sets such that $I(X, \Delta) \cong I(Y, \Delta)$. Then $\hat{X}$ is order isomorphic to $\hat{Y}$. In particular, Δ respects partial order.*

Proof. Let $R = I(X, \Delta)$, $S = I(Y, \Delta)$ and let $\theta\colon R \to S$ be the ring isomorphism. Also let $\{e_\alpha \mid \alpha \in \hat{X}\}$ and $\{f_\beta \mid \beta \in \hat{Y}\}$ be the idempotents for R and S, respectively, as defined in Section 1. The proof relies on the well-behaved structure of $R/J(R)$ and $S/J(S)$. Indeed if $f \in R$ and $\alpha \in \hat{X}$ then $e_\alpha f - f e_\alpha \in J(R)$ since for every $\alpha' \in \hat{X}$, $e_{\alpha'}(e_\alpha f - f e_\alpha)e_{\alpha'} = 0 \in J(R)$. Hence each $e_\alpha + J(R)$ is a central idempotent of $R/J(R)$. Furthermore, as rings, we have

$$\begin{aligned}(e_\alpha R e_\alpha + J(R))/J(R) &\cong e_\alpha R e_\alpha / e_\alpha J(R) e_\alpha \\ &\cong M_n(\Delta)/M_n(J(\Delta)) \cong M_n(\Delta/J(\Delta))\end{aligned}$$

where $n = \text{card}\,\alpha$, because $e_\alpha R e_\alpha$ and $M_n(\Delta)$ are isomorphic via $e_\alpha f e_\alpha \to (A_{ij})$ where $A_{ij} = f(i, j)$ for $i, j \in \alpha$, and this isomorphism takes $e_\alpha J(R) e_\alpha$ onto $M_n(J(\Delta))$. Since ideals of $M_n(\Delta/J(\Delta))$ are of the form $M_n(K)$ where K is an ideal of $\Delta/J(\Delta)$ [1, exercise 1.8], we know that $M_n(\Delta/J(\Delta))$ is indecomposable and therefore that $e_\alpha + J(R)$ is a primitive central idempotent of $R/J(R)$ (see [1, Theorem 7.9]). In fact, the $e_\alpha + J(R)$ for $\alpha \in \hat{X}$ are the only primitive central idempotents in $R/J(R)$. For indeed, if $f + J(R)$ is a primitive central idempotent of $R/J(R)$, then $e_\alpha f e_\alpha$ cannot belong to $J(R)$ for every $\alpha \in \hat{X}$ since $f \notin K(R)$. Hence for some $\alpha \in \hat{X}$, $(e_\alpha + J(R))(f + J(R)) \neq 0$; but in any ring, two primitive central idempotents with nonzero product must be equal. Therefore

$$\{e_\alpha + J(R) \mid \alpha \in \hat{X}\} \quad \text{and} \quad \{f_\beta + J(S) \mid \beta \in \hat{Y}\}$$

are precisely the primitive central idempotents of $R/J(R)$ and $S/J(S)$, respectively. Since $\theta\colon R \to S$ is an isomorphism, there is a set bijection $\sigma\colon \hat{X} \to \hat{Y}$ such that if $\sigma(\alpha) = \beta$ then

$$\theta(e_\alpha) + J(S) = f_\beta + J(S).$$

Thus if $\sigma(\alpha) = \beta$, we have

$$S\theta(e_\alpha)/J(S)\theta(e_\alpha) \cong S f_\beta / J(S) f_\beta$$

as left S-modules. But then (see [1, Proposition 17.18])

$$S\theta(e_\alpha) \cong S f_\beta \quad \text{and} \quad \theta(e_\alpha) S \cong f_\beta S$$

as left and right S-modules respectively. Now let $\alpha \leq \alpha'$ in $\hat{X}$ and let $\beta = \sigma(\alpha)$ and $\beta' = \sigma(\alpha')$. Then $e_\alpha R e_{\alpha'} \neq 0$ and hence $\theta(e_\alpha) S \theta(e_{\alpha'}) \neq 0$. The above two isomorphisms apply, first to give $\theta(e_\alpha) S f_{\beta'} \neq 0$ and then that $f_\beta S f_{\beta'} \neq 0$. Therefore $\beta \leq \beta'$. Similarly σ^{-1} is order preserving, hence $\sigma\colon \hat{X} \to \hat{Y}$ is the required order isomorphism. ∎

The hypothesis of Theorem 1 does not imply that Δ respects pre-order, even if Δ is simple, as the following example shows: Let V be a vector space of countably infinite dimension over a field F. Let

$$I = \{f \in \text{End}_F\, V \mid \dim(\text{Im}\, f) \text{ is finite}\}.$$

Then I is the unique maximal ideal of $\text{End}_F\, V$ [1, exercise 14.13]. Hence $\Delta = (\text{End}_F\, V)/I$ is a simple ring. Since $V \oplus V \cong V$ we have $\text{End}_F\, V \cong \text{End}_F\, (V \oplus V)$. Thus if I' is the unique maximal ideal of $\text{End}_F\, (V \oplus V)$, then

$$\Delta \cong (\text{End}_F\, V)/I \cong \text{End}_F\, (V \oplus V)/I' \cong M_2(\text{End}_F\, V)/M_2(I) \cong M_2(\Delta).$$

So if $X = \{1, 2\}$ with $1 \leq 2$ and $2 \leq 1$ and $Y = \{1\}$, then $I(X, \Delta) \cong M_2(\Delta) \cong \Delta \cong I(Y, \Delta)$, and the simple ring Δ does not respect pre-order.

THEOREM 2. *Let Δ be a ring that is indecomposable modulo $J(\Delta)$. Then Δ respects pre-order if and only if $M_m(\Delta) \cong M_n(\Delta)$ implies $m = n$ for any positive integers m and n.*

Proof. ($\Leftarrow$) Let X and Y be locally finite pre-ordered sets and let $R = I(X, \Delta)$ and $S = I(Y, \Delta)$. If θ: $R \to S$ is an isomorphism, then by the proof of Theorem 1, there is an order isomorphism σ: $\hat{X} \to \hat{Y}$ such that if $\sigma(\alpha) = \beta$ then $S\theta(e_\alpha) \cong Sf_\beta$ as left S-modules. By [1, Theorem 4.15], $\theta(e_\alpha)S\theta(e_\alpha) \cong f_\beta Sf_\beta$ as rings. Since $\theta(e_\alpha)S\theta(e_\alpha) \cong e_\alpha Re_\alpha \cong M_n(\Delta)$ where $n = \text{card } \alpha$, and $f_\beta Sf_\beta \cong M_m(\Delta)$ where $m = \text{card } \beta$, we have, by hypothesis, card α = card β. Thus σ lifts to an order isomorphism σ': $X \to Y$.

($\Rightarrow$) Let $M_m(\Delta) \cong M_n(\Delta)$ and let $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$ with pre-orderings such that $\hat{X}$ and $\hat{Y}$ each have only one element. Since $M_m(\Delta) \cong I(X, \Delta)$ and $M_n(\Delta) \cong I(Y, \Delta)$, by hypothesis X and Y are order isomorphic, that is, $m = n$. ∎

COROLLARY 3. *If Δ is a ring that is simple artinian modulo $J(\Delta)$, then Δ respects pre-order.*

Proof. Let $M_m(\Delta) \cong M_n(\Delta)$. Then $M_m(\Delta/J(\Delta)) \cong M_n(\Delta/J(\Delta))$ and since $\Delta/J(\Delta)$ is isomorphic to $M_k(D)$ for some positive integer k and division ring D, we have $M_{m+k}(D) \cong M_{n+k}(D)$. By the Jordan-Hölder Theorem, $m + k = n + k$, so $m = n$. Now use Theorem 2. ∎

3. A cancellation result for pre-ordered sets

In Section 4 we shall prove that indecomposable semiperfect rings respect pre-order. To do this, however, we need a result about cancellation of pre-ordered sets. A pre-ordered set X is said to be *connected* if for every $x, x' \in X$ there exists a sequence $x = x_1, \ldots, x_r = x'$ from X such that x_i is comparable to x_{i+1} for $i = 1, \ldots, r - 1$. The purpose of this section is to prove that if X, Y, and Z are locally finite pre-ordered sets and Z is finite and connected, then $X \times Z \cong Y \times Z$ implies $X \cong Y$. (Here $\cong$ means order isomorphism.) We remark that this result follows easily from Theorem 2 of Hashimoto [4] in case X, Y, and Z are partially ordered sets.

Recall that if X is a pre-ordered set, we have defined an equivalence relation $\sim$ on X via $x \sim x'$ if and only if $x \leq x'$ and $x' \leq x$. Elements $\alpha \in \hat{X} = X/\sim$ may also be considered to be pre-ordered subsets of X. If P is a set of ordered pairs, then P_1 will denote the set of first coordinates of pairs in P, and P_2 will denote the set of second coordinates. The following lemma is an adaptation of Lemmas 1, 2, and 3 of Hashimoto [4].

LEMMA 1. *Let X, Y, U, and V be pre-ordered sets and let X be connected. If θ: $X \times U \to Y \times V$ is an order isomorphism, then $\theta(X \times \gamma) = \theta(X \times \gamma)_1 \times \theta(X \times \gamma)_2$ for every $\gamma \in \hat{U}$.*

Proof. First we assume that X, Y, U, and V are partially ordered. Although this is just Hashimoto's result, we shall give the proof for the sake of completeness. Let $u \in U$. Then, to prove that

$$\theta(X \times \{u\}) = \theta(X \times \{u\})_1 \times \theta(X \times \{u\})_2,$$

we need only show that

$$(y_1, v_1), (y_2, v_2) \in \theta(X \times \{u\}) \quad \text{implies} \quad (y_1, v_2) \in \theta(X \times \{u\}).$$

First assume there is a $(y', v') \in \theta(X \times \{u\})$ such that $(y_1, v_1) \geq (y', v') \leq (y_2, v_2)$. Then we have

$$(y_1, v_1) \geq (y_1, v') \geq (y', v') \quad \text{and} \quad (y', v') \leq (y', v_2) \leq (y_2, v_2).$$

By applying θ^{-1} to the above, we see that the second coordinates of $\theta^{-1}(y_1, v')$ and $\theta^{-1}(y', v_2)$ must be u. Hence (y_1, v') and (y', v_2) belong to $\theta(X \times \{u\})$. Let $\theta^{-1}(y_1, v_2) = (x', u')$ and let $\theta^{-1}(y_1, v') = (x'', u)$. Since $(y_1, v_2) \geq (y_1, v')$ we have, upon applying θ^{-1}, $(x', u') \geq (x', u) \geq (x'', u)$. Therefore, we have

$$(y_1, v_2) \geq \theta(x', u) \geq (y_1, v').$$

In a similar way, since $(y_1, v_2) \geq (y', v_2)$, we see that

$$(y_1, v_2) \geq \theta(x', u) \geq (y', v_2).$$

These two inequalities imply that the first coordinate of $\theta(x', u)$ is y_1 and that the second coordinate is v_2, that is, $(y_1, v_2) \in \theta(X \times \{u\})$. Similarly if $(y_1, v_1) \leq (y', v') \geq (y_2, v_2)$ we may show that $(y_1, v_2) \in \theta(X \times \{u\})$. Since X is connected, so is $X \times \{u\}$, and therefore $\theta(X \times \{u\})$ is connected since connectedness is preserved by order isomorphisms. Hence, in general, when (y_1, v_1) and (y_2, v_2) belong to $\theta(X \times \{u\})$, there is a sequence (or its dual)

$$(y_1, v_1) = (y'_1, v'_1) \geq (y'_2, v'_2) \leq (y'_3, v'_3) \geq \cdots$$
$$\leq (y'_{n-2}, v'_{n-2}) \geq (y'_{n-1}, v'_{n-1}) \leq (y'_n, v'_n) = (y_2, v_2)$$

each term of which belongs to $\theta(X \times \{u\})$. Assume by induction that $(y_1, v_2) \in \theta(X \times \{u\})$ when (y_1, v_1) and (y_2, v_2) are connected by shorter sequences. Then (y'_1, v'_{n-2}), (y'_3, v'_n), and (y'_2, v'_{n-1}) all belong to $\theta(X \times \{u\})$. Since

$$(y'_1, v'_{n-2}) \geq (y'_2, v'_{n-1}) \leq (y'_3, v'_n),$$

we have, as shown earlier, $(y_1, v_2) = (y'_1, v'_n) \in \theta(X \times \{u\})$.

Now assume that X, Y, U, and V are pre-ordered sets. For each $\alpha \in \hat{X}$ and $\gamma \in \hat{U}$, it is easy to show that there is a unique $\beta \in \hat{Y}$ and $\delta \in \hat{V}$ such that $\theta(\alpha \times \gamma) = \beta \times \delta$. Thus $\hat{\theta}(\alpha, \gamma) = (\beta, \delta)$ defines an order isomorphism $\hat{\theta}\colon \hat{X} \times \hat{U} \to \hat{Y} \times \hat{V}$. By the above,

$$\hat{\theta}(\hat{X} \times \{\gamma\}) = \hat{\theta}(\hat{X} \times \{\gamma\})_1 \times \hat{\theta}(\hat{X} \times \{\gamma\})_2$$

for every $\gamma \in \hat{U}$, since $\hat{X}$ is connected. What we want to show is that

$$\theta(X \times \gamma) = \theta(X \times \gamma)_1 \times \theta(X \times \gamma)_2.$$

Let (y_1, v_1) and (y_2, v_2) belong to $\theta(X \times \gamma)$, and let $y_1 \in \beta_1$, $y_2 \in \beta_2$, $v_1 \in \delta_1$, and $v_2 \in \delta_2$ where $\beta_1, \beta_2 \in \hat{Y}$ and $\delta_1, \delta_2 \in \hat{V}$. Since (β_1, δ_1) and (β_2, δ_2) must then belong to $\hat{\theta}(\hat{X} \times \{\gamma\})$, (β_1, δ_2) must also belong to $\hat{\theta}(\hat{X} \times \{\gamma\})$. Hence $(y_1, v_2) \in \theta(X \times \gamma)$, which completes the proof of the lemma. ∎

A locally finite pre-ordered set is *trivial* if it consists of only one equivalence class under the relation $\sim$. We continue to use Greek letters α, β, γ, ... to denote trivial pre-ordered sets. Of course, two trivial pre-ordered sets are isomorphic if and only if they have the same cardinality.

LEMMA 2. *If X, Y and γ are locally finite pre-ordered sets and γ is trivial, then $X \times \gamma \cong Y \times \gamma$ implies $X \cong Y$.*

Proof. Let $\theta: X \times \gamma \to Y \times \gamma$ be the order isomorphism. Then for each $\alpha \in \hat{X}$ there is a unique $\beta \in \hat{Y}$ such that $\theta(\alpha \times \gamma) = \beta \times \gamma$ and card α = card β. Letting $\hat{\phi}(\alpha) = \beta$, we obtain an order isomorphism $\hat{\phi}: \hat{X} \to \hat{Y}$. Since $\hat{\phi}(\alpha) = \beta$ implies that card α = card β, this isomorphism lifts to an order isomorphism $\phi: X \to Y$. ∎

A connected locally finite pre-ordered set is said to be *irreducible* if it is not isomorphic to the product of two non-singleton pre-ordered sets. For example, a trivial pre-ordered set is irreducible if and only if its cardinality is prime. If X is a *finite* pre-ordered set, define $gcd(X)$ to be the greatest common divisor of $\{\text{card } \alpha \mid \alpha \in \hat{X}\}$. Let $[n]$ denote a trivial pre-ordered set with n elements. Then clearly $X \cong [\gcd(X)] \times Y$ for some finite pre-ordered set Y with $\gcd(Y) = 1$. From number theory,

$$\gcd\{m_i \mid i = 1, \ldots, r\} \cdot \gcd\{n_j \mid j = 1, \ldots, s\} = \gcd\{m_i n_j \mid i = 1, \ldots, r \quad \text{and} \quad j = 1, \ldots, s\}.$$

Hence if X and Y are finite pre-ordered sets,

$$\gcd(X \times Y) = \gcd(X) \gcd(Y)$$

since $\{\text{card } \gamma \mid \gamma \in \widehat{X \times Y}\} = \{\text{card } \alpha \cdot \text{card } \beta \mid \alpha \in \hat{X} \text{ and } \beta \in \hat{Y}\}$. This device helps us to prove the following.

LEMMA 3. *Let α, Z, U, and V be finite pre-ordered sets where α is trivial and Z is irreducible. Then $\alpha \times Z \cong U \times V$ implies that either U or V is trivial.*

Proof. Since Z is irreducible, $\gcd(Z) = 1$. Also $U \cong [\gcd(U)] \times U'$ and $V \cong [\gcd(V)] \times V'$ for some finite pre-ordered sets U' and V' such that $\gcd(U') = 1$ and $\gcd(V') = 1$. Since $\alpha \times Z \cong U \times V$, we have card α = $\gcd(\alpha \times Z) = \gcd(U) \gcd(V)$. Hence

$$\alpha \times Z \cong U \times V \cong [\gcd(U) \gcd(V)] \times U' \times V' \cong \alpha \times U' \times V'$$

and therefore $Z \cong U' \times V'$ by Lemma 2. Since Z is irreducible, either U' or V' is a singleton set. Thus either $U \cong [\gcd(U)]$ or $V \cong [\gcd(V)]$, that is, either U or V is trivial. ∎

Now we are ready to prove our cancellation result.

THEOREM 4. *Let X, Y, and Z be locally finite pre-ordered sets where Z is finite and connected. Then $X \times Z \cong Y \times Z$ implies $X \cong Y$.*

Proof. First we shall assume in addition that X and Y are connected and Z is irreducible. By Lemma 2 we may assume that Z is not trivial. Let $\theta: X \times Z \to Y \times Z$ be the order isomorphism. Choose any $\alpha \in \hat{X}$ and $\gamma \in \hat{Z}$; then there exists a (unique) $\beta \in \hat{Y}$ and $\delta \in \hat{Z}$ such that $\theta(\alpha \times \gamma) = \beta \times \delta$. Now by Lemma 1,

$$\theta(\alpha \times Z) = \theta(\alpha \times Z)_1 \times \theta(\alpha \times Z)_2.$$

Since θ is an order isomorphism, Lemma 3 implies that either $\theta(\alpha \times Z)_1$ or $\theta(\alpha \times Z)_2$ is trivial. Because $\beta \subseteq \theta(\alpha \times Z)_1$ and $\delta \subseteq \theta(\alpha \times Z)_2$, this means that either $\theta(\alpha \times Z)_1 = \beta$ or $\theta(\alpha \times Z)_2 = \delta$.

First we consider the case $\theta(\alpha \times Z)_1 = \beta$. That is,

$$\theta(\alpha \times Z) = \beta \times \theta(\alpha \times Z)_2 \subseteq \beta \times Z. \tag{1}$$

By Lemma 1 we have $\theta^{-1}(\beta \times Z) = \theta^{-1}(\beta \times Z)_1 \times \theta^{-1}(\beta \times Z)_2$. Hence by Lemma 3, either $\theta^{-1}(\beta \times Z)_1 = \alpha$ or $\theta^{-1}(\beta \times Z)_2 = \gamma$. If the latter is true, then

$$\beta \times \theta(\alpha \times Z)_2 \subseteq \beta \times Z \subseteq \theta(X \times \gamma).$$

But by (1), $\beta \times \theta(\alpha \times Z)_2 = \theta(\alpha \times Z)$ and therefore

$$\beta \times \theta(\alpha \times Z)_2 \subseteq \theta(X \times \gamma) \cap \theta(\alpha \times Z) = \theta(\alpha \times \gamma) = \beta \times \delta.$$

Then we would have $\theta(\alpha \times Z)_2 = \delta$ which contradicts (1), since Z is not trivial. Thus we must have $\theta^{-1}(\beta \times Z)_1 = \alpha$ and

$$\theta^{-1}(\beta \times Z) = \alpha \times \theta^{-1}(\beta \times Z)_2 \subseteq \alpha \times Z. \tag{2}$$

Putting (1) and (2) together, we obtain

$$\theta(\alpha \times Z) = \beta \times Z. \tag{3}$$

Since card Z is finite, this gives us card $\alpha =$ card β. Hence card $\gamma =$ card δ and $\gamma \cong \delta$, because $\theta(\alpha \times \gamma) = \beta \times \delta$. By Lemma 1,

$$\theta(X \times \gamma) = \theta(X \times \gamma)_1 \times \theta(X \times \gamma)_2.$$

Therefore $\beta \times \theta(X \times \gamma)_2 \subseteq \theta(X \times \gamma)$ and so by (3),

$$\beta \times \theta(X \times \gamma)_2 \subseteq \theta(X \times \gamma) \cap \theta(\alpha \times Z) = \theta(\alpha \times \gamma) = \beta \times \delta.$$

Thus $\theta(X \times \gamma)_2 = \delta$ and

$$\theta(X \times \gamma) = \theta(X \times \gamma)_1 \times \delta \subseteq Y \times \delta. \tag{4}$$

Similarly,

$$\theta^{-1}(Y \times \delta) = \theta^{-1}(Y \times \delta)_1 \times \gamma \subseteq X \times \gamma. \tag{5}$$

Then (4) and (5) together gives us $\theta(X \times \gamma) = Y \times \delta$ and since $\gamma \cong \delta$, we have $X \times \gamma \cong Y \times \gamma$. Lemma 2 now gives $X \cong Y$.

Secondly we consider the case $\theta(\alpha \times Z)_2 = \delta$, that is,

$$\theta(\alpha \times Z) = \theta(\alpha \times Z)_1 \times \delta \subseteq Y \times \delta. \tag{6}$$

By Lemma 1, $\theta^{-1}(\beta \times Z) = \theta^{-1}(\beta \times Z)_1 \times \theta^{-1}(\beta \times Z)_2$, and so

$$\alpha \times \theta^{-1}(\beta \times Z)_2 \subseteq \theta^{-1}(\beta \times Z).$$

Then (6) implies that

$$\alpha \times \theta^{-1}(\beta \times Z)_2 \subseteq \theta^{-1}(\beta \times Z) \cap \theta^{-1}(Y \times \delta) = \alpha \times \gamma.$$

Hence $\theta^{-1}(\beta \times Z)_2 = \gamma$ and

$$\theta^{-1}(\beta \times Z) = \theta^{-1}(\beta \times Z)_1 \times \gamma \subseteq X \times \gamma. \tag{7}$$

Again by Lemma 1, we have $\theta(X \times \gamma) = \theta(X \times \gamma)_1 \times \theta(X \times \gamma)_2$ which implies that $\theta(X \times \gamma)_2 = Z$, since by (7) $\theta(X \times \gamma) \supseteq \beta \times Z$. Hence,

$$\theta(X \times \gamma) = \theta(X \times \gamma)_1 \times Z \tag{8}$$

and similarly,

$$\theta^{-1}(Y \times \delta) = \theta^{-1}(Y \times \delta)_1 \times Z. \tag{9}$$

Then by (8), $\theta^{-1}(\theta(X \times \gamma)_1 \times \delta) \subseteq \theta^{-1}(\theta(X \times \gamma)_1 \times Z) = X \times \gamma$ and by, (9),

$$\theta^{-1}(\theta(X \times \gamma)_1 \times \delta) \subseteq \theta^{-1}(Y \times \delta) = \theta^{-1}(Y \times \delta)_1 \times Z.$$

Therefore we obtain

$$\theta^{-1}(\theta(X \times \gamma)_1 \times \delta) \subseteq \theta^{-1}(Y \times \delta)_1 \times \gamma. \tag{10}$$

In the same way, using (9) and (8), we have

$$\theta(\theta^{-1}(Y \times \delta)_1 \times \gamma) \subseteq \theta(X \times \gamma)_1 \times \delta. \tag{11}$$

Hence both (10) and (11) must be equalities. By (8),

$$X \times \gamma \times \delta \cong \theta(X \times \gamma)_1 \times \delta \times Z,$$

and by (9),

$$Y \times \delta \times \gamma \cong \theta^{-1}(Y \times \delta)_1 \times \gamma \times Z.$$

Then $X \times \gamma \times \delta \cong Y \times \delta \times \gamma$ since by (10) or (11), $\theta(X \times \gamma)_1 \times \delta \cong \theta^{-1}(Y \times \delta)_1 \times \gamma$. Because $\gamma \times \delta \cong \delta \times \gamma$ is trivial, Lemma 2 implies that $X \cong Y$. This completes the proof, provided X and Y are connected and Z is irreducible.

If Z is finite and connected then Z is order isomorphic to a finite product $Z_1 \times \cdots \times Z_r$ of irreducible finite connected pre-ordered sets. So by the above proof, $X \times Z_1 \times \cdots \times Z_r \cong Y \times Z_1 \times \cdots \times Z_r$ implies

$$X \times Z_1 \times \cdots \times Z_{r-1} \cong Y \times Z_1 \times \cdots \times Z_{r-1}$$

and so on, until by induction we arrive at $X \cong Y$. Here we are still assuming that X and Y are connected.

Now let X and Y be arbitrary locally finite pre-ordered sets. A *connected component* of X is a maximal connected subset of X. Every element of X belongs to a unique connected component and X is the disjoint union of its connected components. It is clear that an order isomorphism between two pre-ordered sets must take connected components of the first onto connected components of the second in a one-to-one fashion. Let $\{X_\alpha \mid \alpha \in A\}$ and $\{Y_\beta \mid \beta \in B\}$ be the connected components of X and Y respectively. Let Z be a finite connected pre-ordered set such that $X \times Z \cong Y \times Z$. Then $\{X_\alpha \times Z \mid \alpha \in A\}$ and $\{Y_\beta \times Z \mid \beta \in B\}$ are the connected components of $X \times Z$ and $Y \times Z$ respectively and there exists a set bijection $\sigma\colon A \to B$ such that for each $\alpha \in A$, $X_\alpha \times Z \cong Y_{\sigma(\alpha)} \times Z$. Therefore by the above, $X_\alpha \cong Y_{\sigma(\alpha)}$ for every $\alpha \in A$. Since no element of X_α is comparable to any element of $X_{\alpha'}$ when $\alpha \neq \alpha'$, we may assemble these isomorphisms to obtain $X \cong Y$. ∎

For incidence rings we have the following result which, since $M_n(\Delta) \cong I([n], \Delta)$, shows that the property "Δ respects pre-order" is inherited by the matrix rings $M_n(\Delta)$.

COROLLARY 5. *If Z is a finite connected pre-ordered set and Δ respects pre-order then so does $I(Z, \Delta)$.*

Proof. If X is a locally finite pre-ordered set then, as we remarked before Section 1, Proposition 1, $I(X, I(Z, \Delta)) \cong I(X \times Z, \Delta)$. Now we use Theorem 4. ∎

4. Incidence rings over indecomposable semiperfect rings

In this section we shall prove that indecomposable semiperfect rings respect pre-order. Unless otherwise specified, Δ will denote a semiperfect ring with radical N and complete set of primitive orthogonal idempotents $e_1, \ldots, e_n$. Then as a left Δ-module, $\Delta = \Delta e_1 \oplus \cdots \oplus \Delta e_n$. Also each $e_i \Delta e_i$ is a local ring and each $\Delta e_i / N e_i$ is a simple left Δ-module (see [1, proof of Theorem 27.6]). We define the *associated pre-ordered set* of Δ to be the set $Z = \{e_1, \ldots, e_n\}$ with the pre-ordering $e_i \leq e_j$ if and only if there is a sequence $e_i = e_{k_1}, \ldots, e_{k_l} = e_j$ in Z such that $e_{k_1} \Delta e_{k_2} \neq 0, \ldots, e_{k_{l-1}} \Delta e_{k_l} \neq 0$. The associated pre-ordered set Z relates to the ring Δ as follows.

LEMMA 1. *If Δ is indecomposable then Z is a connected pre-ordered set.*

Proof. Define another relation ρ on Z by $e_i \rho e_j$ if and only if there exists an e_k in Z such that $e_k \Delta e_i \neq 0$ and $e_k \Delta e_j \neq 0$. By the Block Decomposition Theorem (see [1, Theorem 7.9]), the transitive closure of ρ must have only one equivalence class. That is, for any $e_i, e_j \in Z$ there exists a sequence $e_i = e_{h_1}, \ldots, e_{h_p} = e_j$ in Z such that $e_{h_1} \rho e_{h_2} \rho \cdots \rho e_{h_p}$. Using the definition of ρ we obtain another sequence $e_{k_1}, \ldots, e_{k_{p-1}}$ in Z such that $e_{k_1} \Delta e_{h_1} \neq 0$, $e_{k_1} \Delta e_{h_2} \neq 0, \ldots,$ $e_{k_{p-1}} \Delta e_{h_{p-1}} \neq 0$, $e_{k_{p-1}} \Delta e_{h_p} \neq 0$. But this just means $e_i = e_{h_1} \geq e_{k_1} \leq e_{h_2} \geq \cdots \leq e_{h_{p-1}} \geq e_{k_{p-1}} \leq e_{h_p} = e_j$. Hence Z is connected. ∎

Let X be a locally finite pre-ordered set and let $R = I(X, \Delta)$. For each $x \in X$ and $i = 1, \ldots, n$ we define an idempotent $e_{xi} \in R$ by

$$e_{xi}(u, v) = \begin{cases} e_i & \text{if } u = v = x \\ 0 & \text{otherwise.} \end{cases}$$

We shall need the following lemma, describing the ordering on $X \times Z$ in terms of the idempotents e_{xi}.

LEMMA 2. *Let Z be the associated pre-ordered set of Δ. Then $(x, e_i) \leq (x', e_{i'})$ in $X \times Z$ if and only if there is a sequence $x = x_1, \ldots, x_r = x'$ in X and a sequence $e_i = e_{i_1}, \ldots, e_{i_r} = e_{i'}$ in Z such that $e_{x_1 i_1} R e_{x_2 i_2} \neq 0, \ldots, e_{x_{r-1} i_{r-1}} R e_{x_r i_r} \neq 0$.*

Proof. It suffices to show that $x \leq x'$ and $e_i \Delta e_{i'} \neq 0$ if and only if $e_{xi} R e_{x'i'} \neq 0$. If $x \leq x'$ and $e_i d e_{i'} \neq 0$ for some $d \in \Delta$, then by letting f be any element of R such that $f(x, x') = e_i d e_{i'}$, we see that $e_{xi} f e_{x'i'}(x, x') = e_i f(x, x') e_{i'} = e_i d e_{i'} \neq 0$. Conversely if $e_{xi} f e_{x'i'} \neq 0$ for some $f \in R$ then, since $e_{xi} f e_{x'i'}(u, v) = 0$ unless $u = x$ and $v = x'$, we have $e_{xi} f e_{x'i'}(x, x') \neq 0$. Thus $e_i f(x, x') e_{i'} \neq 0$ and $x \leq x'$. ∎

We are now ready to prove the main result of this section.

THEOREM 3. *Let Δ be an indecomposable semiperfect ring. Then Δ respects pre-order.*

Proof. Let X and Y be locally finite pre-ordered sets and let $R = I(X, \Delta)$ and $S = I(Y, \Delta)$. We must show that if $\theta\colon R \to S$ is an isomorphism then X and Y are order isomorphic. Let e_{xi} ($x \in X$ and $i = 1, \ldots, n$) be the above defined idempotents of R and let f_{yj} ($y \in Y$ and $j = 1, \ldots, n$) be those for S. The proof relies on the well behaved structure of the left *socle* of $R/J(R)$ and $S/J(S)$ which is, by definition, the sum of the minimal left ideals. For $x \in X$ and $i = 1, \ldots, n$ we know that $e_{xi} R e_{xi}$ is a local ring because $e_{xi} R e_{xi}$ is isomorphic to $e_i \Delta e_i$ via

$$e_{xi} f e_{xi} \mapsto e_i f(x, x) e_i.$$

Thus by [1, Corollary 17.20], $Re_{xi}/J(R)e_{xi}$ is a simple left R-module. Hence $(Re_{xi} + J(R))/J(R) \cong Re_{xi}/J(R)e_{xi}$ is a minimal left ideal of $R/J(R)$. We claim that the left socle of $R/J(R)$ is

$$\operatorname{Soc} R/J(R) = \bigoplus_{x \in X, i = 1, \ldots, n} (Re_{xi} + J(R))/J(R).$$

Here the sum is direct because the $e_{xi} + J(R)$ are orthogonal idempotents of $R/J(R)$. We already have the $\supseteq$ inclusion. To prove the other, let $T/J(R)$ be a minimal left ideal of $R/J(R)$. For some $x \in X$ and $1 \leq i \leq n$, $e_{xi} T \nsubseteq J(R)$ since otherwise we would have $T \subseteq J(R)$ by Section 1, Proposition 4. It then follows that

$$T/J(R) \cong (Re_{xi} + J(R))/J(R).$$

If also $e_{x'i'}T \nsubseteq J(R)$, then we obtain

$$(Re_{xi} + J(R))/J(R) \cong (Re_{x'i'} + J(R))/J(R)$$

which implies that $Re_{xi} \cong Re_{x'i'}$ by [1, Proposition 17.18]. Then $e_{x'i'}Re_{xi} \neq 0$ and $e_{xi}Re_{x'i'} \neq 0$, and by Lemma 2 above, $x \sim x'$. Let $\alpha \in \hat{X}$ be the equivalence class containing x. Since $e_\alpha = \Sigma e_{uj}$ ($u \in \alpha$ and $j = 1, \ldots, n$), we have $e_\alpha T \nsubseteq J(R)$ and $e_\beta T \subseteq J(R)$ if $\beta \neq \alpha$. Hence $e_\alpha T$ equals T modulo $J(R)$ and recalling from Section 2 that e_α is central modulo the radical, we see that

$$\begin{aligned} T/J(R) &= (e_\alpha T + J(R))/J(R) \subseteq (Re_\alpha + J(R))/J(R) \\ &= \bigoplus_{u \in \alpha, j=1,\ldots,n} (Re_{uj} + J(R))/J(R) \end{aligned}$$

and so $T/J(R) \subseteq \oplus (Re_{xi} + J(R))/J(R)$ ($x \in X$ and $i = 1, \ldots, n$), proving our claim. Similarly the left socle of $S/J(S)$ is

$$\operatorname{Soc} S/J(S) = \bigoplus_{y \in Y, j=1,\ldots,n} (Sf_{yj} + J(S))/J(S).$$

Let $\bar{\theta}$: $R/J(R) \to S/J(S)$ be the isomorphism given by $f + J(R) \mapsto \theta(f) + J(S)$. Since $\bar{\theta}$ must take the socle of $R/J(R)$ onto the socle of $S/J(S)$,

$$\bigoplus_{x \in X, i=1,\ldots,n} (S\theta(e_{xi}) + J(S))/J(S) = \bigoplus_{y \in Y, j=1,\ldots,n} (Sf_{yj} + J(S))/J(S).$$

By [1, Exercise 11.11] these two semisimple decompositions are equivalent, that is, letting $Z = \{e_1, \ldots, e_n\}$ be the associated pre-ordered set for Δ, there exists a set bijection σ: $X \times Z \to Y \times Z$ such that

$$(S\theta(e_{xi}) + J(S))/J(S) \cong (Sf_{yj} + J(S))/J(S) \quad \text{when } \sigma(x, e_i) = (y, e_j).$$

These isomorphisms lift to (see [1, Proposition 17.18])

$$S\theta(e_{xi}) \cong Sf_{yj} \text{ and } \theta(e_{xi})S \cong f_{yj}S \quad \text{when } \sigma(x, e_i) = (y, e_j),$$

as left and right S-modules respectively. Hence, as in the proof of Section 2, Theorem 1, when $\sigma(x, e_i) = (y, e_j)$ and $\sigma(x', e_{i'}) = (y', e_{j'})$ we have $e_{xi}Re_{x'i'} \neq 0$ implies that $f_{yj}Sf_{y'j'} \neq 0$. Then by Lemma 2, σ is order preserving. Since $\sigma(x, e_i) = (y, e_j)$ implies that

$$(Re_{xi} + J(R))/J(R) \cong (R\theta^{-1}(f_{yj}) + J(R))/J(R),$$

a similar argument shows that σ^{-1} preserves order. Thus σ: $X \times Z \to Y \times Z$ is an order isomorphism. Now by Lemma 1, Z is connected. Hence X is order isomorphic to Y by Section 3, Theorem 4. ∎

Remarks. The condition that Z be finite and connected in Section 3, Theorem 4 may be weakened to the assumption that Z consists of a finite number of finite connected components which are pairwise order isomorphic to each other. The proof of this follows by grouping the components of $X \times Z$ and $Y \times Z$ into isomorphism classes and applying Section 3, Theorem 4. This allows us to modify the above Theorem 3 to show that if Δ is a finite product of indecomposable semiperfect rings which have pairwise order isomorphic associated pre-ordered sets, then Δ respects pre-order. If those associated pre-ordered sets are only pairwise order isomorphic modulo $\sim$, then Δ respects partial order.

For example, a semisimple ring or a finite product of indecomposable serial rings that are not factors of upper triangular matrix rings respects partial order, because each factor has a singleton associated pre-ordered set modulo $\sim$ (see Murase [6]). We remark that we do not know whether the class of rings that respect partial order is closed under finite products. Hence it is an open question whether (decomposable) artinian rings respect partial order.

Acknowledgement

This paper constitutes a portion of a Ph.D. thesis written under the supervision of Professors K. R. Fuller and V. Camillo to be submitted to the graduate faculty of the University of Iowa. The author wishes to express his gratitude to Professors Fuller and Camillo and to Professor I. Kaplansky for their encouragement and helpful suggestions.

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